Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 5

Open mapping theorem; closed graph theorem. Hand in: February 26, 2010

Let X, Y, Z Banach spaces, $T : X \supseteq \mathcal{D}(T) \to Y$ a linear operator.

(a) Let $S : X \supseteq \mathcal{D}(S) \to Y$ be a linear operator. Then the *operator sum* $S + T$ is defined by

$$\mathcal{D}(S+T) := \mathcal{D}(S) \cap \mathcal{D}(T), \qquad (S+T)x := Sx + Tx.$$

(b) Let $R : Y \supseteq \mathcal{D}(R) \to Z$ be a linear operator. Then the *operator product* or *composition* RT is defined by

$$\mathcal{D}(RT) := \{x \in \mathcal{D}(T) : Tx \in \mathcal{D}(R)\}, \qquad (RT)x := R(Tx).$$

1. Let X, Y be Banach spaces, $T \in L(X,Y)$ and $S : X \supseteq \mathcal{D}(S) \to Y$ a closed linear operator. Show that $S + T$ is a closed linear operator.

2. Let X, Y, Z be Banach spaces, $T : X \supseteq \mathcal{D}(T) \to Y$, $S : Y \supseteq \mathcal{D}(S) \to Z$ closed linear operators. Show:

   (a) If T is continuous, then ST is closed.

   (b) If S is continuously invertible (i. e., $S^{-1} : \mathrm{R}(S) \to Y$ exists and is continuous), then ST is closed.

   The statements hold also if “closed” is replaced by “closable”.

3. Let $X = \ell_2(\mathbb{N})$ and

   $$T : X \supseteq \mathcal{D}(T) \to X, \qquad Tx = (nx_n)_{n\in\mathbb{N}} \quad \text{for} \quad x = (x_n)_{n\in\mathbb{N}}.$$

   Check if T is closed if

   (a) $\mathcal{D}(T) = \{x = (x_n)_{n\in\mathbb{N}} \in \ell_2(\mathbb{N}) \,:\, (nx_n)_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})\}$,

   (b) $\mathcal{D}(T) = d = \{x = (x_n)_{n\in\mathbb{N}} \in \ell_2(\mathbb{N}) \,:\, x_n \neq 0$ for only finitely many $n\}$.

4. Let X be a Banach space, $n \in \mathbb{N}$ and T a densely defined linear operator from X to K^n. Then T is closed if and only $T \in L(X, \mathbb{K}^n)$.