Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 6

Voluntary problem sheet

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1. Let X be a normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a *weak Cauchy sequence* if for every $\varphi \in X'$ the sequence $(\varphi(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$.

   (a) Let $x = (x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Show that x is a weak Cauchy sequence if and only if there exists as dense subset U' of X' such that $(\varphi(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence for every $\varphi \in U'$.

   (b) Every weak Cauchy sequence in X is bounded.

2. Let X be a Banach space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, $(\varphi_n)_{n\in\mathbb{N}} \subseteq X'$, and $x_0 \in X$, $\varphi_0 \in X'$ such that $x_n \xrightarrow{\|\cdot\|} x_0$ and $\varphi_n \xrightarrow{w*} \varphi_0$. Show that $\lim\limits_{n\to\infty} \varphi_n(x_n) = \varphi_0(x_0)$.

3. For $\lambda \in \mathbb{R}$ define $f_\lambda : \mathbb{R} \to \mathbb{C}$, $f_\lambda(s) = \mathrm{e}^{\mathrm{i}\lambda s}$ and let $X = \operatorname{span}\{f_\lambda : \lambda \in \mathbb{R}\}$. Show that

$$\langle f\,, g\rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s)\overline{g(s)}\,\mathrm{d}s$$

   defines an inner product on X. Show that the completion of X is not separable. ($\|f_\lambda - f_{\lambda'}\| =?$)

   Elements in the closure of X are called *almost periodic functions*.

4. Let X be a vector space with a semidefinite sesquilinear form $\langle\cdot\,,\cdot\rangle$. Show that the Cauchy-Schwarz inequality

$$|\langle x\,, y\rangle|^2 \le \langle x\,, x\rangle\langle y\,, y\rangle, \qquad x, y \in X, \qquad (*)$$

   holds. Does equality in $(*)$ imply that x and y are linearly dependent?