Functional Analysis

Problem Sheet 3

Dual space.

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1. In $X = \ell_2(\mathbb{N})$ consider the subspace

$$U = \{(x_n)_{n\in\mathbb{N}} : x_n \neq 0 \text{ for at most finitely many } n\}.$$

Let V be an algebraic complement of U in X, i. e., U is a subspace such that $U + V = X$ and $U \cap V = \{0\}$. Show that

$$\varphi : X \to \mathbb{K}, \quad \varphi(x) = \sum_{n=0}^{\infty} u_n \qquad \text{for } x = u + v \text{ with } u \in U,\ v \in V.$$

is well-defined, linear and not bounded.

2. Let $\ell_\infty(\mathbb{N}, \mathbb{R})$ the set of all bounded sequences in $\mathbb{R}$ with the supremum norm. Show that there exists an $\varphi \in (\ell_\infty(\mathbb{N}, \mathbb{R}))'$ such that

$$\liminf_{n\to\infty} x_n \leq \varphi(x) \leq \limsup_{n\to\infty} x_n, \qquad x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty.$$

3. Let X be a separable normed space and $(x'_n)_{n\in\mathbb{N}}$ a bounded sequence in X'. Then there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ and a $x'_0 \in X'$ such that

$$\lim_{k\to\infty} x'_{n_k}(x) = x'_0(x), \qquad x \in X.$$

Can the statement be proved without the assumption that X is separable?

4. An *isomorphism between normed spaces* X and Y is a linear homeomorphism.
   (a) If $T : X \to Y$ is an [isometric] isomorphism between the normed spaces X and Y, then $T' : Y' \to X'$ is an [isometric] isomorphism. If X and Y are Banach spaces, then the reverse is also true.
   (b) If a normed space Y is isomorphic to a reflexive Banach space X, then Y is a reflexive Banach space.