Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 4

Baire's theorem; uniform boundedness principle. Hand in: February 19, 2010

---

1. (a) Let (M,d) be a complete metric space with infinitely many elements and no isolated points Then M is not countable.
   (b) Every algebraic basis of an infinite dimensional Banach space is uncountable.

2. (a) Let X be a Banach space, Y be a normed space and $(T_n)_{n\in\mathbb{N}} \subseteq L(X,Y)$. Assume that for all $x \in X$ the limit $Tx := \lim\limits_{n\in\mathbb{N}} T_n x$ exists. Then $T \in L(X,Y)$.
   (b) Let X, Y be Banach spaces, Y reflexive, and $(T_n)_{n\in\mathbb{N}} \subseteq L(X,Y)$ such that $(\varphi(T_n x))_{n\in\mathbb{N}}$ converges for every $x \in X$ and $\varphi \in Y'$. Then there exists an $T \in L(X,Y)$ such that $T_n \xrightarrow{w} T$.

3. For a sequence $(s_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ the following is equivalent:
   (a) $\sum\limits_{n=1}^{\infty} s_n$ converges absolutely.
   (b) $\sum\limits_{n=1}^{\infty} s_n t_n$ converges for every sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{K}$ that converges to 0.

4. Let $[a,b] \subseteq \mathbb{R}$, $n \in \mathbb{N}$ and choose $a \le t_1^{(n)} < \cdots < t_n^{(n)} \le b$ and $\alpha_k^{(n)} \in \mathbb{K}$, $k = 1, \dots, n$. For $f \in C([a,b])$ define

$$Q_n(f) := \sum_{k=1}^{n} \alpha_k^{(n)}\ f(t_k^{(n)}).$$

   Show that the following is equivalent:
   (a) $Q_n(f) \to \int_a^b f(t)\,\mathrm{d}t,\ n \to \infty$, for all $f \in C[a,b]$.
   (b) $Q_n(p) \to \int_a^b p(t)\,\mathrm{d}t,\ n \to \infty$, for every polynomial $p : [a,b] \to \mathbb{K}$ and $\sup\limits_{n\in\mathbb{N}} \sum_{k=1}^{n} |\alpha_k^{(n)}| < \infty$.