Universidad de los Andes
MATE-2201

Functional Analysis

Problem Sheet 1

Metric and normed spaces. Hand in: January 29, 2010

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1. **Banach's fixed point theorem.** Let M be a metric space. A map $f : M \to M$ is called a *contraction* if there exists a $\gamma < 1$ such that

$$d(f(x),\, f(y)) \leq \gamma\, d(x,\, y), \qquad x, y \in M.$$

Show that every contraction f on a complete normed space M has exactly one fixed point, that is, there exists exactly one $x_0 \in M$ such that $f(x_0) = x_0$.

2. Let X be a normed space. Show:

(a) Every finite-dimensional subspace of X is closed.

(b) If V is a finite-dimensional subspace of X and W is a closed subspace of X, then

$$V + W := \{v + w : v \in V,\ w \in W\}$$

is a closed subspace of X.

3. Let T be a set and $\ell_\infty(T)$ be the space of all functions $x : T \to \mathbb{K}$ with

$$\|x\|_\infty := \sup\{|x(t)| : t \in T\} < \infty.$$

Show that $(\ell_\infty(T), \|\cdot\|_\infty)$ is a Banach space.

4. Let the sequence spaces d, c_0, c be defined as in Example 1.10.

(a) Show that $(c_0, \|\cdot\|_\infty)$ and $(c, \|\cdot\|_\infty)$ are Banach spaces.

(b) Show that $(d, \|\cdot\|_\infty)$ is a normed space, but that it is not complete.