Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 2

Bounded linear operators; Hahn-Banach theorem. Hand in: February 5, 2010

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1. Let X be a normed space with $\dim X \geq 1$ and S, T linear operators on X such that $ST - TS = \mathrm{id}$. Show that at least one of the operators S and T is unbounded.

   Hint. Show that $ST^{n+1} - T^{n+1}S = (n+1)T^n$.

2. Let $1 \leq p < \infty$. For $z = (z_n)_{n\in\mathbb{N}} \in \ell_\infty$ let $T : \ell_p \to \ell_p$ defined by $(Tx)_n = x_n z_n$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell_p$. Show that $T \in L(\ell_p)$ and find $\|T\|$.

3. Let X be a normed space. Show that X is separable if X' is separable.

4. Show the Hahn-Banach theorem for a complex vector space.

   Suggestion: For a complex vector space X show:

   (a) Let $\varphi : X \to \mathbb{R}$ be an $\mathbb{R}$-linear functional. Then

   $$V_\varphi : X \to \mathbb{C}, \quad V_\varphi(x) := \varphi(x) - \mathrm{i}\varphi(\mathrm{i}x),$$

   is a $\mathbb{C}$-linear functional on X with $\mathrm{Re}V_\varphi = \varphi$.

   (b) Let $\lambda : X \to \mathbb{C}$ a $\mathbb{C}$-linear functional with $\mathrm{Re}\lambda = \varphi$. Then $V_\varphi = \lambda$.

   (c) Let p be a sublinear functional on X and φ, V_φ as above. Then

   $$|\varphi(x)| \leq p(x) \iff |V_\varphi(x)| \leq p(x), \qquad x \in X.$$

   (d) $\|\varphi\| = \|V_\varphi\|$ for φ and V_φ as above.