Functional Analysis

Analysis Series

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Contents

Notation

The letter $\mathbb{K}$ usually denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. The positive real numbers are denoted by $\mathbb{R}_+ := (0, \infty)$.

Chapter 1

Banach spaces

1.1 Metric spaces

We repeat the definition of a metric space.

Definition 1.1. A *metric space* (M, d) is a non-empty set M together with a map

$$d : M \times M \to \mathbb{R}$$

such that for all $x, y, z \in M$:

(i) $d(x, y) = 0 \iff x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

The last inequality is called *triangle inequality*. Usually the metric space (M, d) is denoted simply by M.

Note that the triangle inequality together with the symmetry of d implies

$$d(x, y) \geq 0, \qquad x, y \in M,$$

since $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$.
It is easy to check that

$$|d(x, y) - d(y, z)| \leq d(x, z), \qquad x, y, z \in M.$$

A subset $N \subseteq M$ is called *bounded* if

$$\operatorname{diam} N := \sup\{d(x, y) : x, y \in N\} < \infty.$$

Let $r > 0$ and $x \in M$. Then

$$\begin{aligned} B_r(x) &:= \{y \in M : d(x, y) < r\} =: \text{ open ball with centre } x \text{ and radius } r, \\ K_r(x) &:= \{y \in M : d(x, y) \leq r\} =: \text{ closed ball with centre } x \text{ and radius } r, \\ S_r(x) &:= \{y \in M : d(x, y) = r\} =: \text{ sphere with centre } x \text{ and radius } r. \end{aligned}$$

Examples.

- $\mathbb{R}$ with the $d(x, y) = |x - y|$ is a metric space.
- Let X be a set and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = 0$ for $x = y$ and $d(x, y) = 1$ for $x \neq y$. Then (X, d) is a metric space. d is called the *discrete metric* on X.

Let (M, d) be a metric space. Recall that the metric d induces a topology on M: a set $U \subseteq M$ is open if and only if for every $p \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U$. In particular, the open balls are open and closed balls are closed subsets of M. Let $x \in M$. A subset $U \subseteq M$ is called a *neighbourhood* of x if there exists an open set U_x such that $x \in U_x \subseteq U$.
It is easy to see that the topology generated by d has the Hausdorff property, that is, for every $x \neq y \in M$ there exist neighbourhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$.
Recall that a set $N \subseteq M$ is called *dense* in M if $\overline{N} = M$, where $\overline{N}$ denotes the *closure* of N.

Definition 1.2. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ *converges* to $x \in M$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$, that is,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \quad n \geq N \implies d(x_n, x) < \varepsilon.$$

The limit x is unique. A sequence $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* in M if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \quad m, n \geq N \implies d(x_n, x_m) < \varepsilon.$$

Definition 1.3. A metric space in which every Cauchy sequence is convergent, is called a *complete metric space.*

Definition 1.4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces.

(i) A function $f : X \to Y$ is called *continuous* if and only if $f^{-1}(U)$ is open in X for every U open in Y.

(ii) An bijective function $f : X \to Y$ is called a *homeomorphism* if and only if f and f^{-1} are contiunous.

The following lemma is often useful.

Lemma 1.5. *Let (M, d) be a complete metric space and $N \subseteq M$. Then N is closed in M if and only if $(N, d|_M)$ is complete.*

Remarks.

- Every convergent sequence is a Cauchy sequence.
- Every Cauchy sequence is bounded. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ is bounded if the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

Not every metric space is complete, but every metric space can be completed in the following sense.

Definition 1.6. Let (M, d_M) and (N, d_N) be metric spaces. A map $f : M \to N$ is called an *isometry* if and only if $d_N(f(x), f(y)) = d_M(x, y)$ for all $x, y \in M$. The spaces M and N are called isometric if there exists a bijective isometry $f : M \to N$.

Note that an isometry is necessarily injective since $x \neq y$ implies $f(x) \neq f(y)$ because $d(f(x), f(y)) = d(x,y) \neq 0$, and that every isometry is continuous.

Theorem 1.7. *Let (M, d) be a metric space. Then there exists a complete metric space $(\widehat{M}, \widehat{d})$ and an isometry $\varphi : M \to \widehat{M}$ such that $\overline{\varphi(M)} = \widehat{M}$. $\widehat{M}$ is called* completion *of M; it is unique up to isometry.*

Proof. Let

$$\mathcal{C}_M := \{(x_n)_{n\in\mathbb{N}} \subseteq M : (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } M\}$$

be the set of all Cauchy sequences in M. We define the equivalence relation $\sim$ on $\mathcal{C}_M$ by

$$x \sim y \qquad \iff \qquad d(x_n, y_n) \to 0,\ n \to \infty$$

for all $x = (x_n)_{n\in\mathbb{N}},\ y = (y_n)_{n\in\mathbb{N}} \in \mathcal{C}_M$. It is easy to check that $\sim$ is indeed a equivalence relation (reflexivity and symmetry follow directly from properties (i) and (ii) of the definition of a metric and transitivity of $\sim$ is a consequence of the triangle inequality).

Let $\widehat{M} := \mathcal{C}_M / \sim$ the set of all equivalence classes. The equivalence class containing $x = (x_n)_{n\in\mathbb{N}}$ is denoted by $[x]$. On $\widehat{M}$ we define

$$\widehat{d} : \widehat{M} \times \widehat{M} \to \mathbb{R}, \qquad \widehat{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n). \tag{1.1}$$

We have to show that $\widehat{d}$ is well-defined.
Let $(x_n)_{n\in\mathbb{N}} \in [x]$ and $(y_n)_{n\in\mathbb{N}} \in [y]$. Then

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &\leq |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \\ &\leq d(x_n, x_m) + d(y_n, y_m) \to 0, \qquad m, n \to \infty. \end{aligned}$$

Since $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $\mathbb{R}$, the limit in (1.1) exists.
Moreover, for $(\tilde{x}_n)_{n\in\mathbb{N}} \in [x]$ and $(\tilde{y}_n)_{n\in\mathbb{N}} \in [y]$ it follows that

$$\begin{aligned} |d(x_n, y_n) - d(\tilde{x}_n, \tilde{y}_n)| &\leq |d(x_n, y_n) - d(\tilde{x}_n, y_n)| + |d(\tilde{x}_n, y_n) - d(\tilde{x}_n, \tilde{y}_n)| \\ &\leq d(x_n, \tilde{x}_n) + d(y_n, \tilde{y}_n) \to 0, \qquad n \to \infty. \end{aligned}$$

Hence $\widehat{d}$ is well-defined.
Let

$$\varphi : M \to \widehat{M}, \qquad \varphi(x) = [(x)_{n\in\mathbb{N}}].$$

We will show that $(\widehat{M}, \widehat{d})$ is a complete metric space, that φ is an isometry and that $\overline{\varphi(M)} = \widehat{M}$ in several steps.
Step 1: $(\widehat{M}, \widehat{d})$ is a metric space.
Proof. Let $[x], [y], [z] \in \widehat{M}$. Then

- $0 = \widehat{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n) \quad \iff \quad x \sim y \quad \iff \quad [x] = [y].$

- $\widehat{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \widehat{d}([y],[x])$.
- $\widehat{d}([x],[y]) = \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n, z_n) + d(z_n, y_n) = \widehat{d}([x],[z]) + \widehat{d}([z],[y])$.

Step 2: φ is an isometry.
Proof. This follows immediately from the definition.

Step 3: $\overline{\varphi(M)} = \widehat{M}$.
Proof. Let $(x_n)_{n\in\mathbb{N}} \in [x] \in \widehat{M}$ and $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$, $m, n \ge N$. Let $z := x_N \in M$. Then

$$\widehat{d}(\varphi(z), [x]) = \lim_{n\to\infty} d(x_N, x_n) \le \frac{\varepsilon}{2} < \varepsilon.$$

Next we show that $(\widehat{M}, \widehat{d})$ is complete. Let $(\hat{x}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widehat{M}$. Since $\varphi(M)$ is dense in $\widehat{M}$ there exists a sequence $z = (z_n)_{n\in\mathbb{N}} \subseteq M$ such that

$$\widehat{d}(\widehat{x}_n, z_n) < \frac{1}{n}, \qquad n \in \mathbb{N}.$$

The sequence z is a Cauchy sequence in M because

$$\begin{aligned} d(z_n, z_m) = \widehat{d}(\varphi(z_n),\, \varphi(z_m)) &\le \widehat{d}(\varphi(z_n),\, \hat{x}_n) + \widehat{d}(\hat{x}_n,\, \hat{x}_m) + \widehat{d}(\hat{x}_m,\, \varphi(z_m)) \\ &< \frac{1}{n} + \widehat{d}(\hat{x}_n,\, \hat{x}_m) + \frac{1}{m} \to 0, \qquad m, n \to \infty. \end{aligned}$$

The sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ converges to $[z]$ because

$$\widehat{d}(\hat{x}_n, z) \le \widehat{d}(\hat{x}_n, \varphi(z_n)) + \widehat{d}(\varphi(z_n), z) < \frac{1}{n} + \lim_{m\to\infty} d(z_n, z_m) \to 0, \qquad n \to \infty.$$

We have shown that $\varphi(M)$ is a dense subset of the complete metric space $(\widehat{M}, \widehat{d})$ and that φ is an isometry.

Finally, we have to show that $\widehat{M}$ is unique (up to isometry). Let (N, d_N) be complete metric space and $\psi : M \to N$ an isometry such that $\overline{\psi(M)} = N$. Then the map

$$T : \varphi(M) \to \psi(M), \quad T(\varphi(x)) = \psi(x)$$

can be extended to a surjective isometry $\overline{T} : \overline{\varphi(M)} = \widehat{M} \to N$ by

$$\overline{T}x = \overline{T}(\lim_{n\to\infty} x_n) := \lim_{n\to\infty} Tx_n$$

for $x = \lim_{n\to\infty} x_n$ with $x_n \in \varphi(M)$, $n \in \mathbb{N}$. □

Examples.

- $\mathbb{C}^n$ with $d(x,y) = \max\{|x_j - y_j| : j = 1, \dots, n\}$ is a complete metric space.
- $\mathbb{C}^n$ with $d(x,y) = \sqrt{|x_1 - y_1|^2 + \dots + |x_n - y_n|^2}$ is a complete metric space.

- Let $C([a,b])$ be the set of all continuous functions on the interval $[a,b]$. For $f,g \in C([a,b])$ let

$$d_1(f,g) := \max\{|f(x) - g(x)| : x \in [a,b]\},$$
$$d_2(f,g) := \int_a^b |f(x) - g(x)| \, \mathrm{d}x.$$

Then d_1 and d_2 are metrics on $C([a,b])$. $(C([a,b]), d_1)$ is complete, $(C([a,b]), d_2)$ is not complete.

Remark. The completion of $(C([a,b]), d_2)$ is $L_1(a,b)$ (the set of all Lebesgue integrable functions on (a,b)).

Definition 1.8. A metric space is called *separable* if it contains a countable dense subset.

Proposition 1.9. *Let (M,d) be a separable metric space and $N \subseteq M$. Then N is separable.*

Proof. We have to show that there exists a countable set $B \subseteq N$ such that $\overline{B} \supseteq N$ where the closure is taken with respect to the metric on M. By assumption on M there exists a countable set $A := \{x_n : n \in \mathbb{N}\} \subseteq M$ such that $\overline{A} = M$. Let $J := \{(n,m) \in \mathbb{N} \times \mathbb{N} : \exists y \in N \text{ with } d(x_n, y) < \frac{1}{m}\}$. For every $(n,m) \in J$ choose a $y_{n,m} \in N$ and define $B := \{y_{n,m} : (n,m) \in J\}$. Obviously, B is a countable subset of N. To show that B is dense in N it suffices to show that for every $y \in N$ and $k \in \mathbb{N}$ there exists a $b \in B$ such that $d(b,y) < \frac{1}{k}$. By definition of A there exists a $x_n \in A$ such that $d(x_n, y) < \frac{1}{2k}$. In particular, $(n, 2k) \in J$. It follows that $d(y_{n,2k}, y) \le d(y_{n,2k}, x_n) + d(x_n, y) < \frac{1}{k}$. □

1.2 Normed spaces

Definition 1.10. Let X be a vector space over $\mathbb{K}$. A *norm* on X is a map

$$\|\cdot\| : X \to \mathbb{R}$$

such that for all $x, y \in X$, $\alpha \in \mathbb{K}$

(i) $\|x\| = 0 \iff x = 0$,
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$,
(iii) $\|x + y\| \le \|x\| + \|y\|$.

Remarks.
- Note that the implication $\Leftarrow$ in (i) follows from (ii) because $\|0\| = \|2 \cdot 0\| = 2\|0\|$.
- Note that $\|x\| \ge 0$ for all $x \in X$ because $0 = \|x - x\| \le 2\|x\|$. The last inequality follows from the triangle inequality (iii) and (ii) with $\alpha = -1$.

Remark. A function $[\,\cdot\,] : X \to \mathbb{R}$ which satisfies only (ii) and (iii) of Definition 1.10 is called a *seminorm*. As seen in the remark above for norms, a seminorm is non-negative and satisfies $[0] = 0$.

Remark. A norm on X induces a metric on X by setting

$$d(x,y) := \|x - y\|, \qquad x, y \in X.$$

Hence a norm induces a topology on X via the metric and we have the concept of convergence etc. on a normed space.

Definition 1.11. A complete normed space is called a *Banach space*.

Obviously, every subspace of a normed space is a normed space by restriction of the norm. A subspace of a Banach space is a Banach space if and only if it is closed.

Proposition 1.12. *Let X be a normed space. Then the following is equivalent:*

(i) *X is complete.*

(ii) *Every absolutely convergent series in X converges in X.*

Proof. Exercise 1.2. □

Example 1.13 (Quotient space). Let X be a Banach space and $M \subseteq X$ a closed subspace. On X we have the equivalence relation

$$x \sim y \qquad \Longleftrightarrow \qquad x - y \in M.$$

For $x \in X$ we denote the equivalence class of X/M containing x by $[x]$. Then X/M is a vector space if we set

$$[x] + [y] := [x + y], \quad \alpha[x] := [\alpha x], \qquad x, y \in X,\ \alpha \in \mathbb{K}.$$

For $x \in X$ let $\operatorname{dist}(x, M) := \inf\{\|x - m\| : m \in M\}$.

- $(X/M, \|\cdot\|_\sim)$ is a normed space with

$$\|\cdot\|_\sim : X/M \to \mathbb{R}, \quad \|[x]\|_\sim := \operatorname{dist}(x, M).$$

Proof. First we show that $\|\cdot\|_\sim$ is well-defined. For $x, y \in X$ with $x - y \in M$ we find

$$\begin{aligned} \operatorname{dist}(x, M) = \inf\{\|x - m\| : m \in M\} &= \inf\{\|y - (\overbrace{y - x}^{\in M} + m)\| : m \in M\} \\ &= \inf\{\|y - m\| : m \in M\} = \operatorname{dist}(y, M). \end{aligned}$$

Property (ii) in the definition of a norm is easily checked. For property (iii) let $[x], [y] \in X/M$. Then

$$\begin{aligned} \|[x] + [y]\|_\sim = \|[x + y]\|_\sim &= \inf\{\|x + y - m\| : m \in M\} \\ &= \inf\{\|x - m_x + y - m_y\| : m_x, m_y \in M\} \\ &\leq \inf\{\|x - m_x\| : m_x \in M\} + \inf\{\|y - m_y\| : m_y \in M\} \\ &= \|[x]\|_\sim + \|[y]\|_\sim. \end{aligned}$$

It is clear that $[x] = 0$ implies $\|[x]\|_\sim = 0$. Now assume that $\|[x]\|_\sim = 0$. We have to show that $x \in M$. By definition of dist there exists a sequence $(m_n)_{n\in\mathbb{N}}$ such that $\|x - m_n\| \to 0$, that is, $(m_n)_{n\in\mathbb{N}}$ converges to x. Since M is closed, it follows that $x \in M$. □

- Let X be a Banach space and M a closed subspace. Then X/M is Banach space with the norm defined in Example 1.13.

  Proof. We already saw that X/M is normed space. It remains to prove completeness. Let $([x_n])_{n\in\mathbb{N}}$ be a Cauchy sequence.
  First we show that we can assume $\|[x_n]-[x_m]\|_\sim \leq 2^{-n}$ for all $m\geq n$: Choose $N_1\in\mathbb{N}$ such that $\|[x_{N_1}]-[x_m]\|_\sim \leq 2^{-1}$ for all $m\geq N_1$. Next choose $N_2>N_1$ such that $\|[x_{N_2}]-[x_m]\|_\sim \leq 2^{-2}$ for all $m\geq N_2$. Continuing this process, we obtain a subsequence with the desired property. Since a Cauchy sequence converges if and only if it contains a convergent subsequence, it suffices to prove convergence of the subsequence constructed above.
  By definition of the quotient norm we can assume that $\|x_n-x_{n+1}\|\leq \|[x_n-x_{n+1}]\|_\sim + 2^{-n} < 2^{1-n}$. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence in X because for all $n>m$

  $$\|x_n-x_m\| = \Big\|\sum_{j=m}^{n-1} x_{n+1}-x_n\Big\| \leq \sum_{j=m}^{n-1}\|x_{n+1}-x_n\| < 2\sum_{j=m}^{n-1}2^{-j}.$$

  Therefore $x:=\lim_{n\to\infty}x_n$ exists and

  $$\|[x_n]-[x]\|_\sim = \|[x_n-x]\|_\sim \leq \|x_n-x\| \to 0, \quad n\to\infty. \qquad \square$$

Remark 1.14. (i) In the proof above we used that, by definition of $\|\cdot\|_\sim$, for every $x\in X$ and every $\varepsilon>0$ there exists an $\widetilde{x}\in[x]$ such that $\|\widetilde{x}\| < \|[x]\|_\sim+\varepsilon$. Equivalently, there exists an $m\in M$ such that $\|x+m\|<\|[x]\|_\sim+\varepsilon$.

(ii) Obviously, $\|x\|\geq\|[x]\|_\sim$ for every $x\in X$.

Examples 1.15. (i) *Finite dimensional normed spaces.* $\mathbb{C}^n$ and $\mathbb{R}^n$ are complete normed spaces with

$$\|\cdot\|_\infty:\mathbb{K}^n\to\mathbb{R},\quad \|x\|_\infty=\max\{|x_j|: j=1,\dots,n\}.$$

Let $1\leq p<\infty$. Then $\mathbb{C}^n$ and $\mathbb{R}^n$ are complete normed spaces with

$$\|\cdot\|_p:\mathbb{C}^n\to\mathbb{R},\quad \|x\|_p=\Big(\sum_{j=1}^n|x_j|^p\Big)^{\frac{1}{p}}.$$

The triangle inequality $\|x+y\|_p\leq\|x\|_p+\|y\|_p$ is called the *Minkowski inequality* (see Section 1.3).

(ii) Let T be a set and define

$$\ell_\infty(T):=\{x:T\to\mathbb{K}\text{ bounded map}\}.$$

Obviously, $\ell_\infty(T)$ is a vector space. Let

$$\|x\|_\infty:=\sup\{|x(t)|:t\in T\},\qquad x\in\ell_\infty,$$

be the *supremum norm.* Then $(\ell_\infty(T),\|\cdot\|_\infty)$ is a Banach space.

Proof. Exercise 1.4. □

(iii) *Sequence spaces.*

- $\ell_\infty := \ell_\infty(\mathbb{N})$ is a Banach space.
- For $1 \le p < \infty$ let

$$\ell_p := \ell_p(\mathbb{N}) := \Big\{ (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} \;:\; \sum_{n=1}^{\infty} |x_n|^p < \infty \Big\}$$

and

$$\|x\|_p := \Big(\sum_{n=1}^{\infty} |x_n|^p \Big)^{\frac{1}{p}}, \qquad x \in \ell_p.$$

With the usual component-by-component addition and multiplication with a scalar, ℓ_p is a vector space and $(\ell_p, \|\cdot\|_p)$ is a Banach space.

Proof. First we show that ℓ_p is a vector space. For $\alpha \in \mathbb{K}$ and $x, y \in \ell_p$ we have

$$\sum_{n=1}^{\infty} |\alpha x_n|^p = |\alpha|^p \sum_{n=1}^{\infty} |x_n|^p < \infty$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n + y_n|^p &\le \sum_{n=1}^{\infty} \big(2 \max\{|x_n|, |y_n|\}\big)^p = 2^p \sum_{n=1}^{\infty} \big(\max\{|x_n|, |y_n|\}\big)^p \\ &\le 2^p \sum_{n=1}^{\infty} |x_n|^p + |y_n|^p = 2^p (\|x\|_p^p + \|y\|_p^p) < \infty.\end{aligned}$$

Hence ℓ_p is a $\mathbb{K}$-vector space. Properties (i) and (ii) in the definition of a norm are easily verified. The triangle inequality is the Minkowski inequality (see Section 1.3).
To show that $(\ell_p, \|\cdot\|_p)$ is complete, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ_p. Set $x_n = (x_{n,m})_{m\in\mathbb{N}}$. Then the sequence of the m-th components is a Cauchy sequence in $\mathbb{K}$ because

$$|x_{n,m} - x_{k,m}| < \|x_n - x_k\|_p, \qquad m \in \mathbb{N}.$$

Since $\mathbb{K}$ is complete, the limit $y_m := \lim\limits_{n\to\infty} x_{n,m}$ exists. Let $y := (y_m)_{m\in\mathbb{N}}$. We will show that $y \in \ell_p$ and that $x_n \xrightarrow{\|\cdot\|_p} y$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|x_n - x_k\| < \varepsilon$ for all $k, n \ge N$. For every $M \in \mathbb{N}$

$$\sum_{j=1}^{M} |x_{n,j} - x_{k,j}|^p \le \|x_n - x_k\|_p^p < \varepsilon^p.$$

Taking the limit $k \to \infty$ on the left hand side yields

$$\sum_{j=1}^{M} |x_{n,j} - y_j|^p < \varepsilon^p.$$

Taking the limit $M \to \infty$ on the left hand side finally gives

$$\sum_{j=1}^{\infty} |x_{n,j} - y_j|^p \leq \varepsilon^p < \infty,$$

in particular, $x_n - y \in \ell_p$. Since ℓ_p is a vector space, we obtain $y = x_n + (y - x_n) \in \ell_p$ and $\|x_n - y\|_p \leq \varepsilon$. That $(x_n)_{n\in\mathbb{N}}$ converges to y follows from the inequality above since ε can be chosen arbitrarily. $\square$

(iv) $\mathscr{L}_p$ *spaces*: See measure theory.

(v) *Subspaces of* ℓ_∞. Let

$$\begin{aligned} d &:= \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \ x_n \neq 0 \text{ for at most finitely many } n\}, \\ c_0 &:= \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \ \lim_{n\to\infty} x_n = 0\}, \\ c &:= \{x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{K} : \ \lim_{n\to\infty} x_n \text{ exists}\}, \end{aligned}$$

Obviously, the inclusions $d \subsetneq c_0 \subsetneq c \subsetneq \ell_\infty$ hold. Moreover, it can be shown that c_0 and c are closed subspaces of ℓ_∞ and that d is a non-closed subspace of ℓ_∞. In particular, $(c_0, \|\cdot\|_\infty)$ and $(c, \|\cdot\|_\infty)$ are Banach spaces, $(d, \|\cdot\|_\infty)$ is not a Banach space (see Exercise 1.5).

(vi) *Spaces of continuous functions.* For metric space T (e. g. an interval in $\mathbb{R}$) let

$$\begin{aligned} C(T) &:= \{f : T \to \mathbb{K} : f \text{ is continuous}\}, \\ B(T) &:= \{f : T \to \mathbb{K} : f \text{ is bounded}\}, \\ BC(T) &:= C(T) \cap B(T). \end{aligned}$$

For $f \in B(T)$ let

$$\|f\|_\infty := \sup\{|f(t)| : t \in T\}.$$

In Analysis 1 it was shown that $(B(T), \|\cdot\|_\infty)$ and $(BC(T), \|\cdot\|_\infty)$ are Banach spaces. Note that $C(T) = BC(T)$ for a compact metric space T.

(vii) *Spaces of differentiable functions.* Let $[a, b]$ a real interval. We can define several norms on the vector space

$$C^1([a, b]) := \{f : [a, b] \to \mathbb{K} : f \text{ is continuously differentiable}\}.$$

- $(C^1([a, b]), \|\cdot\|_\infty)$ is not a Banach space.

  Proof. For $n \in \mathbb{N}$ let $f_n : [-1, 1] \to \mathbb{K},\ f_n(t) := (t^2 + n^{-2})^{\frac{1}{2}}$. Then the f_n converge to $g : [-1, 1] \to \mathbb{K},\ g(t) = |t|$ in the $\|\cdot\|_\infty$-norm. But $g \notin C^1([a, b])$. Hence $C^1([a, b])$ is not closed as a subspace of the Banach space $C([a, b])$, so it is not a Banach space. $\square$

- For $f \in C^1([a,b])$ let
$$\|f\|_{(1)} := \|f\|_\infty + \|f'\|_\infty.$$
Then $(C^1([a,b]), \|\cdot\|_{(1)})$ is a Banach space. Note that the right hand side is finite because by assumption f' is continuous.

  Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(C^1([a,b]), \|\cdot\|_{(1)})$. Then there exist $x, y \in C([a,b])$ such that $x_n \to x$ and $x_n' \to y$ in the supremum norm. A well-known theorem in analysis implies $x' = y$, hence $x_n \to x$ in $\|\cdot\|_{(1)}$. □

  In the following, $C^1([a,b])$ will always be considered to be equipped with the norm $\|\cdot\|_{(1)}$ unless stated otherwise.

Theorem 1.16. *Let X be a Banach space, Y a closed subspace and N a finite dimensional subspace of X. Then $Y + N$ is a closed subspace. In particular, every finite-dimensional subspace is closed.*

Proof. Obviously, $Y + N$ is a subspace of X. To proof that it is closed, we proceed by induction. Therefore we can assume without restriction that $\dim N = 1$. Let $z \in X$ such that $N = \{\lambda z : \lambda \in \mathbb{K}\}$ and $(x_n)_{n\in\mathbb{N}} = (y_n + a_n z)_{n\in\mathbb{N}}$ a Cauchy sequence in $Y + N$.
*C*ase 1. $(a_n)_{n\in\mathbb{N}}$ is bounded. Then it contains a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Then the sequence $(y_{n_k})_{k\in\mathbb{N}} = (x_{n_k} - a_{n_k} z)_{k\in\mathbb{N}}$ converges because it is the sum of two convergent sequences.
*C*ase 2. $(a_n)_{n\in\mathbb{N}}$ is unbounded. Then there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} |a_{n_k}| = \infty$. Since $(x_{n_k})_{k\in\mathbb{N}}$ is bounded, it follows that
$$\left\| z + \frac{1}{a_{n_k}} y_{n_k} \right\| = \left\| \frac{1}{a_{n_k}} x_{n_k} \right\| \to 0, \qquad n \to \infty.$$
Hence $d(z, Y) = 0$. Since Y is closed, this implies $z \in Y$, therefore $N + Y = Y$ is closed in X.
Finally, choosing $Y = \{0\}$ shows that every finite-dimensional subspace is closed. □

Note that the sum of two closed subspaces is not necessarily closed, see as the following example shows. Another example can be found in [Hal98, § 15].

Example. In ℓ_1 consider the subspaces
$$U := \{(x_n)_{n\in\mathbb{N}} \in \ell_1 : x_{2n} = 0,\ n \in \mathbb{N}\}$$
$$V := \{(x_n)_{n\in\mathbb{N}} \in \ell_1 : x_{2n-1} = n x_{2n},\ n \in \mathbb{N}\}.$$
Obvioulsy, U and V are closed subspaces of ℓ_1. Let e_n be the nth unit vector in ℓ_1. Let $m \in \mathbb{N}$. Then $\mathrm{e}_{2m-1} \in U \subseteq V + U$ and $\mathrm{e}_{2m} = (\mathrm{e}_{2m} + \frac{1}{m}\,\mathrm{e}_{2m-1}) - \frac{1}{m}\,\mathrm{e}_{2m-1} \in V + U$. Since $\operatorname{span}\{\mathrm{e}_n : n \in \mathbb{N}\}$ is a dense subset of ℓ_1, it follows that $\overline{V + U} = \ell_1$.
Now we will show that $V + U \neq \ell_1$. Let
$$x = (x_n)_{n\in\mathbb{N}}, \qquad x_n = \begin{cases} \frac{1}{n^2}, & n \text{ even},\\ 0, & n \text{ odd}. \end{cases}$$

Clearly $x \in \ell_1$. Suppose that there exist $v = (v_n)_{n\in\mathbb{N}} \in V$, $u = (u_n)_{n\in\mathbb{N}} \in U$ such that $x = v + u$. It follows for all $m \in \mathbb{N}$

$$\frac{1}{(2m)^2} = x_{2m} = v_{2m} + u_{2m} = v_{2m},$$
$$0 = x_{2m-1} = v_{2m-1} + u_{2m-1} = mv_{2m} + u_{2m-1} = \frac{1}{4m^2} + u_{2m-1},$$

impliying that $u_{2m-1} = -\frac{1}{4m^2}$, $m \in \mathbb{N}$, hence $u \neq \ell_1$. Therefore $x \neq V + U$.

Definition 1.17. Let X be a normed space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on X. They are called *equivalent norms* if there exist $m, M > 0$ such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1, \qquad x \in X. \tag{1.2}$$

Theorem 1.18. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X. The the following are equivalent:*

(i) *$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

(ii) *A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges with respect to $\|\cdot\|_1$ if and only if it converges with respect to $\|\cdot\|_2$ and in this case the $\|\cdot\|_1$-limit and the $\|\cdot\|_2$-limit are equal.*

(iii) *A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to 0 with respect to $\|\cdot\|_1$ if and only if it converges with respect to $\|\cdot\|_2$.*

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is clear.

"(iii) $\Longrightarrow$ (i)": Obviously it suffices to show the existence of $M \in \mathbb{R}$ such that (1.2) is true. Assume no such M exists. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\|_1 = 1$ and $\|x_n\|_2 > n\|x_n\|_1 = n$. Let $y_n := n^{-1}x_n$. Then $y_n \xrightarrow{\|\cdot\|_1} 0$, so by assumption also $y_n \xrightarrow{\|\cdot\|_2} 0$. This contradicts $\|y_n\|_2 > 1$ for all $n \in \mathbb{N}$. □

The theorem above implies in particular, that the topologies generated by equivalent norms coincide. Moreover, the identity map id : $(X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is uniformly continuous for equivalent norms.

Example 1.19. On $C^1([a,b])$ define the norm

$$\|f\|_{(2)} := \sup\{\max\{|x(t)|, |x'(t)|\} : t \in [a,b]\}.$$

and let $\|\cdot\|_{(1)}$ be as in Example 1.15 (7). It is not hard to see that

$$\|x\|_{(1)} \leq \|x\|_{(2)} \leq 2\|x\|_{(1)}, \qquad x \in C^1([a,b]).$$

Theorem 1.20. *All norms on $\mathbb{K}^n$ are equivalent.*

Proof. Let $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ be a basis of $\mathbb{K}^n$. For $x = \sum_{j=1}^n \alpha_n \,\mathrm{e}_n$ define

$$\|x\|_2 := \Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{\frac{1}{2}}.$$

Obviously, $\|\cdot\|_2$ is a norm on X and it suffices to show that every norm on X is equivalent to $\|\cdot\|_2$. Let $\|\cdot\|$ be a norm on X and $x = \sum_{j=1}^n \alpha_n \,\mathrm{e}_n$. Using triangle inequality for $\|\cdot\|$ and Hölder's inequality, we obtain

$$\|x\| = \Big\|\sum_{j=1}^n \alpha_j \,\mathrm{e}_j\Big\| \leq \sum_{j=1}^n |\alpha_j| \|\,\mathrm{e}_j\,\| \leq \Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^n \|\,\mathrm{e}_j\,\|^2\Big)^{\frac{1}{2}} = C\,\|x\|_2 \tag{1.3}$$

with constant $C := \Big(\sum_{j=1}^n \|\,\mathrm{e}_j\,\|^2\Big)^{\frac{1}{2}}$ independent of x.

Note that $\|\cdot\|_2 : X \to \mathbb{R}$ is continuous, hence $S := \{x \in X : \|x\|_2 = 1\}$ is closed being the preimage of the closed set $\{1\}$ in $\mathbb{R}$. In addition, S is bounded, therefore S is compact by the theorem of Heine-Borel. Now consider the map $T : (X, \|\cdot\|_2) \to \mathbb{R}, Tx = \|x\|$. By (1.3), T is uniformly continuous, so its restriction to the compact set S has a minimum m and a maximum M. Since $\|\cdot\|$ is a norm, $m > 0$ (otherwise there would exist an $x \in S$ with $\|x\| = 0$, thus $x = 0$ but $0 \notin S$). Therefore

$$m\|x\|_2 = m \leq \|x\| \leq M = M\|x\|_2, \qquad x \in S,$$

and by the homogeneity of the norms

$$m\|x\|_2 \leq \|x\| \leq M\|x\|_2, \qquad x \in X.$$

□

The theorem above implies that all norms a a finite-dimensional $\mathbb{K}$-vector space are equivalent. Moreover, it follows that every finite normed space is complete because $\mathbb{K}^n$ with the Euclidean norm is complete and that a subset of a finite dimensional normed space is compact if and only if it is bounded and closed (Theorem of Heine-Borel for $\mathbb{K}^n$ with the Euclidean metric). In particular, the unit ball in a finite dimensional space is compact.

This is no longer true in infinite dimensional normed spaces. In fact, the unit ball is compact if and only if the dimensions of the space is finite. For the proof we use the following theorem which is also of independent interest, as it shows that in a certain sense quotient spaces can work as a substitute for the orthogonal complement in inner product spaces (see 4.2).

Theorem 1.21 (Riesz's lemma). *Let X be a normed space, $Y \subseteq X$ a closed subspace with $Y \neq X$ and $\varepsilon > 0$. Then there exists an $x \in X$ such with $\|x\| = 1$ and $\operatorname{dist}(x, Y) > 1 - \varepsilon$.*

Proof. If $Y = \{0\}$ or $\varepsilon \geq 1$, the assertion is clear. Now assume $0 < \varepsilon < 1$. Note that in this case $\frac{1}{1-\varepsilon} > 1$. Since Y is closed and different from X, the quotient space X/Y is not trivial. Hence there exists an $\xi \in X$ such that $\|[\xi]\|_\sim = 1$. By Remark 1.14 there exists $y \in Y$ such that

$$1 = \|[\xi]\|_\sim \leq \|\xi + y\| < \frac{1}{1-\varepsilon}.$$

Let $x = \|\xi + y\|^{-1}(\xi + y)$. Obviously, $\|x\| = 1$ and for every $z \in Y$

$$\|x - z\| = \|\xi + y\|^{-1} \Big\|\xi + \underbrace{y - \|\xi + y\|z}_{\in Y}\Big\| \geq \|\xi + y\|^{-1} \|[\xi]\|_{\sim} = \|\xi + y\|^{-1} > 1 - \varepsilon.$$

Hence $d(x, Y) = \inf\{\|x - z\| : z \in Y\} > 1 - \varepsilon$. □

Theorem 1.22. *For a normed space X the following are equivalent:*

(i) $\dim X < \infty$,

(ii) $B_X := \{x \in X : \|x\| \leq 1\}$ *is compact.*

(iii) *Every bounded sequence in X contains a convergent subsequence.*

Proof. "(i) $\Longrightarrow$ (ii)" follows from Theorem 1.20.

"(ii) $\Longrightarrow$ (i)": Assume that B_X is compact. Then there are $x_1, \dots, x_n \in X$ with $\|x_j\| \leq 1$, $j = 1, \dots, n$, such that

$$B_X \subseteq \bigcup_{j=1}^{n} B_{\frac{1}{2}}(x_j). \tag{1.4}$$

Let $U = \operatorname{span}\{x_1, \dots, x_n\}$. If $U \neq X$, then, by Riesz's lemma, there exists an $x \in X$ such that $\|x\| = 1$ and $\operatorname{dist}(x, U) > \frac{1}{2}$, in contradiction to (1.4). Therefore $\dim X = \dim U \leq n$.

"(ii) $\Longrightarrow$ (iii)": If B_X is compact, then obviously for every $\alpha \geq 0$ also $\alpha B_X := \{\alpha x : x \in B_X\}$ is compact. Since every bounded sequence is a subset of some αB_X, it must contain a convergent subsequence.

"(iii) $\Longrightarrow$ (i)": Assume that $\dim X = \infty$. Choose $x_1 \in X$ with $\|x_1\| = 1$ and set $U_1 := \operatorname{span}\{x_1\} \neq X$. By Riesz's lemma there exists an $x_2 \in X$ with $\|x_2\| = 1$ and $\operatorname{dist}(x_2, U_1) > \frac{1}{2}$, in particular $\|x_1 - x_2\| > \frac{1}{2}$. Set $U_2 := \operatorname{span}\{x_1, x_2\} \neq X$. Continuing this way, we obtain a sequence $x = (x_n)_{n\in\mathbb{N}} \subseteq X$ with $\|x_n - x_m\| > \frac{1}{2}$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Therefore, the sequence x does not contain a convergent subsequence, hence B_X is not compact (Recall that a compact metric space is sequentially compact). □

Let X be a vector space and Λ a set. A family $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ is called *linearly independent* if every finite subset is linearly independent. A *Hamel basis* (or an *elgebraic basis*) of X is a family $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ that is linearly independent and such that every element $x \in X$ is a (finite!) linear combination of the x_λ. The existence of a Hamel basis can be shown with Zorn's lemma.

Definition 1.23. Let X be a normed space. A family $(x_n)_{n\in\mathbb{N}}$ is a *Schauder basis* of X if every $x \in X$ can be written uniquely as

$$\sum_{n=1}^{\infty} \alpha_n x_n \qquad \text{with } \alpha_n \in \mathbb{K}.$$

Definition 1.24. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. A subset $Y \subseteq X$ is said to be a total subset of X if

$$\overline{\operatorname{span}(Y)} = X,$$

that is, if the set of all linear combinations of elements of Y is dense in X.

Theorem 1.25. *A normed space $(X, \|\cdot\|)$ is separable if and only if it contains a countable total subset.*

Proof. Let A be a dense countable subset of X. Then obviously $\overline{\operatorname{span} A} = X$, that is, A is a total subset of X.
Now assume that A is countable total subset of X. Let $B := \{\lambda a_n : n \in \mathbb{N},\, \lambda \in \widetilde{\mathbb{Q}}\}$ where $\widetilde{\mathbb{Q}} := \mathbb{Q}$ if X is a $\mathbb{R}$-vector space and $\widetilde{\mathbb{Q}} := \mathbb{Q} + \mathrm{i}\mathbb{Q}$ if X is a $\mathbb{C}$-vector space. In both cases B is countable. We will show that $\overline{B} = X$. Let $x \in X$ and $\varepsilon > 0$. Since A is a total subset of X, there exist $a_1, \dots, a_n \in A$ and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ such that

$$\|x - \sum_{j=1}^{n} \lambda_j a_j\| < \frac{\varepsilon}{2}.$$

Since $\widetilde{Q}$ is dense in $\mathbb{K}$, there exist $\mu_1, \dots, \mu_n \in \widetilde{\mathbb{Q}}$ such that

$$|\mu_j - \lambda_j| < \frac{\varepsilon}{2}\Big(\sum_{j=1}^{n} \|a_j\|\Big)^{-1}, \qquad j = 1, \dots, n.$$

Then $y := \sum_{j=1}^{n} \mu_j a_j \in \operatorname{span} A$ and

$$\begin{aligned}\|x - y\| &\le \Big\|x - \sum_{j=1}^{n} \lambda_j a_j\Big\| + \Big\|y - \sum_{j=1}^{n} \lambda_j a_j\Big\| < \frac{\varepsilon}{2} + \Big\|\sum_{j=1}^{n} |\mu_j - \lambda_j| a_j\Big\| \\ &\le \frac{\varepsilon}{2} + \max_{j=1}^{n} |\mu_j - \lambda_j| \sum_{j=1}^{n} \|a_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$ □

Note that every normed space with a Schauder basis is separable, but not every separable normed space has a Schauder basis.

Examples 1.26. (i) ℓ_p is separable for $1 \le p < \infty$.

Proof. Let $e_n := (0, \dots, 0, 1, 0 \dots)$ be the nth unit vector in ℓ_p. We will show that $\{e_n : n \in \mathbb{N}\}$ is a total subset of ℓ_p. Let $x = (x_n)_{n\in\mathbb{N}} \subseteq \ell_p$. Then

$$\Big\|x - \sum_{j=1}^{n} x_j e_j\Big\|_p = \Big\|\sum_{j=n+1}^{\infty} x_j e_j\Big\|_p \to 0, \quad n \to \infty.$$ □

(ii) ℓ_∞ is not separable.

Proof. Recall that the set $A := \{(x_n)_{n\in\mathbb{N}} : x_n \in \{0,1\}\}$ i s not countable. Obviously, $A \subseteq \ell_\infty$. Let B be a dense subset of ℓ_∞. Then for every $x \in A$ there exists an $b_x \in B$ such that $\|x - b_x\|_\infty < \frac{1}{2}$. Since $\|x-y\|_\infty = 1$ for $x \neq y \in A$, it follows that B has at least the cardinality of A, that is, there exists no countable dense subset of ℓ_p. □

(iii) $C[a,b]$ is separable since by the theorem of Weierstraß the set of polynomials

$$\{[a,b] \to \mathbb{R},\, x \mapsto x^n : n \in \mathbb{N}\}$$

is a total subset of $C[a,b]$.

1.3 Hölder and Minkowski inequality

In this section we prove Hölder's inequality and Minkowski's inequality. For the proof we use Young's inequality.

Theorem 1.27. *Let $p, q \in (1,\infty)$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then for all $a, b \geq 0$:

$$ab \leq \frac{1}{p}\, a^p + \frac{1}{q}\, b^q. \tag{1.5}$$

Proof. If $ab = 0$, then inequality (1.5) is clear. Now assume $ab > 0$. Since the logarithm is concave and $\frac{1}{p} + \frac{1}{q} = 1$ is follows that

$$\ln\Big(\frac{1}{p}\, a^p + \frac{1}{q}\, b^q\Big) \geq \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) = \ln(a) + \ln(b) = \ln(ab).$$

Application of the monotonically increasing function $\exp : \mathbb{R} \to \mathbb{R}$ yields (1.5). □

Theorem 1.28 (Hölder's inequality). *Let $1 \leq p \leq \infty$ and $q = \frac{p}{p-1}$, i. e.,*

$$\frac{1}{p} + \frac{1}{q} = 1$$

(setting $\frac{1}{\infty} = 0$). If $x \in \ell_p$ and $y \in \ell_q$, then $z = (x_n y_n)_{n\in\mathbb{N}} \in \ell_1$ and

$$\|z\|_1 \leq \|x\|_p\, \|y\|_q. \tag{1.6}$$

Proof. If $x = 0$ or $y = 0$ then the inequality (1.6) clearly holds. Also the cases $p = 1$ and $p = \infty$ are clear.
Now assume $x, y \neq 0$ and $1 < p < \infty$. The Young inequality (1.6) with

$$a = \frac{|x_j|}{\|x\|_p}, \quad b = \frac{|y_j|}{\|y\|_q}$$

yields

$$\frac{|x_j|\,|y_j|}{\|x\|_p\,\|y\|_q} \;\le\; \frac{1}{p}\,\frac{|x_j|^p}{\|x\|_p^p} + \frac{1}{q}\,\frac{|y_j|^q}{\|y\|_q^q}\,.$$

Taking the sum over gives

$$\frac{1}{\|x\|_p\,\|y\|_q}\;\sum_{j=1}^{\infty}|x_jy_j| \;\le\; \underbrace{\frac{1}{p}\;\frac{1}{\|x\|_p^p}\underbrace{\sum_{j=1}^{\infty}|x_j|^p}_{=\|x\|_p^p}}_{=1} + \underbrace{\frac{1}{q}\;\frac{1}{\|y\|_q^q}\underbrace{\sum_{j=1}^{\infty}|y_j|^q}_{=\|y\|_q^q}}_{=1} \;=\; \frac{1}{p}+\frac{1}{q} \;=\; 1. \qquad \square$$

In the special case $p = q = 2$ we obtain the Cauchy-Schwarz inequality.

Corollary 1.29 (Cauchy-Schwarz inequality). *For $x = (x_n)_{n\in\mathbb{N}},\ y = (y_n)_{n\in\mathbb{N}} \in \ell_2$ the Hölder inequality implies*

$$|\langle x\,,y\rangle| := \Big|\sum_{j=1}^{\infty} x_j\overline{y_j}\Big| \le \|x\|_2\,\|y\|_2.$$

Theorem 1.30 (Minkowski inequality). *For $1 \le p \le \infty$ and $x, y \in \ell_p$ Minkowski's inequality holds:*

$$\|x + y\|_p \;\le\; \|x\|_p + \|y\|_p. \tag{1.7}$$

Proof. If $x + y = 0$ then (1.7) clearly holds. Also the cases $p = 1$ and $p = \infty$ are easy to check. Now assume $x+y \neq 0$ and $1 < p < \infty$. Let $q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q} = 1$. The triangle inequality in $\mathbb{K}$ and Hölder's inequality (1.6) yield for all $M \in \mathbb{N}$:

$$\begin{aligned}
\sum_{j=1}^{M}|x_j+y_j|^p &= \sum_{j=1}^{M}|x_j+y_j|\cdot|x_j+y_j|^{p-1} \\
&\le \sum_{j=1}^{M}|x_j|\,|x_j+y_j|^{p-1} + \sum_{j=1}^{M}|y_j|\,|x_j+y_j|^{p-1} \\
&\le \Big(\sum_{j=1}^{M}|x_j|^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^{M}|x_j+y_j|^{\overbrace{(p-1)q}^{p}}\Big)^{\frac{1}{q}} + \Big(\sum_{j=1}^{M}|y_j|^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^{M}|x_j+y_j|^{\overbrace{(p-1)q}^{p}}\Big)^{\frac{1}{q}} \\
&\le \Big(\|x\|_p + \|x\|_p\Big)\Big(\sum_{j=1}^{M}|x_j+y_j|^p\Big)^{\frac{1}{q}}.
\end{aligned}$$

Note that $\Big(\sum_{j=1}^{M}|x_j+y_j|^p\Big)^{\frac{1}{q}} \neq 0$ for M large enough. Hence the above inequality yields

$$\Big(\sum_{j=1}^{M}|x_j+y_j|^p\Big)^{\frac{1}{p}} \;\le\; \|x\|_p + \|x\|_p$$

using $p - \frac{p}{q} = p\big(1 - \frac{1}{q}\big) = 1$. Taking the limit $M \to \infty$ finally proves (1.7). $\square$

Chapter 2

Bounded maps; the dual space

2.1 Bounded linear maps

Definition 2.1. Let X, Y be normed spaces over the same field $\mathbb{K}$. The set of all linear continuous maps $X \to Y$ is denoted by $L(X,Y)$, i. e.,

$$L(X,Y) = \{T : X \to Y : T \text{ linear and continuous}\}$$

and $L(X) := L(X,X)$.

Recall that the following is equivalent:

(i) $T : X \to Y$ is continuous

(ii) $\lim\limits_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n$ for every convergent sequence $(x_n)_{n\in\mathbb{N}} \in X$

(iii) $\forall\, x_0 \in X\ \forall\, \varepsilon > 0\ \exists\, \delta > 0 :\ \|x - x_0\| < \delta \implies \|Tx - Tx_0\| < \varepsilon$

(iv) $U \subseteq Y$ open $\implies T^{-1}(U) = \{x \in\ X : f(x) \in U\}$ open in X.

Definition 2.2. Let X, Y be normed spaces over the same field $\mathbb{K}$. For a linear map $T : X \to Y$ define the *operator norm*

$$\|T\| := \sup\{\|Tx\| : x \in X,\ \|x\| = 1\}.$$

If $\|T\| < \infty$ then T is called a *bounded linear operator* and $\|T\|$ is the *operator norm* of $\|T\|$.

Remark 2.3. (i) For a continuous linear map $T : X \to Y$

$$\|Tx\| \leq \|T\|\, \|x\|, \qquad x \in X.$$

Proof. The inequality is obvious for $x = 0$ or $\|x\| = 1$. For $x \in X \setminus \{0\}$ let $\widetilde{x} = \|x\|^{-1}x$. By definition of $\|T\|$ we find $\|Tx\| = \|x\|\, \|T\widetilde{x}\| \leq \|x\|\, \|T\|$. Note that the inequality is also true if T is unbounded and $x \neq 0$. □

(ii) The following is easy to check:

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : x \in X,\ \|x\| = 1\}\\ &= \sup\{\|Tx\| : x \in X,\ \|x\| \le 1\}\\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X,\ x \neq 0\right\}\\ &= \inf\{M \in \mathbb{R} : \forall\, x \in X\ \|Tx\| \le M\|x\|\}.\end{aligned}$$

Remark 2.4. (i) For S, $T \in L(X, Y)$ and $\lambda \in \mathbb{K}$ we define

$$(\lambda T + S) : X \to Y, \quad (\lambda T + S)x := \lambda Tx + Sx.$$

Since the sum and composition of continuous functions is continuous, and ($\lambda T + S$ obviously is linear, $L(X, Y)$ is a vector space.

It will be shown in Theorem 2.6 that $\|\cdot\|$ is indeed a norm. Note the the operator norm depends on the norms on X and Y. This is can be made explicit using the notation $\|T\|_{L(X,Y)}$, or similar notation.

(ii) Let X, Y, Z be normed spaces and $T \in L(X, Y)$, $S \in L(Y, Z)$. Then

$$ST : X \to Z, \quad STx := S(Tx).$$

Obviously, $ST \in L(X, Z)$ as composition of continuous linear functions and $\|ST\| \le \|S\|\,\|T\|$ because by Remark 2.3

$$\|STx\| \le \|S\|\,\|Tx\| \le \|S\|\,\|T\|\,\|x\|, \qquad x \in X.$$

In particular, $L(X)$ is an algebra.

Theorem 2.5. *Let X, Y be normed spaces, $T : X \to Y$ linear. The following is equivalent:*

(i) *T is continuous.*

(ii) *T is continuous in 0.*

(iii) *T is bounded.*

(iv) *T is uniformly continuous.*

Proof. The implications (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) are obvious.

“ (ii) $\Longrightarrow$ (iii)”: Assume that T is not bounded. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| = 1$ and $\|Tx_n\| > n$ for all $n \in \mathbb{N}$. Let $y_n := n^{-1}x_n$. Then $y_n \to 0$ but $\|Ty_n\| > 1$ for all $n \in \mathbb{N}$ in contradiction to the continuity of T in 0. □

Theorem 2.6. *Let X, Y be normed spaces.*

(i) *$L(X, Y)$ is a normed space.*

(ii) *If Y is Banach space, then $L(X, Y)$ is a Banach space.*

Proof. (i) In Remark 2.4 we have seen that $L(X,Y)$ is a vector space. From definition of the operator norm it is clear that $\|T\| = 0$ if and only if $T = 0$ and that $\|\lambda T\| = |\lambda|\,\|T\|$ for all $\lambda \in \mathbb{K}$. To prove the triangle inequality let $S, T \in L(X,Y)$ and $x \in X$.

$$\|(S+T)x\| = \|Sx + Tx\| \le \|Sx\| + \|Tx\| \le \|S\| + \|T\|.$$

Taking the supremum over all $x \in X$ with $\|x\| = 1$ yields $\|S+T\| \le \|S\| + \|T\|$.

(ii) Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L(X,Y)$. For $x \in X$, the sequence $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y because

$$\|T_n x - T_m x\| \le \|T_n - T_m\|\,\|x\|.$$

Since Y is complete, we can define

$$T : X \to Y, \quad Tx := \lim_{n\to\infty} T_n x.$$

It is easy to check that T is linear. That T is bounded and $T_n \to T$ follows as in Example 1.13(2): For $\varepsilon > 0$ exists an $N \in \mathbb{N}$ such that

$$\|T_n - T_m\| < \frac{\varepsilon}{2}, \qquad n, m \ge N.$$

In particular, for all $x \in X$ it follows for $n \ge N$ that

$$\|Tx - T_n x\| \le \|Tx - T_m x\| + \|T_m x - T_n x\| \le \|Tx - T_m x\| + \frac{\varepsilon}{2}, \quad m \in \mathbb{N}. \tag{2.1}$$

Taking the limit $m \to \infty$ on the right hand side yields $\|Tx - T_n x\| \le \frac{\varepsilon}{2} < \varepsilon$. It follows that $T - T_n$ is a bounded linear map. Since $L(X,Y)$ is a vector space, also $T = T_n + (T - T_n)$ is a bounded linear map. In addition, (2.1) shows that $T_n \to T$, $n \to \infty$. □

Examples 2.7. In the following examples, the linearity of the operator under consideration is easy to check.

(i) Let X be a normed space. Then the identity $\mathrm{id} : X \to X$ is bounded and $\|\mathrm{id}\| = 1$.

(ii) Let $1 \le p \le \infty$. The left shift and the right shift on ℓ_p are defined by

$$\begin{aligned} R &: \ell_p \to \ell_p, \qquad (x_1, x_2, x_3 \dots)_{n\in\mathbb{N}} \mapsto (0, x_1, x_2, \dots), \\ L &: \ell_p \to \ell_p, \qquad (x_1, x_2, x_3 \dots)_{n\in\mathbb{N}} \mapsto (x_2, x_3, \dots). \end{aligned}$$

Obviously, R and L are well-defined and linear. Moreover, R is an isometry because $\|Rx\|_p = \|x\|_p$; in particular $\|R\| = 1$.
The left shift is not an isometry because, e. g., $\|L(1,0,0,\dots)\|_p = \|0\|_p = 0 < 1 = \|(1,0,0,\dots)\|_p$.
It is easy to see that $\|Lx\|_p \le \|x\|_p$, $x \in \ell_p$, implying that $\|L\| \le 1$. Since $\|L(0,1,0,0\dots)\|_p = \|(1,0,0\dots)\|_p = \|(0,1,0,0\dots)\|_p$ we also have $\|L\| \ge 1$, so that altogether $\|L\| = 1$.
Note that $LR = \mathrm{id}_{\ell_p}$ but $RL \ne \mathrm{id}_{\ell_p}$.

(iii) $T : C^1([0,1], \|\cdot\|_{C^1}) \to C([a,b], \|\cdot\|_\infty)$, $Tx = x'$ with $\|x\|_{C^1} := \|x\|_\infty + \|x'\|_\infty$. The operator T is bounded and $\|T\| = 1$.

Proof. The operator T is bounded with $\|T\| \le 1$ because $\|Tx\|_\infty = \|x'\|_\infty \le \|x\|_\infty + \|x'\|_\infty \le \|x\|_{C^1}$ for all $x \in X$.
To proof that $\|T\| \ge 1$ let $x_n : [0,1] \to \mathbb{R}$, $x_n(t) := \frac{1}{n}\exp(-nt)$. Obviously, $x_n \in C^1([0,1])$, $\|x_n\|_{C^1} = \frac{1}{n} + 1$ and $\|Tx_n\|_\infty = 1$. It follows that

$$\begin{aligned}\|T\| &= \sup\left\{\tfrac{\|Tx\|_\infty}{\|x\|_{C^1}} : x \in C^1([0,1]) \setminus \{0\}\right\} \ge \sup\left\{\tfrac{\|Tx_n\|_\infty}{\|x_n\|_{C^1}} : n \in \mathbb{N}\right\} \\ &= \sup\left\{\tfrac{1}{1+\frac{1}{n}} : n \in \mathbb{N}\right\} = 1.\end{aligned}$$

□

(iv) $T : C^1([0,1], \|\cdot\|_\infty) \to C([a,b], \|\cdot\|_\infty)$, $Tx = x'$ is not bounded.

Proof. As in the example above let $x_n : [0,1] \to \mathbb{R}$, $x_n(t) := \frac{1}{n}\exp(-nt)$. It follows that

$$\begin{aligned}\sup\left\{\tfrac{\|Tx\|_\infty}{\|x\|_\infty} : x \in C^1([0,1]) \setminus \{0\}\right\} &\ge \sup\left\{\frac{\|Tx_n\|_\infty}{\|x_n\|_\infty} : n \in \mathbb{N}\right\} \\ &= \sup\left\{\tfrac{1}{\frac{1}{n}} : n \in \mathbb{N}\right\} = \infty\end{aligned}$$

□

Lemma 2.8. *let X, Y be normed spaces, X finite-dimensional. Then every linear map $T : X \to Y$ is bounded.*

Proof. Let $\mathrm{e}_1, \dots, \mathrm{e}_n$ be a basis of X. Since on X all norms are equivalent, we can assume that

$$\Big\| \sum_{j=1}^n \alpha_j \, \mathrm{e}_j \Big\| = \sum_{j=1}^n |\alpha_j|.$$

Let $M := \max\{\|T\,\mathrm{e}_j\| : j = 1, \dots, n\}$. Then T is bounded with $\|T\| \le M$ because for $x = \sum_{j=1}^n \alpha_j \, \mathrm{e}_j \in X$

$$\|Tx\|_Y = \Big\| \sum_{j=1}^n \alpha_j T\,\mathrm{e}_j \Big\|_Y \le \sum_{j=1}^n |\alpha_j| \, \|T\,\mathrm{e}_j\|_Y \le M \sum_{j=1}^n |\alpha_j| = M\|x\|_X.$$

□

Theorem 2.9. *Let X, Y be normed spaces, Y a Banach space. Let $D \subseteq X$ be a dense subspace of X and $T \in L(D, Y)$. Then there exists exactly one continuous extension $\widehat{T} : X \to Y$ of T. The extension is bounded with $\|\widehat{T}\| = \|T\|$.*

Proof. For $x \in X$ choose a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ which converges to x. The sequence is a Cauchy sequence in D, hence, by the uniform continuity of T, $(Tx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, and therefore it converges in Y because Y is complete. Let $(\xi_n)_{n\in\mathbb{N}}$ be another Cauchy sequence in D which converges to x. By what was said before, $(T\xi_n)$ converges in Y. Then $\lim\limits_{n\to\infty} \|Tx_n - T\xi_n\| = \lim\limits_{n\to\infty} \|T(x_n - \xi_n)\| \le \lim\limits_{n\to\infty} \|T\| \, \|(x_n - \xi_n)\| = \|T\| \lim\limits_{n\to\infty} \|(x_n - \xi_n)\| = 0$, the following operator is well defined:

$$\widetilde{T} : X \to Y, \qquad \widetilde{T}x := \lim_{n\to\infty} Tx_n \quad \text{for any } (x_n)_{n\in\mathbb{N}} \subseteq D \text{ which converges to } x.$$

It is not hard to see that $\widetilde{T}$ is a linear extension of T and that $\|\widetilde{T}\| \geq \|T\|$. To see that indeed equality holds, we only need to observe that by definition of $\widetilde{T}$

$$\overline{\{\|Tx\| : x \in D, \|x\| = 1\}} = \overline{\{\|\widetilde{T}x\| : x \in X, \|x\| = 1\}},$$

hence the suprema of both sets without the closure are equal (and equal to the supremum of the closed sets). Since $\widetilde{T}$ is linear and bounded by $\|T\|$, it is continuous.
Assume that S is an arbitrary continuous extension of T. For $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ which converges to x we find

$$Sx = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \widehat{T}x_n = \widehat{T}x.$$

Therefore, $\widehat{T}$ is the unique continuous extension of T. □

Finally we give a criterion for the invertibility of a bounded linear operator.

Theorem 2.10 (Neumann series). *Let X be a normed space and $T \in L(X)$ such that $\sum_{n=0}^{\infty} T^n$ converges. Then $\mathrm{id} - T$ is invertible in $L(X)$ and*

$$(\mathrm{id} - T)^{-1} = \sum_{n=0}^{\infty} T^n. \tag{2.2}$$

In particular, if X is a Banach space and $\|T\| < 1$, then $\mathrm{id} - T$ is invertible and

$$\|(\mathrm{id} - T)^{-1}\| \leq (1 - \|T\|)^{-1}.$$

Proof. The proof is analogous to the proof for the convergence of the geometric series. We define the partial sums $S_m := \sum_{n=0}^{m} T^n$, $m \in \mathbb{N}_0$. Then

$$(\mathrm{id} - T)S_m = S_m(\mathrm{id} - T) = \mathrm{id} - T^{m+1}, \qquad m \in \mathbb{N}_0. \tag{2.3}$$

Note that:

(i) $T^m \to 0$ for $m \to \infty$ because $\sum_{m=0}^{\infty} T^m$ converges.
(ii) $S_m \to \sum_{n=0}^{\infty} T^n$ for $m \to \infty$ by assumption.
(iii) For fixed $R \in L(X)$ the maps $L(X) \to L(X)$, $S \mapsto RS$ and $S \mapsto SR$ respectively are continuous.

Hence taking the limit $m \to \infty$ in (2.3) gives

$$(\mathrm{id} - T) \sum_{n=0}^{\infty} T^n = \Big(\sum_{n=0}^{\infty} T^n \Big)(\mathrm{id} - T) = \mathrm{id}$$

implying that $\mathrm{id} - T$ is invertible and that (2.2) holds.
Now assume that X is a Banach space and that $\|T\| < 1$. Then $\sum_{n=0}^{\infty} T^n$ converges in norm because $\|T^n\| \leq \|T\|^n$. In particular, $\left(\sum_{j=0}^{m} T^j\right)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L(X)$. Since $L(X)$ is complete by assumption on X and Theorem 2.6 the series converges. By the first part of the proof, $\mathrm{id} - T$ is invertible and formula (2.2) holds. □

Application 2.11 (Volterra integral equation). Let $k \in C([0,1]^2)$ and $y \in C([0,1])$. We ask if the equation

$$x(s) - \int_0^s k(s,t)x(t)\ \mathrm{d}t = y(s), \qquad s \in [0,1]. \tag{2.4}$$

has solution $x \in C([0,1])$. If a solution exists, is it unique? Can the norm of the solution be estimated in terms of y?

Solution. Note that equation (2.4) can be written as an equation in the Banach space $C([0,1])$:

$$x - Kx = y$$

where

$$K : C([0,1]) \to C([0,1]), \quad (Kx)(s) := \int_0^s k(s,t)x(t)\ \mathrm{d}t,\ s \in [0,1].$$

Obviously, K is a well-defined linear operator and for all $x \in C([0,1]),\ s \in [0,1]$

$$\begin{aligned} |Kx(s)| &= \left| \int_0^s k(s,t)x(t)\ \mathrm{d}t \right| \le \int_0^s |k(s,t)|\,|x(t)|\ \mathrm{d}t \le s\,\|k\|_\infty\,\|x\|_\infty, \\ |K^2x(s)| &= \left| \int_0^s k(s,t) \int_0^t k(t,t_1)x(t_1)\ \mathrm{d}t_1\ \mathrm{d}t \right| \le \|k\|_\infty^2\,\|x\|_\infty \int_0^s \int_0^t\ \mathrm{d}t_1\ \mathrm{d}t \\ &= \|k\|_\infty^2\,\|x\|_\infty \frac{s^2}{2}. \end{aligned}$$

Repeating this process, it follows that

$$|K^n x(s)| \le \frac{s^n}{n!}\,\|k\|_\infty^n\,\|x\|_\infty, \qquad s \in [0,1],\ x \in C([0,1]),\ n \in \mathbb{N},$$

which shows that $\|K^n\| \le \frac{\|k\|_\infty}{n!}$. In particular, $\sum_{n=0}^\infty K^n$ converges so that $\mathrm{id} - K$ is invertible by Theorem 2.10. Hence equation (2.4) has exactly one solution $x \in C([0,1])$, given by

$$x = \sum_{n=0}^\infty K^n y.$$

Moreover, $\|x\|_\infty = \left\| \sum_{n=0}^\infty K^n y \right\|_\infty \le \sum_{n=0}^\infty \|K^n\|\,\|y\|_\infty \le \sum_{n=0}^\infty \frac{\|k\|_\infty}{n!}\|y\|_\infty = \mathrm{e}\,\|k\|_\infty \|y\|_\infty.$ □

2.2 The dual space and the Hahn-Banach theorem

Definition 2.12. Let X be a normed space. $X' := L(X,\mathbb{K})$ is the *dual space* of X; elements in the dual space are called *functionals*.

Note that in general the *algebraic* dual space, i. e., the space of all linear maps $X \to \mathbb{K}$ in general is larger than the *topological* dual space defined above.
Theorem 2.6 implies immediately:

Proposition 2.13. *The dual space of a normed space X with the norm*

$$\|x'\| = \sup\{|x'(x)| : x \in X, \|x\| \leq 1\}, \qquad x' \in X',$$

is a Banach space.

Definition 2.14. Let X be normed space. $p : X \to \mathbb{R}$ is a a *seminorm* if

(i) $p(\lambda x) = |\lambda| p(x), \quad \lambda \in \mathbb{K},\ x \in X,$

(ii) $p(x + y) \leq p(x) + p(y), \quad x, y \in X.$

A seminorm p is called *bounded* if there exists an $M \in \mathbb{R}$ such that

$$p(x) \leq M\|x\|, \qquad x \in X.$$

If p satisfies only

(i') $p(\lambda x) = \lambda p(x), \quad \lambda \geq 0,\ x \in X$

instead of (i), then it is called a *sublinear functional*.

Remark. Observe that $p(x) \geq 0$ for every $x \in X$ and every sublinear functional p. Moreover, note that every seminorm is a sublinear functional.

Examples.

- Every norm on X is a seminorm.
- Every linear functional $\varphi : X \to \mathbb{K}$ induces a seminorm by

$$X \to \mathbb{R}, \qquad x \mapsto |\varphi(x)|.$$

- On the space of all real valued bounded sequences $\ell_\infty(\mathbb{N}, \mathbb{R})$ we have the sublinear functional

$$\ell_\infty(\mathbb{N}, \mathbb{R}) \to \mathbb{R}, \qquad x \mapsto \limsup_{n\to\infty} x_n$$

  which is not a seminorm.

The next fundamental theorem shows that every normed space admits non-trivial functionals (except when $X = \{0\}$).

Theorem 2.15 (Hahn-Banach theorem for normed spaces over $\mathbb{R}$). *Let X be normed space over $\mathbb{R}$ and let $p : X \to \mathbb{R}$ be a sublinear functional. Let $Y \subseteq X$ a subspace and let $\varphi_0 : Y \to \mathbb{R}$ be a linear function on Y with*

$$-p(-y) \leq \varphi_0(y) \leq p(y), \qquad y \in Y.$$

Then φ_0 has an extension to a linear function $\varphi : X \to \mathbb{R}$ which satisfies

$$-p(-x) \leq \varphi(x) \leq p(x), \qquad x \in X. \tag{2.5}$$

Proof. For $Y = X$ there is nothing to show. Now assume $Y \neq X$. We distinguish between the real and the complex case.

We divide the proof in two steps.

Step 1. Let $z_0 \in X \setminus Y$ and $Z := \operatorname{span}\{z_0, Y\}$. We will show that φ_0 can be extended to some $\psi \in Z'$ such that (2.8) holds for all $z \in Z$.

Obviously, every linear extension of ψ must be of the form

$$\psi_c(y + \lambda z_0) = \varphi_0(y) + \lambda c, \qquad \lambda \in \mathbb{R},\ y \in Y$$

for some $c \in \mathbb{R}$. We have to find c such that

$$-p(-(y + \lambda z_0)) \le \psi_c(y + \lambda z_0) \le p(y + \lambda z_0), \qquad y \in Y,\ \lambda \in \mathbb{R}. \tag{2.6}$$

By assumption on φ_0

$$\varphi_0(x) - \varphi_0(y) = \varphi_0(x - y) \le p(x - y) \le p(x + z_0) + p(-(y + z_0)), \qquad y, x \in Y,$$

implying

$$-\varphi_0(y) - p(-(y + z_0)) \le -\varphi_0(x) + p(x + z_0), \qquad y, x \in Y,$$

so that

$$a := \sup\{-\varphi_0(x) - p(x + z_0) : x \in Y\} \le \inf\{-\varphi_0(x) + p(x + z_0) : x \in Y\} =: b.$$

Now let $c \in [a, b]$ arbitrary. We show that ψ_c is an extension of φ_0 as desired. Let $z = y + \lambda z_0 \in Z$ with $y \in Y$ and $\lambda \in \mathbb{R}$.

We have to show (2.6). For $\lambda = 0$ equation (2.6) clearly holds. Now let $\lambda \neq 0$. Note that by definition of a and b, we have that

$$-\varphi_0(\tfrac{1}{\lambda}x) - p(-(\tfrac{1}{\lambda}x + z_0)) \le a \le c \le b \le \varphi_0(\tfrac{1}{\lambda}x) + p(\tfrac{1}{\lambda}x + z_0)$$

Hence, for $\lambda > 0$,

$$\begin{aligned}
\psi_c(y + \lambda z_0) &= \varphi_0(y) + \lambda c \le \varphi_0(y) + \lambda b \le \lambda p(\tfrac{1}{\lambda}x + z_0) = p(y + \lambda z_0),\\
\psi_c(y + \lambda z_0) &= \varphi_0(y) + \lambda c \ge \varphi_0(y) + \lambda a \ge -\lambda p(0(\tfrac{1}{\lambda}x + z_0)) = -p(-(y + \lambda z_0)).
\end{aligned}$$

If $\lambda < 0$, then we can write $\psi_c(y + \lambda z_0) = -\psi_c(-y + |\lambda| z_0)$ and the above inequalities show that

$$\begin{aligned}
\psi_c(y + \lambda z_0) &\ge -p(-y + |\lambda| z_0) = -p(-(y + \lambda z_0)),\\
\psi_c(y + \lambda z_0) &\le p(-(-y + |\lambda| z_0)) = p(y + \lambda z_0).
\end{aligned}$$

In summary, we have $-p(-z) \le \psi(z) \le p(z)$ for all $z \in Z$.

Step 2. Let Φ be the set of all proper extensions φ of φ_0 that satisfy $-p(-x) \le \varphi(x) \le p(x)$ for all $x \in \mathcal{D}(\varphi)$ (the domain of φ). By Step 1, Φ is not empty and partially ordered by

$$\varphi_1 < \varphi_2 \iff \varphi_2 \text{ is an extension of } \varphi_1.$$

Every totally ordered subset Φ_0 has the upper bound

$$\mathcal{D}(f) = \bigcup_{\psi \in \Phi_0} \mathcal{D}(\psi), \qquad f(x) = \psi(x) \text{ for } x \in \mathcal{D}(\psi).$$

By Zorn's lemma, Φ contains a maximal element φ. This φ is defined on X because otherwise, by Step 1, it would not be maximal. Therefore φ is an extension of φ_0 as desired. ☐

The above theorem can be formulated easily for seminorms instead of sublinear functionals.

Theorem (Hahn-Banach theorem for normed spaces over $\mathbb{R}$ for seminorms). *As in Theorem , Let X be normed space over $\mathbb{R}$, $Y \subseteq X$ a subspace and let $\varphi_0 : Y \to \mathbb{R}$ be a linear function on Y with*

$$|\varphi_0(y)| \le p(y), \qquad y \in Y$$

where $p : X \to \mathbb{R}$ is a seminorm on X. Then φ_0 has an extension to a linear function $\varphi : X \to \mathbb{R}$ which satisfies

$$|\varphi(x)| \le p(x), \qquad x \in X. \tag{2.7}$$

Proof. This follows immediately from Theorem 2.2 because $p(-x) = p(x)$ for all $x \in X$. Therefore $-p(-x) \le \alpha \le p(x)$ is equivalent to $|\alpha| \le p(x)$. □

Next we prove the version of the Hahn-Banach theorem for seminorms of the case of a complex normed space. (Note that the inequality (2.7) does not make sense in a complex normed space.)

Theorem 2.16 (Hahn-Banach theorem for normed spaces over $\mathbb{R}$). *Let X be normed space and $p : X \to \mathbb{R}$ a seminorm. Let $Y \subseteq X$ a subspace and $\varphi_0 : Y \to \mathbb{K}$ a linear function on Y with*

$$|\varphi_0(y)| \le p(y), \qquad y \in Y.$$

Then φ_0 has an extension to a linear function $\varphi : X \to \mathbb{K}$ which satisfies

$$|\varphi(x)| \le p(x), \qquad x \in X. \tag{2.8}$$

Proof. The case $\mathbb{K} = \mathbb{R}$ was shown above. Now assume that $\mathbb{K} = \mathbb{C}$. Consider X as a vector space over $\mathbb{R}$ and define the functional

$$V_0 : Y \to \mathbb{R}, \qquad V_0(y) = \operatorname{Re}(\varphi(y)).$$

It is $\mathbb{R}$-linear because for all $x, y \in Y$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} V_0(\alpha x + y) &= \operatorname{Re}(\varphi_0(\alpha x + y)) = \operatorname{Re}(\alpha\varphi_0(x) + \varphi_0(y)) = \alpha \operatorname{Re}(\varphi_0(x)) + \operatorname{Re}(\varphi_0(y)) \\ &= \alpha V_0(x) + V_0(y). \end{aligned}$$

In addition, V_0 is bounded by the sublinear functional p

$$|V_0(y)| = |\operatorname{Re}(\varphi_0(y))| \le |\varphi_0(y)| \le p(y), \qquad y \in \ Y.$$

By what we have already shown, there exists an $\mathbb{R}$-linear extension $V : X \to \mathbb{R}$ of V_0 with $|V(x)| \le p(x)$, $x \in X$. Now define

$$\varphi : X \to \mathbb{C}, \qquad \varphi(x) = V(x) - \mathrm{i}V(\mathrm{i}x).$$

The function φ has the following properties:

(i) *φ is an extension of φ_0.* To see this, let $y \in Y$.

$$\begin{aligned}\varphi(y) &= V_0(y) - \mathrm{i}V_0(\mathrm{i}y) = \mathrm{Re}(\varphi_0(y)) - \mathrm{i}\,\mathrm{Re}(\varphi_0(\mathrm{i}y)) = \mathrm{Re}(\varphi_0(y)) - \mathrm{i}\,\mathrm{Re}(\mathrm{i}\varphi_0(y)) \\ &= \mathrm{Re}(\varphi_0(y)) + \mathrm{i}\,\mathrm{Im}(\varphi_0(y)) = \varphi_0(y).\end{aligned}$$

(ii) *φ is $\mathbb{C}$-linear.* To show this, let $x, y \in X$ and $\zeta = a + ib$ with $a, b \in \mathbb{R}$.

$$\begin{aligned}\varphi(x+y) &= V(x+y) - \mathrm{i}V(\mathrm{i}(x+y)) = V(x) + V(y) - \mathrm{i}V(\mathrm{i}x) - \mathrm{i}V(\mathrm{i}y) \\ &= \varphi(x) + \varphi(y),\end{aligned}$$

$$\begin{aligned}\varphi(\zeta x) &= \varphi(ax) + \varphi(\mathrm{i}bx) = V(ax) - \mathrm{i}V(\mathrm{i}ax) + V(\mathrm{i}bx) - \mathrm{i}V(\mathrm{i}^2 bx) \\ &= a[V(x) - \mathrm{i}V(\mathrm{i}x)] + b[V(\mathrm{i}x) + \mathrm{i}V(x)] \\ &= (a + \mathrm{i}b)[V(x) - \mathrm{i}V(\mathrm{i}x)] = \zeta\varphi(x).\end{aligned}$$

(iii) *φ is bounded by p.* To prove this, let $x \in X$ and $\alpha \in \mathbb{R}$ such that

$$|\varphi(x)| = \mathrm{e}^{\mathrm{i}\alpha}\,\varphi(x) = \mathrm{Re}\left(\varphi(\mathrm{e}^{\mathrm{i}\alpha}\,x)\right) = V\big((\mathrm{e}^{\mathrm{i}\alpha}\,x)\big) \le p\big((\mathrm{e}^{\mathrm{i}\alpha}\,x)\big) = p(x).$$

In conclusion, φ is a $\mathbb{C}$-linearextension of φ_0 which is bounded by p as desired. □

Remark. If in the Hahn-Banach theorem we consider only real normed spaces and replace the seminorm p by a sublinear functional such that $\varphi_0(y) \le q(y)$ for all $y \in Y$, then φ_0 can be extented to a functional $\varphi : X \to \mathbb{K}$ such that $-q(x) \le \varphi(x) \le q(x)$ for all $x \in X$, see [Rud91, Theorem 3.2].

The Hahn-Banach theorem has some important corollaries.

Corollary 2.17. *Let X be a normed space, $Y \subseteq X$ a subspace and $\varphi_0 \in Y'$. Then there exists an extension $\varphi \in X'$ of φ_0 such that $\|\varphi\| = \|\varphi_0\|$.*

Proof. The map $p : X \to \mathbb{R}$, $p(x) = \|\varphi_0\|\,\|x\|$ is a sublinear functional on X and $|\varphi_0(y)| \le \|\varphi_0(y)\|\,\|y\| = p(y)$ for all $y \in Y$. By the Hahn-Banach theorem, φ_0 can be extended to a $\varphi \in X'$ with $|\varphi(x)| \le p(x) = \|\varphi_0\|\,\|x\|$, so that $\|\varphi\| \le \|\varphi_0\|$. On other hand $\|\varphi\| \ge \|\varphi_0\|$ holds because φ_0 is a restriction of φ. □

The next corollary shows that X' does not consist only of the trivial functional and that it separates points in X.

Corollary 2.18. *Let X be a normed space, $x \in X$, $x \ne 0$. Then there exists a $\varphi \in X'$ such that $\varphi(x) = \|x\|$. In particular for all $x, y \in X$:*

(i) $x = 0 \iff \forall \varphi \in X' \;\; \varphi(x) = 0$,

(ii) $x \ne y \implies \exists\, \varphi \in X' \;\; \varphi(x) \ne \varphi(y)$.

Proof. Let $Y := \mathrm{span}\{x\}$ and $\varphi_0 \in Y'$ defined by $\varphi_0(\lambda x) = \lambda\|x\|$. Then $\varphi_0(x) = \|x\|$ and $\|\varphi_0\| = 1$. By Corollary 2.17 there exists an extension $\varphi \in X'$ of φ_0 with the desired properties. Statement (i) is clear; (ii) follows when (i) is applied to $x - y$. □

Corollary 2.19. *Let X, Y be a normed spaces.*

(i) $\|x\| = \sup\{\varphi(x) : \varphi \in X',\ \|\varphi\| = 1\}$, $x \in X$.

(ii) *For $T : X \to Y$ linear*

$$\|T\| = \sup\{\varphi(Tx) : x \in X,\ \|x\| = 1,\ \varphi \in Y',\ \|\varphi\| = 1\}.$$

Proof. (i) For all $\varphi \in X'$ with $\|\varphi\| = 1$: $\|x\| = \|\varphi\|\,\|x\| \geq |\varphi(x)|$, hence $\|x\| \geq \sup\{\varphi(x) : \varphi \in X',\ \|\varphi\| = 1\}$. To show that in fact we have equality, we recall that by Corollary 2.18 there exists a $\varphi \in X'$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$. Hence the formula in (i) is proved. Note the the supremum is in fact a maximum.

(ii) Let $M := \sup\{\varphi(Tx) : x \in X, \|x\| = 1,\ \varphi \in Y', \|\varphi\| = 1\}$. We have to show $M = \|T\|$. Obviously, $M = \infty$ if and only if $\|T\| = \infty$. Now assume $\|T\| < \infty$. Let $\varepsilon > 0$. Then there exists an $x \in X$ with $\|x\| = 1$ and $\|Tx\| \geq \|T\| - \varepsilon$. Choose a $\varphi \in X'$ such that $\|\varphi\| = 1$ and $\varphi(Tx) = \|Tx\|$. Then $M \geq \varphi(Tx) = \|T\| - \varepsilon$. Since ε is arbitrary, it follows that $M \geq \|T\|$. The revers inequality follows from

$$\varphi(Tx) \leq \|\varphi\|\,\|Tx\| \leq \|\varphi\|\,\|T\|\,\|x\| = \|T\|, \quad x \in X,\ \|x\| = 1,\ \varphi \in X',\ \|\varphi\| = 1.$$ □

Corollary 2.20. *Let X be a normed space, $Y \subseteq X$ a closed subspace. For every $x_0 \in X \setminus Y$ exists $\varphi \in X'$ such that $\varphi|_Y = 0$ and $\varphi(x_0) = 1$.*

Proof. Let $\pi : X \to X/Y$ be the canonical projection. Then $\pi(y) = 0$, $y \in Y$, and $\pi(x_0) \neq 0$. Since X is a normed space by Example 1.13, there exists a $\psi \in (X/Y)'$ such that $\varphi(\pi(x_0)) \neq 0$ and $\varphi(\pi(x_0)) = 1$. Obviously $\varphi = \psi \circ \pi \in X'$ and has the desired properties. □

Corollary 2.21. *Let X be a normed space, $Y \subseteq X$ a subspace. Then the following are equivalent:*

(i) $\overline{Y} = X$,

(ii) $\big(\varphi|_Y = 0 \implies \varphi = 0\big)$, $\quad \varphi \in X'$.

Theorem 2.22. *Let X be a normed space.*

$$X' \text{ separable} \implies X \text{ separable}.$$

Proof. By Proposition 1.9 the unit sphere $S_{X'} := \{x' \in X' : \|x'\| = 1\}$ is separable. Choose dense subset $\{x'_n : n \in \mathbb{N}\}$ of $S_{X'}$. and $x_n \in S_X := \{x \in X : \|x\| = 1\}$ with $\|x'_n(x_n)\| > \frac{1}{2}$. Let $U = \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$. We will show $U = X$. Assume this is not true. By Corollary 2.20 there exists an $x' \in S_{X'}$ such that $x' \neq 0$ and $x'|_U = 0$. Let $n \in \mathbb{N}$ such that $\|x'_n - x'\| < \frac{1}{4}$. This leads to the contradiction

$$\frac{1}{2} \leq |x'_n(x_n)| \leq |x'_n(x_n) - x'(x_n)| + |x'(x_n)| \leq \|x'_n - x'\| + |x'(x_n)| < \frac{1}{4}.$$ □

2.3 Examples of dual spaces

Theorem 2.23. (i) *Let $1 \leq p < \infty$ and q such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

with the convention $\frac{1}{\infty} = 0$. *q is called the* Hölder conjugate *of* p.
The following map is an isometric isomorphism:

$$T : \ell_q \to (\ell_p)', \quad (Tx)y = \sum_{n=0}^{\infty} x_n y_n \quad \text{for } x = (x_n) \in \ell_q,\ y = (y_n) \in \ell_p.$$

(ii) *The following map is an isometric isomorphism:*

$$T : \ell_1 \to (c_0)', \quad (Tx)y = \sum_{n=0}^{\infty} x_n y_n \quad \text{for } x = (x_n) \in \ell_1,\ y = (y_n) \in c_0.$$

Proof. (i) Let $1 < p < \infty$. T is well-defined by Hölder's inequality and

$$|(Tx)y| = \Big|\sum_{n=0}^{\infty} x_n y_n\Big| \le \|x\|_q \|y\|_p.$$

Linearity and injectivity of T is clear. The inequality above gives

$$\|Tx\| \le \|x\|_q, \quad x \in \ell_q. \tag{2.9}$$

It remains to show surjectivity of T and that $\|Tx\| \ge \|x\|$, $x \in \ell_q$. To this end, let $y' \in (\ell_p)'$ and set $x_n := y'(\mathrm{e}_n)$, $n \in \mathbb{N}$, where e_n is the nth unit vector in ℓ_p. We will show that $x := (x_n)_{n\in\mathbb{N}} \in \ell_q$ and that $Tx = y'$. For $y' = 0$ this is clear. Now assume that $y' \ne 0$. For $n \in \mathbb{N}$ define

$$t_n := \begin{cases} \frac{|x_n|^q}{x_n}, & x_n \ne 0, \\ 0, & x_n = 0. \end{cases}$$

Using $pq - p = q$ we find

$$\sum_{n=1}^{N} |t_n|^p = \sum_{n=1}^{N} |x_n|^{p(q-1)}, \qquad N \in \mathbb{N}.$$

Hence, for all $N \in \mathbb{N}$,

$$\begin{aligned} \sum_{n=1}^{N} |x_n|^q &= \sum_{n=1}^{N} x_n t_n = \sum_{n=1}^{N} t_n y'(\mathrm{e}_n) = y'\Big(\sum_{n=1}^{N} t_n \mathrm{e}_n\Big) \le \|y'\| \Big\|\sum_{n=1}^{N} t_n \mathrm{e}_n\Big\|_p \\ &= \|y'\|\Big(\sum_{n=1}^{N} |t_n|^p\Big)^{\frac{1}{p}} \le \|y'\|\Big(\sum_{n=1}^{N} |x_n|^q\Big)^{\frac{1}{p}}. \end{aligned}$$

For N large enough, the last factor in the line above is not zero, so, using $1 - \frac{1}{p} = \frac{1}{q}$, we obtain

$$\Big(\sum_{n=1}^{N} |x_n|^q\Big)^{\frac{1}{q}} \le \|y'\|$$

implying that $x \in \ell_q$. Since $(Tx)\,\mathrm{e}_n = x_n\,\mathrm{e}_n = y'\,\mathrm{e}_n$, $n \in \mathbb{N}$, and $\{\mathrm{e}_n : n \in \mathbb{N}\}$ a total subset of ℓ_p, it follows that $Tx = y'$. In particular, with the inequality above, $\|x\|_q \le \|y'\| = \|Tx\|$. Together with (2.9) it follows that $\|Tx\| = \|x\|$, that is, T is an isometry.

The proof for $p = 1$ is similar.

(ii) Well-definedness and injectivity of T are clear. Moreover $\|Tx\| \le \|x\|_1$ for every $x \in \ell_1$ because

$$\Big|\sum_{n=0}^{\infty} x_n y_n\Big| \le \|y\|_\infty \sum_{n=0}^{\infty} |x_n| = \|y\|_\infty \|x\|_1, \qquad y \in c_0,\ x \in \ell_1.$$

To show that T is surjective, let $y' \in (c_0)'$ and let $x_n := y_n(\mathrm{e}_n)$ where e_n is the nth unit vector in c_0. For $n \in \mathbb{N}$ choose $\alpha_n \in \mathbb{R}$ such that $|y'(e_n)| = \exp(\mathrm{i}\alpha_n) y'(e_n)$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} |x_n| &= \sum_{n=0}^{\infty} |y'(\mathrm{e}_n)| = \sum_{n=0}^{\infty} \exp(\mathrm{i}\alpha_n)\, y'(\mathrm{e}_n) = y'\Big(\sum_{n=0}^{\infty} \exp(\mathrm{i}\alpha_n)\ \mathrm{e}_n\Big) \\ &\le \|y'\| \Big\|\sum_{n=0}^{\infty} \exp(\mathrm{i}\alpha_n)\ \mathrm{e}_n\Big\|_\infty = \|y'\|.\end{aligned}$$

Hence $x \in \ell_1$ and $\|x\|_1 \le \|y'\|$. As before, since $\{\mathrm{e}_n : n \in \mathbb{N}\}$ is a total subset of c_0, it follows that $Tx = y'$ and the proof is complete. (Note however, that $\{\mathrm{e}_n : n \in \mathbb{N}\}$ is not dense in ℓ_∞.) □

The theorem above shows that

$$\boxed{\begin{aligned} &(\ell_p)' \cong \ell_q, \qquad 1 \le p < \infty, \\ &(c_0)' \cong \ell_1. \end{aligned}}$$

Remark. Note that $(\ell_\infty)' \not\cong \ell_1$. To see this, assume that $(\ell_\infty)' \cong \ell_1$. Since ℓ_1 is separable, Theorem 2.22 would imply that also ℓ_∞ is separable, in contradiction to Example 1.26.

Other important examples are given without proof in the following theorems.

Theorem 2.24. *Let (Ω, Σ, μ) be a σ-finite measure space. Let $1 \le p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$T : \mathscr{L}_q(\Omega) \to (\mathscr{L}_p(\Omega))', \qquad (Tf)(g) = \int_\Omega f\overline{g}\ \mathrm{d}\mu, \quad f \in \mathscr{L}_q(\Omega),\ g \in \mathscr{L}_p(\Omega),$$

is an isometric isomorphism.

Theorem 2.25 (Riesz's representation theorem). *Let K be a compact metric space and $M(K)$ the set of all regular Borel measures with finite variation, that is $\|\mu\| < \infty$ with*

$$\|\mu\| := \sup\Big\{ \sum_{V \in \mathcal{Z}} |\mu(V)| : \mathcal{Z} \text{ partition of } \mathcal{K} \text{ in pairwise disjoint measurable sets}\Big\}.$$

Let $1 \le p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$T : M(K) \to (C(K))', \qquad (T\mu)(g) = \int_\Omega g\ \mathrm{d}\mu, \quad \mu \in M(K),\ g \in C(K),$$

is an isometric isomorphism.

For a proof, see [Rud87, Theorem 6.19].
The theorems above show that

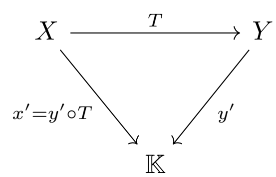
$$\boxed{\begin{aligned} (\mathscr{L}_p)' &\cong \mathscr{L}_q, \qquad 1 \le p < \infty, \\ (C(K))' &\cong M(K). \end{aligned}}$$

2.4 The Banach space adjoint and the bidual

Definition 2.26. Let X, Y be normed spaces and $T \in L(X, Y)$. The *Banach space adjoint* of T is

$$T' : Y' \to X', \qquad (T'y')x := y'(Tx), \quad y' \in Y',\ x \in X.$$

Obviously, T' is linear and continuous as composition of continuous functions, hence $T' \in L(Y', X')$ and the following diagram commutes



Theorem 2.27. *Let X, Y, Z be normed spaces.*

(i) *The map $L(X, Y) \to L(Y', X')$, $T \mapsto T'$, is linear and isometric, that is, $\|T'\| = \|T\|$. In general, it is not surjective.*

(ii) *$(ST)' = T'S'$ for $S \in L(Y, Z)$ and $T \in L(X, Y)$.*

Proof. (i) Linearity of $T \mapsto T'$ is clear. Immediately by the definition of T' we have that

$$\|T'y'\| = \|y' \circ T\| \le \|y'\|\,\|T\|, \qquad y' \in Y',$$

hence $\|T'\| \le \|T\|$. By Corollary 2.19 $\|T\|$ is

$$\|T\| = \sup\{y'(Tx) : x \in X,\ \|x\| = 1,\ y' \in Y',\ \|y'\| = 1\}.$$

For every $\varepsilon > 0$ there exist $x \in X$, $\|x\| = 1$, $y' \in Y'$ such that $\|T\| - \varepsilon < y'(Tx) = (T'y')(x) \le \|T'\|\,\|y'\|\,\|x\| = \|T'\|$, so $\|T\| \le \|T'\|$.
(ii) For all $z' \in Z'$ and $x \in X$ we have $((ST)'z')(x) = z'(ST(x)) = z'(S(Tx)) = (S'z')(Tx) = T'(S'z')x = (T'S')(z')(x)$, hence $(ST)' = T'S'$. □

Example 2.28. Let $1 \le p < \infty$. The adjoint of the left shift

$$L : \ell_p \to \ell_p, \qquad L(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

is the right shift.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $y = (y_n)_{n\in\mathbb{N}} \in l_q \cong (l_p)'$. Then for all $x = (x_n)_{n\in\mathbb{N}} \in l_p$:

$$\begin{aligned}(L'y)x = y(Lx) &= \sum_{n=1}^{\infty} y_n (Lx)_n = \sum_{n=1}^{\infty} y_n x_{n+1} = \sum_{n=2}^{\infty} y_{n-1} x_n = \sum_{n=2}^{\infty} (Ry)_n x_n \\ &= \sum_{n=1}^{\infty} (Ry)_n x_n = (Ry)x. \end{aligned}$$ □

Definition 2.29. Let X be a normed space. $X'' := (X')'$ is the *bidual* of X.

For every $x \in X$ the linear map

$$J_X(x) : X' \to \mathbb{K}, \qquad J_X(x)x' := x'x$$

is linear and bounded by $\|x\|$, hence $J_X(x) \in X''$.

Theorem 2.30. *The map*

$$J_X : X \to X'', \qquad J_X(x)x' = x'x, \quad x' \in X'$$

is a linear isometry. In general, it is not surjective.

Proof. We have seen above that J_X is well-defined, linear and $\|J_X(x)\| \le \|x\|$, $x \in X$. Now let $x \in X$ and choose $\varphi_x \in X'$ such that $\varphi_x(x) = \|x\|$ (Corollary 2.18). It follows that $\|J_X(x)\varphi_x\| = |\varphi_x(x)| = \|x\|$, hence $\|J_X(x)\| \ge 1$. □

The preceding theorem gives another easy proof that every normed space X can be completed (see Theorem 1.7).

Corollary 2.31. *Every normed space is isometrically isomorphic to a dense subspace of a Banach space.*

Proof. By the theorem above, X is isometrically isomorphic to $J_X(X) \subseteq X''$. Since X'' is complete (Theorem 2.6), the closure $\overline{J_X(X)}$ is a Banach space. □

Definition 2.32. A Banach space is called *reflexive* if J_X is surjective.

Examples 2.33. (i) Every finite-dimensional normed space is reflexive.

(ii) ℓ_p is reflexive for $1 < p < \infty$ by Theorem 2.23.

(iii) c_0 and ℓ_1 are not reflexive.

Note that there are non-reflexive Banach spaces X such that $X \cong X''$ (but J_X is not surjective).

Lemma 2.34. *Let X, Y be normed spaces and $T \in L(X, Y)$. Then $T'' \circ J_X = J_Y \circ T$, that is, the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ {\scriptstyle J_X}\downarrow & & \downarrow{\scriptstyle J_Y} \\ X'' & \xrightarrow[\ T''\ ]{} & Y'' \end{array}$$

Proof. For $x \in X$ and $y' \in Y'$

$$[T''(J_X(x))](y') = (J_X(x))(T'y') = T'y'x = y'(Tx) = (J_Y(Tx))y' = [(J_Y \circ T)(x)]y'.$$

□

If X and Y are identified with subspaces of X'' and Y'' via the canonical maps J_X and J_Y, then T'' is an extension of T. Note that with this identification $S \in L(Y', X')$ is adjoint operator of some $T \in L(X, Y)$ if and only if $S'(X) \subseteq Y$.

Lemma 2.35. *Let X be a normed space. Then $J'_X \circ J_{X'} = \mathrm{id}_{X'}$.*

Proof. Note that $J_{X'} : X' \to X'''$ and $J'_X : X''' \to X'$. For $x \in X$, $x' \in X'$

$$[(J'_X \circ J_{X'})x'](x) = [J_{X'}x'](J_X(x)) = [J_X x]x' = x'x.$$ □

Theorem 2.36. (i) *Every closed subspace of a reflexive normed space is reflexive.*

(ii) *A Banach space X is reflexive if and only if X' is reflexive.*

Proof. (i) Let U be a closed subspace of a reflexive normed space X and let $u'' \in U''$. We have to find a $u \in U$ such that $J_X(u) = u''$. Let $x''_0 : X' \to K$, $x''_0(x') = u''(x'|_U)$. Obviously, x''_0 is linear and bounded because

$$|x''_0(x')| = |u''(x'|_U)| \le \|u''\|\|x'|_U\| \le \|u''\|\|x'\|,$$

hence $x''_0 \in X''$. Since X is reflexive there exists an $x_0 \in X$ such that $J_X(x_0) = x''_0$. Assume that $x_0 \notin U$. Since U is closed, there exists a $\varphi \in X'$ such that $\varphi|_U = 0$ and $\varphi(x_0) = 1$ (Corollary 2.20). On the other hand $\varphi(x_0) = 0$ by choice of x_0 because

$$x'(x_0) = x''_0(x') = J_X(x_0)x' = u''(x'|_U), \qquad x' \in X',$$

Therefore $x_0 \in U$. It remains to be shown that $J_U(x_0) = u''$, that is

$$u''(u') = u'(x_0), \qquad u' \in U'.$$

Let $u' \in U'$ and choose an arbitrary extension $\varphi \in X'$ (Corollary 2.17). By definition of x_0 it follows that

$$u''(u') = u''(\varphi|_U) = x''_0(\varphi) = \varphi(x_0) = u'(x_0).$$

(ii) Let X be reflexive. We have to show that $J_{X'} : X' \to X'''$ is surjective. Let $x'''_0 \in X'''$. The map $x'_0 : X \to \mathbb{K}$, $x'_0(x) = x'''_0(J_X(x))$ is linear and bounded, hence $x'_0 \in X'$. We will show that

$J_{X'}(x_0') = x_0'''$. Let $x'' \in X''$. Since X is reflexive, there exists an $x \in X$ such that $J_X(x) = x''$. Therefore

$$J_{X'}(x_0')x'' = x''(x_0') = J_X(x)(x_0') = x_0'x = x_0'''(J_X(x)) = x_0'''(x''),$$

hence indeed $J_{X'}(x_0') = x_0'''$.

Now assume that X' is reflexive. By what was is already proved, X'' is reflexive. Since X is a closed subspace of X'' via the canonical map J_X, X is reflexive by part (i) of the theorem. □

Corollary 2.37. *A reflexive normed space X is separable if and only if X' is separable.*

Proof. That separability of X' implies separability of X was shown in Theorem 2.22. If X is separable and reflexive, then also X'' is separable. By Theorem 2.36 X' is reflexive, so we can again apply Theorem 2.22 to obtain that X' is separable. □

Definition 2.38. Let X be a normed space. A sequence $(x_n)_{n\in\mathbb{N}}$ *converges weakly* to $x_0 \in X$ if and only if

$$\lim_{n\to\infty} x'(x_n) = x'(x_0), \qquad x' \in X'.$$

Notation: $x_n \xrightarrow{w} x$ or $w\text{-}\lim_{n\to\infty} x_n = x$.

If it should be emphasised that a sequence converges with respect to the norm in the given Banach space, then the sequence is called *norm convergent*. Sometimes the notion *strongly convergent* is used. Note, however, that in spaces of linear operators the term “strong convergence” has another meaning (see Defintion 3.12).

The next remark shows that strong convergence is indeed stronger than weak convergence.

Remarks 2.39. (i) If the weak limit of a sequence exists, then it is unique, because, by the Hahn-Banach theorem, the dual space separates points (Corollary 2.18).

(ii) Every convergent sequence is weakly convergent with the same limit.

(iii) A weakly convergent sequence is not necessarily convergent. Consider for example the sequence of the unit vectors $(\mathrm{e}_n)_{n\in\mathbb{N}}$ in c_0. Let $\varphi \in c_0' \cong \ell_1$. Then $\lim_{n\in\mathbb{N}} \varphi(\mathrm{e}_n) = 0$ but the sequence of the unit vectors does not converge in norm.

Example 2.40. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C([0,1])$. Then the following is equivalent:

(i) $(x_n)_{n\in\mathbb{N}}$ converges weakly to $y \in C[(0,1)]$.

(ii) $(x_n)_{n\in\mathbb{N}}$ converges pointwise to $y \in C[(0,1)]$.

Proof. “(i) $\implies$ (ii)” It is easy to see that for every $t_0 \in [0,1]$ the point evaluation $x \mapsto x(t_0)$ is a bounded linear functional. Hence for all $t \in [0,1]$ the sequence $(x_n(t)_{n\in\mathbb{N}}$ converges to some $y(t)$. By assumption, $[0,1] \to \mathbb{K},\ t \mapsto y(t)$ belongs to $C([0,1])$.

“(ii) $\implies$ (i)” follows from Riesz's representation theorem (Theorem 2.25) and the Lebesgue convergence theorem (see A.19). □

Theorem 2.41. *Every bounded sequence in a reflexive normed space contains a weakly convergent subsequence.*

Proof. Let X be a reflexive normed space and $x = (x_n)_{n\in\mathbb{N}} \subseteq X$ be a bounded sequence.

First we assume that X is separable. By theorem 2.37, also X' is separable. Let $\{\varphi_n : n \in \mathbb{N}\}$ be a dense subset of X'. We will construct a subsequence $y = (y_n)_{n\in\mathbb{N}}$ of x such that for every $j \in \mathbb{N}$ the sequence $(\varphi_j(y_n))_{n\in\mathbb{N}}$ converges. The sequence $(\varphi_1(x_n))_{n\in\mathbb{N}}$ is bounded, so it contains a convergent subsequence

$$(\varphi_1(x_{n_1,1}),\ \varphi_1(x_{n_1,2}),\ \varphi_1(x_{n_1,3}),\ \dots)$$

Now the sequence $(\varphi_2(x_{n_1,j}))_{j\in\mathbb{N}}$ is bounded, so it contains a convergent subsequence

$$(\varphi_2(x_{n_2,1}),\ \varphi_2(x_{n_2,2}),\ \varphi_2(x_{n_2,3}),\ \dots)$$

Continuing like this, we obtain a sequence of subsequences $x_{n_m} = (x_{n_m,j})_{j\in\mathbb{N}}$, $m \in \mathbb{N}$ such that $(\varphi_m(x_{n_m,j}))_{j\in\mathbb{N}}$ converges. Now the "diagonal sequence" y with $y_m := x_{n_m,m}$ has the desired property.

Now we will show that y is weakly convergent. Let $x' \in X'$ and $\varepsilon > 0$. Choose an $k \in \mathbb{N}$ such that $\|x' - \varphi_k\| < \frac{\varepsilon}{4M}$ where $M := \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$. Let $N \in \mathbb{N}$ such that $|\varphi_k(y_n) - \varphi_k(y_m)| < \frac{\varepsilon}{2}$, $m, n \geq N$. It follows for $m, n \geq N$:

$$\begin{aligned} |x'(y_n) - x'(y_m)| &\leq |x'(y_n) - \varphi_k(y_n)| + |\varphi_k(y_n) - \varphi_k(y_m)| + |\varphi_k(y_m) - x'(y_m)| \\ &\leq 2M\|x' - \varphi_k\| + |\varphi_k(y_n) - \varphi_k(y_m)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $(x'(y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$, hence it converges. To show that $(y_n)_{n\in\mathbb{N}}$ converges weakly, define the map

$$\psi : X' \to \mathbb{K}, \quad \psi(x') = \lim_{n\to\infty} x'(y_n).$$

By what is already shown, ψ is well-defined and linear. It is also bounded because

$$|\psi(x')| = \big|\lim_{n\to\infty} x'(y_n)\big| = \lim_{n\to\infty} |x'(y_n)| \leq \lim_{n\to\infty} \|x'\|\,\|(y_n)\| \leq M\|x'\|.$$

Hence $\psi \in X''$. Since X is reflexive, there exists a $y_0 \in X$ such that $x'(y_0) = \psi(x') = \lim\limits_{n\to\infty} x'(y_n)$. Hence $(y_n)_{n\in\mathbb{N}}$ converges weakly to y_0.

Now assume that X is not separable. Let $Y := \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$ where $(x_n)_{n\in\mathbb{N}}$ is the bounded sequence in X chosen at the beginning of the proof. Y is separable (Theorem 1.25) and reflexive (Theorem 2.36). Hence, by the first step of the proof, there exists a subsequence $(y_n)_{n\in\mathbb{N}} \subseteq Y$ of $(x_n)_{n\in\mathbb{N}}$ and a y_0 such that $y_n \xrightarrow{w} y_0$ in Y. Let $x' \in X'$. Then $x'|_{Y'} \in Y'$, hence $\lim\limits_{n\to\infty} x'(y_n) = \lim\limits_{n\to\infty} x'|_{Y'}(y_n) = x'|_{Y'}(y_0) = x'(y_0)$. Therefore we also have $y_n \xrightarrow{w} y_0$ in X. □

Chapter 3

Linear operators in Banach spaces

3.1 Baire's theorem

Theorem 3.1 (Baire-Hausdorff). *Let (X, d) be a complete metric space and $(A_n)_{n\in\mathbb{N}}$ be a family of open dense subsets of X. Then $\bigcap_{n=1}^{\infty} A_n$ is dense in X.*

Taking complements, it is easily seen that the theorem above implies

Theorem. *Let (X, d) be a complete metric space and $(B_n)_{n\in\mathbb{N}}$ be a family of closed subsets of X such that $\bigcup_{n=1}^{\infty} B_n$ contains an open subset. Then at least one of the sets B_n contains a non-empty open subset.*

Proof of Theorem 3.1. For $r > 0$ and $x \in X$ let $B(x, r) := \{\xi \in X : \|x - \xi\| < r\}$. We have to show that any open ball in X has non-empty intersection with $\bigcap_{n\in\mathbb{N}} A_n$. Let $\varepsilon > 0$ and $x_0 \in X$.
A_1 is open and dense in X, hence $A_1 \cap B(x_0, \varepsilon)$ is open and not empty. Hence there exist $\varepsilon_1 \in (0, 2^{-1}\varepsilon)$ and $x_1 \in A_1$ such that $B(x_1, \varepsilon_1) \subseteq A_1 \cap B(x_0, \varepsilon)$, hence

$$\overline{B(x_1, \tfrac{\varepsilon_1}{2})} \subseteq B(x_1, \varepsilon_1) \subseteq A_1 \cap B(x_0, \varepsilon).$$

A_2 is open and dense in X, hence $A_2 \cap B(x_1, \frac{\varepsilon_1}{2})$ is open and not empty. Hence there exist $\varepsilon_2 \in (0, 2^{-2}\varepsilon)$ and $x_2 \in A_2$ such that $B(x_2, \varepsilon_2) \subseteq A_2 \cap B(x_1, \frac{\varepsilon_1}{2})$, hence

$$\overline{B(x_2, \tfrac{\varepsilon_2}{2})} \subseteq B(x_2, \varepsilon_2) \subseteq A_2 \cap B(x_1, \tfrac{\varepsilon_1}{2}) \subseteq A_2 \cap A_1 \cap B(x_0, \varepsilon_1).$$

In this way we obtain sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < 2^{-n}\varepsilon$ and

$$\overline{B(x_n, \tfrac{\varepsilon_n}{2})} \subseteq B(x_n, \varepsilon_n) \subseteq A_n \cap B(x_{n-1}, \varepsilon_{n-1}) \subseteq A_{n-1} \cap \dots A_2 \cap A_1 \cap B(x_0, \varepsilon_1). \qquad (3.1)$$

Observe that $x_n \in B(x_N, \frac{\varepsilon_N}{2})$ for $N \in \mathbb{N}$ and $n \geq N$. This implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X because, for fixed $N \in \mathbb{N}$ and all $n, m > N$ we obtain $d(x_m, x_n) \leq d(x_m, x_N) + d(x_n, x_N) < 2^{-N+1}$. Since X is complete, $y := \lim\limits_{n\to\infty} x_n$ exists and $x_0 \in \overline{B(x_N, \varepsilon_N)}$ for every $N \in \mathbb{N}$ because for fixed N, we have that $x_n \in B(x_N, \frac{\varepsilon_N}{2})$ if $n \geq N$. Hence (3.1) implies

$$y \in \overline{B(x_N, \tfrac{\varepsilon_N}{2})} \subseteq B(x_{N-1}, \varepsilon_{N-1}) \subseteq A_{N-1} \cap \dots A_2 \cap A_1 \cap B(x_0, \varepsilon_1), \qquad N \geq 2,$$

so $y \in \bigcap_{n\in\mathbb{N}} A_n \cap B(x_0, \varepsilon)$. □

Definition 3.2. Let (X, d) be a metric space.

- $A \subseteq X$ is called *nowhere dense* in X, if $\overline{A}$ does not contain an open set.
- $A \subseteq X$ is of *first category* if it is the countable union of nowhere dense sets.
- $A \subseteq X$ is of *second category* if it is not of first category.

Note that A is nowhere dense if and only if $X \setminus \overline{A}$ is dense in X.
An equivalent formulation of

Theorem 3.3 (Baire's category theorem). *A complete metric space is of second category in itself.*

Examples 3.4. $\mathbb{Q}$ is of first category in $\mathbb{R}$. $\mathbb{R}$ is of second category in $\mathbb{R}$.

3.2 Uniform boundedness principle

Definition 3.5. Let (X, d) be a metric space. A family $\mathcal{F} = (f_\lambda)_{\lambda\in\Lambda}$ of maps $X \to \mathbb{R}$ is called *uniformly bounded* if there exists an $M \in \mathbb{R}$ such that

$$|f_\lambda(x)| \le M, \qquad x \in X,\ \lambda \in \Lambda.$$

The next theorem shows that a family of pointwise bounded continuous functions on a complete metric space is necessarily uniformly continuous on a certain ball.

Theorem 3.6 (Uniform boundedness principle). *Let X be a complete metric space, Y a normed space and $\mathcal{F} \subseteq C(X, Y)$ a family of continuous functions which is pointwise bounded, i. e.,*

$$\forall\, x \in X \quad \exists\, C_x \ge 0 \quad \forall\, f \in \mathcal{F} \quad \|f(x)\| < C_x.$$

Then there exists an $M \in \mathbb{R}$, $x_0 \in X$ and $r > 0$ such that

$$\forall\, x \in B_r(x_0) \quad \forall\, f \in \mathcal{F} \quad \|f(x)\| < M. \tag{3.2}$$

Proof. For $n \in \mathbb{N}$ let

$$A_n := \bigcap_{f\in\mathcal{F}} \{x \in X : \|f(x)\| \le n\}.$$

Note that for every $n \in \mathbb{N}$ the set $\{x \in X : \|f(x)\| \le n\}$ is closed because f and $\|\cdot\|$ are continuous. Since all A_n are intersections of closed sets, they are closed. Let $x \in X$. Since $\mathcal{F}$ is pointwise bounded, there exists an $n_x \in \mathbb{N}$ such that $x \in A_{n_x}$, hence $X \subseteq \cup_{n\in\mathbb{N}} A_n$. By Baire's theorem exists an $N \in \mathbb{N}$, $x_0 \in X$, $r > 0$ such that $B_r(x_0) \subseteq A_N$, that is, (3.2) is satisfied with $M = N$. □

The Banach-Steinhaus theorem is obtained in the special case of linear bounded functions.

Theorem 3.7 (Banach-Steinhaus theorem). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subseteq L(X,Y)$ a family of continuous linear functions which is pointwise bounded, i. e.,*

$$\forall\, x \in X \quad \exists\, C_x \geq 0 \quad \forall\, f \in \mathcal{F} \quad \|f(x)\| < C_x.$$

Then there exists an $M \in \mathbb{R}$ such that

$$\|f\| < M, \qquad f \in \mathcal{F}.$$

Proof. By the uniform boundedness principle there exists an open ball $B_r(x_0) \subseteq X$ and an $M' \in \mathbb{R}$ such that $\|f(x)\| < M'$ for all $x \in B_r(x_0)$ and $f \in \mathcal{F}$. For $x \in X$ with $\|x\| = 1$ and $f \in \mathcal{F}$ we find

$$\begin{aligned}\|f(x)\| &= \frac{1}{r}\,\|f(rx)\| = \frac{1}{r}\,\|f(x_0) - f(x_0 - rx)\| \\ &\leq \frac{1}{r}\,(\|f(x_0)\| + \|f(\underbrace{x_0 - rx}_{\in B_r(x_0)})\|) \leq \frac{2M'}{r} =: M,\end{aligned}$$

showing that $\mathcal{F}$ is uniformly bounded by M. □

Corollary 3.8. *Let X be a normed space and $A \subseteq X$. Then the following are equivalent:*

(i) *A is bounded.*
(ii) *For every $x' \in X'$ the set $\{x'(a) : a \in A\}$ is bounded.*

Proof. "(i) $\Longrightarrow$ (ii)" is clear.
"(ii) $\Longrightarrow$ (i)" The family $(J_X(a))_{a\in A} \subseteq X''$ is pointwise bounded by assumption. By the Banach-Steinhaus theorem there exists a $M \in \mathbb{R}$ such that

$$\|a\| = \|J_X(a)\| \leq M, \qquad a \in A.$$

Hence A is bounded. □

Corollary 3.9. *Every weakly convergent sequence in a normed space is bounded.*

Proof. Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in X. By hypothesis, for every $x' \in X'$ the set $\{x'(x_n) : n \in \mathbb{N}\}$ is bounded. Therefore, by Corollary 3.8, the set $\{x_n : n \in \mathbb{N}\}$ is bounded. □

The following theorem follows directly from Theorem 2.41 and Corollary 3.9.

Theorem 3.10. *Let $(X, \|\cdot\|)$ be a normed space, $(x_n)_{n\in\mathbb{N}}$ and $x_0 \in X$. Then the following is equivalent:*

(i) $x_0 = w\text{-}\lim\limits_{n\to\infty} x_n$.
(ii) *$(x_n)_{n\in\mathbb{N}}$ is bounded and there exists a total subset $M' \subseteq X'$ such that*

$$\lim_{n\to\infty} f(x_n) = f(x_0), \qquad f \in M'.$$

Corollary 3.11. *Let X be Banach space and $A' \subseteq X'$. Then the following is equivalent:*

(i) *A' is bounded.*

(ii) *For all $x \in X$ the set $\{a'(x) : a' \in A'\}$ is bounded.*

Proof. The implication "(i) $\Longrightarrow$ (ii)" is clear. The other direction follows directly from the Banach-Steinhaus theorem. □

Note that for "(ii) $\Longrightarrow$ (i)" the assumption that X is a Banach space is necessary. For example, let $d = \{x = (x_n)_{n\in\mathbb{N}} : x_n \neq 0 \text{ for at most finitely many } n\} \subseteq \ell_\infty$. d is a non-complete normed space (see Example 1.15 (5)). For $m \in \mathbb{N}$ define the linear function $\varphi_m : d \to \mathbb{K}$ by $\varphi_m(\mathrm{e}_n) = m\delta_{m,n}$ where $\delta_{m,n}$ is the Kronecker delta. Obviously $\varphi_m \in d'$ and $\|\varphi_m\| = m$, hence the family (φ_m) is not bounded in d', but for every fixed $x \in d$ the set $\{\varphi_m(x) : m \in M\}$ is.

Definition 3.12. Let X, Y be normed spaces, $(T_n)_{n\in\mathbb{N}} \in L(X,Y)$ a sequence of bounded linear operators and $T \in L(X,Y)$.

(i) $(T_n)_{n\in\mathbb{N}}$ *converges* to T, denoted by $\lim\limits_{n\to\infty} T_n = T$, if and only if

$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

(ii) $(T_n)_{n\in\mathbb{N}}$ *converges strongly* to T, denoted by $s\text{-}\lim\limits_{n\to\infty} T_n = T$ or $T_n \xrightarrow{s} T$, if and only if

$$\lim_{n\to\infty} \|T_n x - Tx\| = 0, \qquad x \in X.$$

(iii) $(T_n)_{n\in\mathbb{N}}$ *converges weakly* to T, denoted by $w\text{-}\lim\limits_{n\to\infty} T_n = T$ or $T_n \xrightarrow{w} T$, if and only if

$$\lim_{n\to\infty} |\varphi(T_n x) - \varphi(Tx)| = 0, \qquad x \in X,\ \varphi \in Y'.$$

Remark. (i) The limits are unique if they exist.

(ii) Convergence in norm implies strong convergence and the limits are equal. Strong convergence implies weak convergence and the limits are equal.

The reverse implications are not true:

- Let $X = \ell_2(\mathbb{N})$, $T_n : X \to X$, $T_n x = (x_1, \dots, x_n, 0, \dots)$ for $x = (x_m)_{m\in\mathbb{N}}$. Then T converges strongly to id but $\|T_n - \mathrm{id}\| = 1$ for all $n \in \mathbb{N}$, so that $(T_n)_{n\in\mathbb{N}}$ does not converge to id in norm.
- Let $X = \ell_2(\mathbb{N})$, $T_n : X \to X$, $T_n x = (0, \dots, 0, x_1, x_2, \dots)$ (n leading zeros) for $x = (x_m)_{m\in\mathbb{N}}$. Then T converges weakly to 0 but $\|T_n x\| = 1$ for all $n \in \mathbb{N}$, so that $(T_n)_{n\in\mathbb{N}}$ does not converge strongly to 0.

Proposition 3.13. *Let X be a Banach space, Y be a normed space and $(T_n)_{n\in\mathbb{N}} \subseteq L(X,Y)$ such that for all $x \in X$ the limit $Tx := \lim\limits_{n\in\mathbb{N}} T_n x$ exists. Then $T \in L(X,Y)$.*

Proof. It is clear that T is well-defined and linear. By the uniform boundedness principle, there exists an $C \in \mathbb{R}$ such that $\|T_n\| < C$ for all $n \in \mathbb{N}$. Now let $x \in X$ with $\|x\| = 1$. Then $\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \sup_{n\in\mathbb{N}} \|T_n\| \, \|x\| \le C$ which implies that $T \in L(X,Y)$. □

We finish this section with a result on strong convergence of positive operators on a space of continuous functions. An operator T on a function space is called *positivity preserving* if $Tf \ge 0$ for every $f \ge 0$ in the domain of T.

Theorem 3.14 (Korovkin). *Let $X = C[0,2\pi]$ the space of the continuous functions on $[0,2\pi]$ and let $x_j \in X$ with $x_0(t) = 1$, $x_1(t) = \cos(t)$, $x_2(t) = \sin(t)$ for $t \in [0,2\pi]$. Let $(T_n)_{n\in\mathbb{N}} \subseteq L(X)$ be a sequence of positivity preserving operators such that $T_n x_j \to x_j$ for $n \to \infty$ and $j = 0,1,2$. Then $(T_n)_{n\in\mathbb{N}}$ converges strongly to* id*, that is, $T_n x \to x$ for all $x \in X$.*

Proof. We define the auxiliary functions

$$y_t(s) = \sin^2 \frac{t-s}{2}, \qquad t, s \in [0,2\pi].$$

Note that $y_t(s) = \frac{1}{2}(1 - \cos(s)\cos(t) - \sin(s)\sin(t))$, hence $y_t \in \operatorname{span}\{x_0, x_1, x_2\}$, in particular $T_n y_t \to y_t$ for $n \to \infty$.

Now fix $x \in X$ and $\varepsilon > 0$. Since x is uniformly continuous there exists a $\delta > 0$ such that for all $s, t \in [0,2\pi]$

$$y_t(s) = \sin^2 \frac{t-s}{2} < \delta \quad \Longrightarrow \quad |x(t) - x(s)| < \varepsilon.$$

Setting $\alpha = \frac{2\|x\|_\infty}{\delta}$ we obtain that

$$|x(t) - x(s)| \le \varepsilon + \alpha y_t(s), \qquad s, t \in [0,2\pi],$$

because either s, t are such that $y_t(s) < \delta$, then $|x(t) - x(s)| < \delta$ by definition of δ; or $y_t(s) \ge \delta$, then $|x(t) - x(s)| \le 2\|x\|_\infty = \alpha\delta \le \alpha y_t(s)$. Hence we have that

$$\begin{aligned} & -\varepsilon - \alpha y_t(s) \le x(t) - x(s) \le \varepsilon + \alpha y_t(s), && s, t \in [0,2\pi] \\ \Longrightarrow \quad & -\varepsilon x_0 - \alpha y_t \le x(t)x_0 - x \le \varepsilon x_0 + \alpha y_t, && t \in [0,2\pi] \end{aligned}$$

and since T_n is positive and y_t is a positive function

$$-\varepsilon T_n x_0 - \alpha T_n y_t \le x(t) T_n x_0 - T_n x \le \varepsilon T_n x_0 + \alpha T_n y_t, \qquad t \in [0,2\pi].$$

Since $T_n x_0 \to x_0$ and $T_n y_t \to \frac{1}{2}(1 - \cos(t)x_1 - \sin(t)x_2)$ for $n \to \infty$, we can find $N \in \mathbb{N}$ large enough such that $\varepsilon T_n x_0 + \alpha T_n y_t < \varepsilon x_0 + \alpha y_t + \varepsilon$ for all $n \ge N$, hence

$$|x(t) T_n x_0 - T_n x| \le \varepsilon x_0 + \alpha y_t + \varepsilon, \qquad t \in [0,2\pi], \ n \ge N.$$

Hence $x T_n x_0 - T_n x_0$ converges to 0 in norm in X because by the inequality above

$$|x(t)(T_n x_0)(t) - (T_n x)(t)| \le \varepsilon + \alpha y_t(t) + \varepsilon = 2\varepsilon, \qquad t \in [0,2\pi], \ n \ge N.$$

That $T_n x \to x$ follows now from

$$\|x - x T_n x_0\|_\infty + \|x T_n x_0 - T_n\|_\infty \le \|x\| \, \|x_0 - T_n x_0\|_\infty + \|x T_n x_0 - T_n\|_\infty. \qquad \square$$

Fourier Series

Definition 3.15. Let $x : \mathbb{R} \to \mathbb{R}$ a 2π-periodic integrable function. The Fourier series of x is

$$S(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)),$$

where

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} x(s) \cos(ks) \, \mathrm{d}s, \qquad k \in \mathbb{N}_0,$$
$$b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} x(s) \sin(ks) \, \mathrm{d}s, \qquad k \in \mathbb{N}.$$

Note that the Fourier series is a formal series only. In the following we will prove theorems on convergence of the Fourier series.

First we will use methods from Analysis 1 to show that for a continuously differentiable periodic function its Fourier series converges uniformly to the function. Next we will use the uniform boundedness principle to show that there exist continuous functions whose Fourier series does not converge pointwise everywhere. Finally, the Korovkin theorem implies that the arithmetic means of the partial sums of the Fourier series of a periodic function converges uniformly to the function.

For a given 2π-periodic function and $n \in \mathbb{N}$ we define the nth partial sum

$$s_n(x,t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kt) + b_k \sin(kt)). \tag{3.3}$$

Lemma 3.16.

$$s_n(x,t) = \frac{1}{\pi} \int_{-\pi}^{\pi} x(s+t) D_n(s) \, \mathrm{d}s \quad \textit{with} \quad D_n(s) = \begin{cases} \frac{\sin((n+\frac{1}{2})s)}{2\sin(\frac{s}{2})}, & s \neq 0, \\ n + \frac{1}{2} & s = 0. \end{cases} \tag{3.4}$$

D_n is called Dirichlet kernel. *D_n is continuous and*

$$\frac{1}{\pi} \int_{-\pi}^{\pi} D_n(s) \, \mathrm{d}s = 1. \tag{3.5}$$

Proof. Using the trigonometric identity $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ and that x is 2π-periodic we obtain

$$\begin{aligned} s_n(x,t) &= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kt) + b_k \sin(kt)) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x(s) \Big(\frac{1}{2} + \sum_{k=1}^{n} (\cos(ks)\cos(kt) + \sin(ks)\sin(kt)) \Big) \, \mathrm{d}s \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x(s) \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos(k(s-t)) \Big) \, \mathrm{d}s \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x(s+t) \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos(ks) \Big) \, \mathrm{d}s. \end{aligned}$$

Now we calculate for $s \neq 0$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(ks) = \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{n}(\mathrm{e}^{\imath s} + \mathrm{e}^{-\mathrm{i}ks}) = \frac{1}{2}\sum_{k=-n}^{n} \mathrm{e}^{\imath ks} = \frac{\mathrm{e}^{-\mathrm{i}ns}}{2}\sum_{k=0}^{2n}\mathrm{e}^{\imath ks}$$
$$= \frac{\mathrm{e}^{-\mathrm{i}ns}}{2}\,\frac{\mathrm{e}^{\imath 2ns}-1}{\mathrm{e}^{\imath s}-1} = \frac{1}{2}\,\frac{\mathrm{e}^{\imath(n+\frac{1}{2})s} - \mathrm{e}^{-\imath(n+\frac{1}{2})s}}{\mathrm{e}^{\imath s/2} - \mathrm{e}^{-\imath s/2}} = \frac{\sin((n+\frac{1}{2})s)}{2\sin\frac{s}{2}} = D_n(s).$$

Note that $\lim\limits_{s\to 0} D_n(s) = n + \frac{1}{2} = \frac{1}{2} + \sum_{k=1}^{n}\cos(0)$. For the proof of (3.5) let $x = 1$ a constant function on $\mathbb{R}$. Then, by (3.3),

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(s)\,\mathrm{d}s = s_n(x,t) = x(t) = 1. \qquad \square$$

Theorem 3.17. *Let $x : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic continuously differentiable function. Then the Fourier series of x converges uniformly to x.*

Proof. Let $x : \mathbb{R} \to \mathbb{R}$ a 2π-periodic continuously differentiable function. Let $\varepsilon > 0$ and $h \in (0,\pi)$ such that $h < \frac{\varepsilon}{\pi\|x'\|_\infty}$. Using (3.4) and (3.5) it follows that

$$|x(s) - s_n(x,t)| = \Big|\,\frac{1}{\pi}\int_{-\pi}^{\pi}(x(s+t) - x(t))D_n(s)\,\mathrm{d}s\,\Big|$$
$$\leq \frac{1}{\pi}\Big(\underbrace{\Big|\int_{-\pi}^{-h}\dots\,\mathrm{d}s\Big|}_{=:A_n(t)} + \underbrace{\int_{-h}^{h}|\dots|\,\mathrm{d}s}_{=:B_n(t)} + \underbrace{\Big|\int_{h}^{\pi}\dots\,\mathrm{d}s\Big|}_{=:C_n(t)}\Big).$$

We have to show that $A_n(t)$, $B_n(t)$ and $C_n(t)$ tend to 0 for $n \to \infty$ uniformly in t. Using the mean value theorem and that $\frac{\pi}{2}\sigma \leq \sin(\sigma)$ for $\sigma \in [0,\pi/2]$ we obtain

$$B_n(t) = \int_{-h}^{h}\frac{|x(s+t) - x(t)|}{2\sin|\frac{s}{2}|}\underbrace{|\sin((n+\tfrac{1}{2})s)|}_{\leq 1}\,\mathrm{d}s \leq \int_{-h}^{h}\frac{\|x'\|\,|s|}{2\sin|\frac{s}{2}|}\,\mathrm{d}s$$
$$\leq 2h\|x'\|_\infty\frac{\pi}{2} < \frac{\varepsilon}{2}.$$

Define the auxiliary function

$$f_t(s) = \frac{x(s+t) - x(t)}{2\sin(\frac{s}{2})}, \qquad s \in [h,\pi],\ t \in [0,\pi].$$

The functions f_t are continuously differentiable and $\|f_t\|_\infty \leq \frac{2\|x\|_\infty}{2\sin(h/2)} =: M_1$, $\|f_t'\|_\infty \leq \frac{\|x'\|_\infty}{2\sin(h/2)} =: M_2$. Note that the bounds do not depend on t. Integrating by parts, we find

$$C_n(t) = \Big|\int_{h}^{\pi} f_t(s)\sin((n+\tfrac{1}{2})s)\,\mathrm{d}s\,\Big|$$
$$= \Big|-\frac{\cos((n+\frac{1}{2})s)}{n+\frac{1}{2}}f_t(s)\Big|_h^\pi + \int_h^\pi \frac{\cos((n+\frac{1}{2})s)}{n+\frac{1}{2}}\,f_t'(s)\,\mathrm{d}s\,\Big|$$
$$\leq \frac{1}{n+\frac{1}{2}}\,(2M_1 + (\pi - h)M_2) =: \frac{M}{n+\frac{1}{2}}.$$

Note that M' does not depend on t. When we choose N such that $\frac{M}{n+\frac{1}{2}} < \frac{\varepsilon}{2}$ we obtain finally $|x(s) - s_n(x,t)| < \varepsilon$ for all $t \in \mathbb{R}$, that is, $\|x - s_n(x,\cdot)\|_\infty < \varepsilon$. □

Theorem 3.18. *There exists a 2π-periodic continuous function x whose Fourier series does not converge everywhere pointwise to x.*

Proof. We identify the 2π-periodic functions on $\mathbb{R}$ with

$$X := \big\{x \in C([-\pi,\pi]) \,:\, x(-\pi) = x(\pi)\big\}.$$

Clearly $(X, \|\cdot\|_\infty)$ is a Banach space.
Note that for fixed $t \in [-\pi,\pi]$ and $n \in \mathbb{N}$

$$s_n(\,\cdot\,,t) : X \to \mathbb{K}$$

is linear and bounded, hence an element in X'.
Assume that for every $x \in X$ its Fourier series converges pointwise to x. Then for every $x \in X$ and $t \in [-\pi,\pi]$ the sequence $(s_n(x,t))_{n\in\mathbb{N}}$ is bounded (because it converges to $x(t)$). By the uniform boundedness principle there exists C_t such that $\|s_n(\,\cdot\,,t)\| \le C_t$ for all $n \in \mathbb{N}$. In particular, we have

$$\|s_n(\,\cdot\,,0)\| \le C_0, \qquad n \in \mathbb{N}.$$

It is easy to see that

$$\|s_n(x,0)\| = \frac{1}{\pi}\Big|\int_{-\pi}^{\pi} x(s)D_n(s)\,\mathrm{d}s\Big| \le \frac{1}{\pi}\|x\|_\infty \int_{-\pi}^{\pi} |D_n(s)|\,\mathrm{d}s$$

hence $\|s_n(\,\cdot\,,0)\| \le \int_{-\pi}^{\pi} |D_n(s)|\,\mathrm{d}s$. On the other hand, the function $y(s) = \operatorname{sign}(D_n(s))$ can be approximated by continuous functions y_m with $\|y_m\| = 1$ such that

$$\|s_n(y_m,0)\| = \frac{1}{\pi}\int_{-\pi}^{\pi} x(s)D_n(s)\,\mathrm{d}s \to \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sign}(D_n(s))D_n(s)\,\mathrm{d}s = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(s)|\,\mathrm{d}s$$

so that finally we obtain

$$\|s_n(\,\cdot\,,0)\| = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(s)|\,\mathrm{d}s < C_0, \qquad n \in \mathbb{N}.$$

However $\|s_n(\,\cdot\,,0)\| \to \infty$ for $n \to \infty$ because

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_n(s)|\,\mathrm{d}s \;&=\; 2\int_0^{\pi} \frac{|\sin((n+\frac{1}{2})s)|}{2\sin\frac{s}{2}}\,\mathrm{d}s \;\ge\; 2\int_0^{\pi} \frac{|\sin((n+\frac{1}{2})s)|}{s}\,\mathrm{d}s \\
&=\; 2\int_0^{\pi(n+\frac{1}{2})} \frac{|\sin\sigma|}{\sigma}\,\mathrm{d}\sigma \;\ge\; 2\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi} \frac{|\sin\sigma|}{\sigma}\,\mathrm{d}\sigma \\
&\ge\; 2\sum_{k=0}^{n-1} \frac{1}{\pi(k+1)}\int_{k\pi}^{(k+1)\pi} |\sin\sigma|\,\mathrm{d}\sigma \;=\; 4\pi\sum_{k=0}^{n-1}\frac{1}{\pi(k+1)}. \\
&=\; 4\sum_{k=1}^{n}\frac{1}{k}.
\end{aligned}$$

Hence the theorem is proved. □

Finally we show that the arithmetic mean of the partial sums of the Fourier series of a continuous function converge.

Theorem 3.19 (Fejér). *As before let*

$$X := \{x \in C([-\pi,\pi]) \,:\, x(-\pi) = x(\pi)\}$$

and let $T_n \in L(X)$ defined by

$$T_n x = \frac{1}{n}\sum_{k=0}^{n-1} s_n(x,\,\cdot\,).$$

Then $(T_n)_{n\in\mathbb{N}}$ converges strongly to id *(i. e. $T_n x \to x$ for $n \to \infty$, $x \in X$).*

Proof. Note that the T_n are well-defined and that for all $x \in X$ and $t \in [-\pi,\pi]$

$$T_n x(t) = \frac{1}{n}\sum_{k=0}^{n-1}\int_{-\pi}^{\pi} x(s+t)\, D_k(s)\,\mathrm{d}s = \frac{1}{n\pi}\int_{-\pi}^{\pi}\frac{x(s+t))}{2\sin\frac{s}{2}}\sum_{k=0}^{n-1}\sin((k+\frac{1}{2})s)\ \mathrm{d}s.$$

We simplify the sum in the integrand:

$$\begin{aligned}\sum_{k=0}^{n-1}\sin((k+\tfrac{1}{2})s) &= \operatorname{Im}\sum_{k=0}^{n-1}\mathrm{e}^{\mathrm{i}(k+\frac{1}{2})s} = \operatorname{Im}\Big(\mathrm{e}^{\mathrm{i}\frac{s}{2}}\sum_{k=0}^{n-1}\mathrm{e}^{\mathrm{i}ks}\Big) = \operatorname{Im}\Big(\mathrm{e}^{\mathrm{i}\frac{s}{2}}\,\frac{\mathrm{e}^{\mathrm{i}ns}-1}{\mathrm{e}^{\mathrm{i}s}-1}\Big)\\ &= \operatorname{Im}\frac{\mathrm{e}^{\mathrm{i}ns}-1}{\mathrm{e}^{\mathrm{i}s/2}-\mathrm{e}^{-\mathrm{i}s/2}} = \operatorname{Im}\frac{\mathrm{e}^{\mathrm{i}ns/2}(\mathrm{e}^{\mathrm{i}ns/2}-\mathrm{e}^{\mathrm{i}ns/2})}{\mathrm{e}^{\mathrm{i}s/2}-\mathrm{e}^{-\mathrm{i}s/2}}\\ &= \operatorname{Im}\frac{2\mathrm{i}(\cos(ns/2)+\mathrm{i}\sin(ns/2))\sin(ns/2)}{2\mathrm{i}\sin(s/2)} = \frac{\sin^2(ns/2)}{\sin(s/2)}.\end{aligned}$$

If we define the *Fejér kernel*

$$F_n(s) := \begin{cases}\frac{1}{2n}\frac{\sin^2(ns/2)}{\sin(s/2)}, & s \neq 0,\\ \frac{1}{2n} & s = 0,\end{cases}$$

we can write $T_n x$ as

$$T_n x(t) = \frac{1}{\pi}\int -\pi^{\pi} F_n(s)x(s+t)\ \mathrm{d}s.$$

Note that all F_n are positive functions, hence the T_n are positive operators. To show the theorem, it suffices to show that $T_n x_j \to x_j$ for $x_0(t) = 1$, $x_1(t) = \cos(t)$, $x_2(t) = \sin(t)$ (Korovkin theorem). Using (3.3) it follows that $s_k(x_0,\,\cdot\,) = x_0$ for all $k \in \mathbb{N}_0$ and that

$$\begin{aligned}&s_0(x_1,\,\cdot\,) = s_0(x_2,\,\cdot\,) = 0,\\ &s_k(x_2,\,\cdot\,) = x_1, \quad s_k(x_1,\,\cdot\,) = x_2, \qquad k \in \mathbb{N}.\end{aligned}$$

Since $T_n x_0 = x_0$, $T_n x_j = \frac{n-1}{n}x_j$ for $j = 1, 2$ and $n \in \mathbb{N}$ the theorem is proved. ☐

3.3 The open mapping theorem

Definition 3.20. A map f between metric spaces X and Y is called *open* if the image of an open set in X is an open set in Y.

Note that an open map does not necessarily map closed sets to closed sets. For example, the projection $\pi : \mathbb{R}\times\mathbb{R}\to\mathbb{R},\ \pi((s,t)) = s$, is open. The set $A := \{(s,t)\in\mathbb{R}\times\mathbb{R} : s\geq 0, st\geq 2\}$ is closed in $\mathbb{R}\times\mathbb{R}$ but $\pi(A) = (0,\infty)$ is open in $\mathbb{R}$.

Lemma 3.21. *Let X, Y be Banach spaces and $T\in L(X,Y)$ such that*

$$B_Y(0,r)\subseteq\overline{T(B_X(0,1))}.$$

for some $r>0$. Then for every $\varepsilon\in(0,1)$

$$B_Y(0,(1-\varepsilon)r)\subseteq T(B_X(0,1)).$$

Here $B_X(x_0,r) := \{x\in X : \|x-x_0\|<r\}$ and $B_Y(y_0,r) := \{y\in Y : \|y-y_0\|<r\}$ are open balls in X and Y respectively.

The lemma says that if $T(B_X(0,1))$ is dense in $B_Y(0,r)$, then, for any $0<\rho<r$, the ball $B_Y(0,\rho)$ is contained in $T(B_X(0,1))$.

Proof. Note that the assertion is equivalent to

$$B_Y(0,r)\subseteq(1-\varepsilon)^{-1}T(B_X(0,1)) = T(B_X(0,(1-\varepsilon)^{-1})).$$

Fix $\varepsilon>0$ and $y_0\in B_Y(0,r)$. We have to show that there exists an $x_0\in X$ with $\|x_0\|<(1-\varepsilon)^{-1}$ and $y_0 = T(x_0)$. By assumption, $B_Y(0,r)\subseteq\overline{T(B_X(0,1))}$. Hence there exists an $x_1\in B_X(0,1)$ such that $\|y_0-Tx_1\|<\varepsilon r$. By scaling, we know that $T(B_X(0,\varepsilon))$ is dense in $B_Y(0,\varepsilon r)$. Since $y_0-Tx_1\in B_Y(0,\varepsilon r)$, there exists an $x_2\in B_X(0,\varepsilon)$ such that $\|y_0-Tx_1-Tx_2\|<\varepsilon^2 r$. Since $T(B_X(0,\varepsilon^2))$ is dense in $B_Y(0,\varepsilon^2 r)$, there exists an $x_3\in B_X(0,\varepsilon^2)$ such that $\|y_0-Tx_1-Tx_2-Tx_3\|<\varepsilon^3 r$. Continuing in this way, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\|x_n\|<\varepsilon^{n-1},\qquad \|y_0-\sum_{k=1}^n Tx_k\|<r\varepsilon^n,\qquad n\in\mathbb{N}. \tag{3.6}$$

It follows that $x_0 := \sum_{k=1}^\infty x_k$ exists and lies in $B(0,(1-\varepsilon)^{-1})$ because $\sum_{k=1}^\infty\|x_k\|<\sum_{k=1}^\infty r\varepsilon^{k-1} = r(1-\varepsilon)^{-1}$. Since T is continuous, we know that

$$T(x_0) = T\Big(\sum_{k=1}^\infty x_k\Big) = \sum_{k=1}^\infty Tx_k.$$

By (3.6) it follows that $\sum_{k=1}^n Tx_k$ converges to y_0 for $n\to\infty$. Hence $Tx_0 = y_0$ and the statement is proved. $\square$

In the proof of the open mapping theorem we use the following fact.

Remark. Let $T : X \to Y$ be a linear map between normed spaces X and Y and assume that $T_X(B(0,1))$ is dense in $B_Y(y,\delta)$ for some $y \in Y$ and $\delta > 0$. Then $T_X(B(0,1))$ is dense in $B_Y(0,\delta)$.

Proof. Obviously it suffices to show that $T(B_X(0,2))$ is dense in $B_Y(0,2\delta)$. Since T is linear, it follows immediately that $T_X(B(0,1))$ is dense in $B_Y(-y,\delta)$. Let $z \in B_Y(0,2\delta)$ and $\varepsilon > 0$. Note that $y - z/2 \in B_Y(y,\delta)$ and $-y - z/2 \in B_Y(-y,\delta)$. Choose $x_1, x_2 \in B_X(0,1)$ such that $\|Tx_1 - (y - z/2)\| < \varepsilon/2$ and $\|Tx_2 - (-y - z)\| < \varepsilon/2$. Since $x_1 + x_2 \in B_X(0,2)$ and

$$\|T(x_1 + x_2) - z\| \leq \|Tx_1 - (y - z/2)\| + \|Tx_2 - (-y - z/2)\| < \varepsilon,$$

it follows that $z \in \overline{T(B_X(0,2))}$ because ε can be chosen arbitrarily small. □

Theorem 3.22 (Open mapping theorem). *Let X, Y be Banach spaces and $T \in L(X,Y)$. Then T is open if and only if it is surjective.*

Proof. If T is open, then it is obviously surjective.
Now assume that T is surjective. We use the notation of the preceding lemma. By assumption

$$Y = \bigcup_{k=1}^{\infty} \overline{T(B_X(0,k))}.$$

Since Y is complete, by Baire's category theorem there must exist an $n \in \mathbb{N}$ and $y \in Y$ and $\varepsilon > 0$ such $B_Y(y,\varepsilon) \subseteq \overline{T(B_X(0,n))}$, in other words, $T(B_X(0,1))$ is dense in $B_Y(y/n, \varepsilon/n)$. By the remark above $T(B_X(0,1))$ is dense in $B_Y(0,\varepsilon/n)$, so by Lemma 3.21 $B_Y(0,\delta) \subseteq T(B_X(0,1))$ for all $\delta < \varepsilon/n$.

Now let $U \subseteq X$ be an open set and $u \in U$. Then there exists an open ball $B_X(0,\varepsilon)$ such that $u + B_X(0,\varepsilon) \subseteq U$. By what was shown above, there exists an $\delta > 0$ such that $Tu + B_Y(0,\delta) \subseteq Tu + T(B_X(0,\varepsilon)) = T(u + B_X(0,\varepsilon)) \subseteq T(U)$. □

The open mapping theorem has the following important corollaries.

Corollary 3.23 (Inverse mapping theorem). *Let X, Y be Banach spaces and $T \in L(X,Y)$ a bijection. Then T^{-1} exists and is continuous.*

Proof. By the open mapping theorem T is open, so its inverse T^{-1} is continuous. □

Corollary 3.24. *Let X, Y be Banach spaces and $T \in L(X,Y)$ injective. Then $T^{-1} : \operatorname{rg}(T) \to X$ is continuous if and only if $\operatorname{rg}(T)$ is closed.*

Proof. If $\operatorname{rg}(T)$ is closed in Y then it is a Banach space. So by the previous lemma, $T : X \to \operatorname{rg}(T)$ has a continuous inverse. On the other hand, if $T^{-1} : \operatorname{rg}(T) \to X$ is continuous, then T is an isomorphism between X and $\operatorname{rg}(T)$, so $\operatorname{rg}(T)$ is complete, hence closed in Y. □

Corollary 3.25. *Let X be a $\mathbb{K}$-vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ norms on X such that X is complete with respect to both norms. Assume that there exists an $\alpha > 0$ such that $\|x\|_2 \leq \alpha\|x\|_1$ for all $x \in X$. Then the two norms are equivalent.*

Proof. Let $T:(X,\|\cdot\|_1)\to(X,\|\cdot\|_2)$, $Tx=x$. T is surjective and bounded by α, so it is continuous. By the open mapping theorem, its inverse is continuous, hence bounded. The statement follows now from $\|x\|_1=\|T^{-1}x\|_1\le\|T^{-1}\|\,\|x\|_2$, $x\in X$. □

3.4 The closed graph theorem

Let X,Y be normed spaces. Then $X\times Y$ is a normed space with either of the norms

$$\begin{aligned}\|\cdot\|:X\times Y\to\mathbb{R},\qquad \|(x,y)\|&=\|x\|+\|y\|,\\ \|\cdot\|:X\times Y\to\mathbb{R},\qquad \|(x,y)\|&=\sqrt{\|x\|^2+\|y\|^2}.\end{aligned}$$

Note that the two norms defined above are equivalent.

Definition 3.26. Let X,Y be normed spaces, $\mathcal{D}$ a subspace of X and $T:\mathcal{D}\to Y$ linear. T is called *closed* if its graph

$$\mathrm{G}(T):=\{(x,Tx):x\in\mathcal{D}\}\subseteq X\times Y$$

is closed in $X\times Y$. T is *closable* if $\overline{G(T)}$ is the graph of an operator $\overline{T}$. The operator $\overline{T}$ is called the closure of T.

$\mathcal{D}$ is called the *domain of* T, also denoted by $\operatorname{dom}T$. Sometimes the notations $T:X\supseteq\mathcal{D}\to Y$ or $T(X\to Y)$ are used.

Obviously, the graph $\mathrm{G}(T)$ is a subspace of $X\times Y$.

Lemma 3.27. *Let X,Y normed space and $\mathcal{D}\subseteq X$ a subspace. Then $T:X\supseteq\mathcal{D}\to Y$ is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}}\subseteq\mathcal{D}$ the following is true:*

$$\begin{aligned}&(x_n)_{n\in\mathbb{N}} \textit{ and } (Tx_n)_{n\in\mathbb{N}} \textit{ converge}\\ &\qquad\Longrightarrow\quad x_0:=\lim_{n\to\infty}x_n\in\mathcal{D} \textit{ and } \lim_{n\to\infty}Tx_n=Tx_0.\end{aligned}\tag{3.7}$$

Proof. Assume that T is closed and let $(x_n)_{n\in\mathbb{N}}$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ converge. Then $((x_n,Tx_n))_{n\in\mathbb{N}}\subseteq G(T)$ converges in $X\times Y$. Since $G(T)$ is closed, $\lim\limits_{n\to\infty}(x_n,Tx_n)=(x_0,y_0)\in G(T)$. By definition of $G(T)$ this implies $\lim\limits_{n\to\infty}x_n=x_0\in\mathcal{D}(T)$ and $Tx_0=y_0=\lim\limits_{n\to\infty}Tx_n$.

Now assume that (3.7) holds and let $((x_n,Tx_n))_{n\in\mathbb{N}}\subseteq G(T)$ be a sequence that converges in $X\times Y$. Then both $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ converge, hence $x_0:=\lim\limits_{n\to\infty}x_n\in\mathcal{D}$ and $\lim\limits_{n\to\infty}Tx_n=Tx_0$ which shows that $\lim\limits_{n\to\infty}(x_n,Tx_n)=(x_0,Tx_0)\in G(T)$, hence $G(T)$ is closed. □

Lemma 3.28. *Let X,Y normed space and $\mathcal{D}\subseteq X$ a subspace. Then $T:\mathcal{D}\to Y$ is closable if and only if for every sequence $(x_n)_{n\in\mathbb{N}}\subseteq\mathcal{D}$ the following is true:*

$$\lim_{n\to\infty}x_n=0 \;\textit{ and }\; (Tx_n)_{n\in\mathbb{N}} \textit{ converges}\quad\Longrightarrow\quad\lim_{n\to\infty}Tx_n=0.\tag{3.8}$$

The closure $\overline{T}$ of T is given by

$$\begin{aligned} \mathcal{D}(\overline{T}) &= \{x \in X : \exists\ (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D} \text{ with } \lim_{n\to\infty} x_n = x \text{ and } (Tx_n)_{n\in\mathbb{N}} \text{ converges } \}, \\ \overline{T}x &= \lim_{n\to\infty}(Tx_n) \qquad \text{for } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D} \text{ with } \lim_{n\to\infty} x_n = x. \end{aligned} \tag{3.9}$$

Proof. Assume that T is closable. Then $\overline{G(T)}$ is the graph of a linear function. Hence for a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ with $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Tx_n = y$ for some $y \in Y$ it follows that $(0, y) \in \overline{G(T)} = G(\overline{T})$. Hence $y = \overline{T}0 = 0$ because $\overline{T}$ is linear.
Now assume that (3.8) holds and define $\overline{T}$ as in (3.9). $\overline{T}$ is well-defined because for sequences $(x_n)_{n\in\mathbb{N}}$ and $(\tilde{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \tilde{x}_n = x$ such that $(\overline{T}x_n)_{n\in\mathbb{N}}$ and $(\overline{T}\tilde{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ converge, it follows that $(x_n - \tilde{x}_n)_{n\in\mathbb{N}}$ converges to 0. Since $\overline{T}(x_n - \tilde{x}_n) = T(x_n - \tilde{x}_n)$ converges, it follows by assumption that $\lim_{n\to\infty} \overline{T}x_n - \lim_{n\to\infty} \overline{T}\tilde{x}_n = \lim_{n\to\infty} \overline{T}(x_n - \tilde{x}_n) = 0$. Linearity of $\overline{T}$ is clear. By definition, $G(\overline{T})$ is the closure of $G(T)$, so $\overline{T}$ is the closure of T. □

Remarks 3.29. Let X, Y be normed spaces.

(i) Every $T \in L(X, Y)$ is closed.

(ii) If T is closed and injective, then T^{-1} is closed.

Proof. Closedness of $\{(x, Tx) : x \in X\} \subseteq X \times Y$ implies closeness of $\{(T^{-1}y, y) : y \in \mathrm{rg}(T)\} \subseteq X \times Y$. □

(iii) If $T : \mathcal{D} \supseteq X \to Y$ is linear and continuous, then T is closable and $\mathcal{D}(\overline{T}) = \overline{\mathcal{D}(T)}$.

Examples 3.30. (i) A continuous operator that is not closed.
Let X be normed space, $S \in L(X)$ and $\mathcal{D}$ a dense subset of X with $X \setminus \mathcal{D} \neq \emptyset$. (For example, d is dense in c_0.) Then $T := S|_{\mathcal{D}}$ is continuous because it is the restriction of a continuous function, but is not closed. To see this, fix an $x_0 \in X \setminus \mathcal{D}$ and choose a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ which converges to x_0. Then $(Tx_n)_{n\in\mathbb{N}}$ converges (to Sx_0). If T were closed, this would imply that $x_0 \in \mathcal{D}$, contradicting the choice of x_0.

(ii) A closed operator that is not continuous.
Let $X = C([-1, 1])$, $\mathcal{D} = C^1([-1, 1]) \subseteq C([-1, 1])$ and $T : X \supseteq \mathcal{D} \to X,\ Tx = x'$. Then T is closed and not continuous.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ converge. From a well-known theorem in Analysis 1 it follows that $x_0 := \lim_{n\to\infty} x_n$ is differentiable and $Tx_0 = x_0' = (\lim_{n\to\infty} x_n)' = \lim_{n\to\infty} x_n' = \lim_{n\to\infty} Tx_n$.
That T is not continuous was already shown in Example 2.7 (iv) (choose $x_n(t) = \frac{1}{n}\exp(-n(t+1))$). □

(iii) Let $X = \mathscr{L}_2(-1, 1)$, $\mathcal{D} = C^1([0, 1]) \subseteq \mathscr{L}_2([0, 1])$ and $T : X \supseteq \mathcal{D} \to X,\ Tx = x'$. Then T is not closed.

Proof. Let $x_n : [-1,1] \to \mathbb{R},\ x_n(t) = (t^2 + n^{-2})^{\frac{1}{2}}$. Then $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ and $x_n \to g$ for $n \to \infty$ where $g(t) = |t|$, $t \in [-1,1]$. The sequence of the derivatives converges

$$x_n'(t) = \frac{t}{(t^2 + n^{-1})^{\frac{1}{2}}} \ \to\ h(t) = \begin{cases} 1, & t > 0, \\ -1, & t < 0, \\ 0, & t = 0. \end{cases}$$

Obviously $h \in \mathscr{L}_2(-1,1)$. If T were closed, it would follow that $g \in C^1([-1,1])$, a contradiction. □

Definition 3.31. Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace and $T : X \supseteq \mathcal{D} \to Y$ a linear operator. Then

$$\|\cdot\|_T : \mathcal{D} \to \mathbb{R}, \quad \|x\|_T = \|x\| + \|Tx\|$$

is called the *graph norm* of T.

It is easy to see that $\|\cdot\|_T$ is a norm on $\mathcal{D}$. Moreover, the norm defined above is equivalent to the norm $\|x\|_T' = \sqrt{\|x\|^2 + \|Tx\|^2}$ on $\mathcal{D}$. Most of the time, the graph norm defined in Definition 3.31 is easier to use in calculations. However, the norm with the square root is sometimes more useful when operators in Hilbert spaces are considered.

Lemma 3.32. *Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace and $T : X \supseteq \mathcal{D} \to Y$ a closed linear operator. Then*

(i) *$(\mathcal{D}, \|\cdot\|_T)$ is a Banach space.*

(ii) *$\widetilde{T} : (\mathcal{D}, \|\cdot\|_T) \to Y,\ \widetilde{T}x = Tx$, is continuous.*

Proof. (i) To show completeness of $(\mathcal{D}, \|\cdot\|_T)$ let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be a Cauchy sequence with respect to $\|\cdot\|_T$. Then, by definition of the graph norm, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and $(Tx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since X and Y are complete, the sequences converge. Hence, by the closeness of T, $\|\cdot\|\text{-}\lim\limits_{n\to\infty} x_n =: x_0 \in \mathcal{D}$ and $x_n \xrightarrow{\|\cdot\|_T} x_0$.

(ii) The statement follows from $\|\widetilde{T}x\|_Y \le \|x\|_X + \|Tx\|_Y = \|x\|_T$, $x \in \mathcal{D}$. □

Lemma 3.33. *Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace and $T : X \supseteq \mathcal{D} \to Y$ a closed surjective operator. Then T is open. If, in addition, T is injective, then T^{-1} is continuous.*

Proof. By Lemma 3.32 and the open mapping theorem (Theorem 3.22) the operator $i\widetilde{T} : (\mathcal{D}, \|\cdot\|_T) \to Y,\ \widetilde{T}x = Tx$, is open. Let $U \subseteq \mathcal{D}$ open with respect to the norm in X. Then U is also open with respect to the graph norm because obviously $i : (\mathcal{D}, \|\cdot\|_T) \to (\mathcal{D}, \|\cdot\|),\ ix = x$, is bounded, hence continuous. Hence $T(U) = \widetilde{T}(U)$ is open in Y.

Now assume in addition that T is injective. Then $\widetilde{T}^{-1} : Y \to (\mathcal{D}, \|\cdot\|_T)$ is continuous by the inverse mapping theorem. Since i is continuous, also $T^{-1} = (\widetilde{T} \circ i^{-1})^{-1} = i \circ \widetilde{T}^{-1}$ is continuous. □

Lemma 3.34. *Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace and $T : X \supseteq \mathcal{D} \to Y$ a closed injective linear operator such that $T^{-1} : \operatorname{rg}(T) \to X$ is continuous. Then $\operatorname{rg}(T)$ is closed.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\operatorname{rg}(T)$ with $y_0 := \lim_{n\to\infty} y_n$. and $x_n := T^{-1}y_n$, $n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}$ because $\|x_n - x_m\| = \|T^{-1}y_n - T^{-1}y_m\| \le \|T^{-1}\|\,\|y_n - y_m\|$. Hence $(x_n)_{n\in\mathbb{N}}$ converges in X and its limit x_0 belongs to $\mathcal{D}$ and $y_0 = \lim_{n\to\infty} y_n = Tx_0 \in \operatorname{rg}(T)$ because T is closed. □

Theorem 3.35 (Closed graph theorem). *Let X, Y be Banach spaces and $T : X \to Y$ be a closed linear operator. Then T is bounded.*

Proof. Note that the projections

$$\begin{aligned} \pi_1 &: G(T) \to X, && \pi_1(x, Tx) = x,\\ \pi_2 &: G(T) \to Y, && \pi_2(x, Tx) = Tx \end{aligned}$$

are continuous and that π_1 is bijective. By assumption the graph $G(T)$ is closed in $X \times Y$, hence a Banach space, so π_1 is open by the open mapping theorem (Theorem 3.22). Hence $T = \pi_2 \circ \pi_1^{-1}$ is continuous. □

Lemma 3.36. *Let X, Y be Banach spaces, $\mathcal{D} \subseteq X$ a subspace and $T : \mathcal{D} \to Y$ linear. Then the following are equivalent:*

(i) *T is closed and $\mathcal{D}(T)$ is closed.*

(ii) *T is closed and T is continuous.*

(iii) *$\mathcal{D}(T)$ is closed and T is continuous.*

Proof. (i) $\Longrightarrow$ (ii) follows from the closed graph theorem because by assumption $\mathcal{D}$ is Banach space. (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are clear. □

Example 3.37. An everywhere defined linear operator that is not closed.

Let X be an infinite dimensional Banach space and $(x_\lambda)_{\lambda\in\Lambda}$ an algebraic basis of X. Without restriction we can assume $\|x_\lambda\| = 1$, $\lambda \in \Lambda$. Choose $\mathbb{N} \to \Lambda$, $n \mapsto \lambda_n$ be an injection. Then the operator

$$T : X \to X, \qquad T(x) = \sum_{n\in\mathbb{N}} n\, c_{\lambda_n} x_{\lambda_n} \quad \text{for} \quad x = \sum_{\lambda\in\Lambda} c_{\lambda_n} x_{\lambda_n} \in X,$$

is well-defined. Assume that T is closed. By the closed graph theorem T must be bounded, but $\|Tx_{\lambda_n}\| = \|nx_{\lambda_n}\| = n$ while $\|x_{\lambda_n}\| = 1$, $n \in \mathbb{N}$ contradicting the boundedness of T.

3.5 Projections in Banach spaces

Definition 3.38. Let X be a vector space. $P : X \to X$ is called a *projection* (on $\operatorname{rg}(P)$) if $P^2 = P$.

Note that if P is a projection, then also $\operatorname{id} - P$ is a projection because $(\operatorname{id} - P)^2 = \operatorname{id} - 2P + P^2 = \operatorname{id} - P$.

Lemma 3.39. *Let X be a normed space and $P \in L(X)$ a projection. Then the following holds:*

(i) *Either $P = 0$ or $\|P\| \geq 1$.*

(ii) *$\ker(P)$ and $\operatorname{rg}(P)$ are closed.*

(iii) *X is isomorphic to $\ker P \oplus \operatorname{rg}(P)$.*

Proof. (i) Note that $\|P\| = \|P^2\| \leq \|P\|^2$, hence $0 \leq \|P\| - \|P\|^2 = \|P\|(1 - \|P\|)$.

(ii) Since P is continuous, $\ker(P) = P^{-1}(\{0\})$ is closed. To see that $\operatorname{rg}(P)$ is closed, it suffices to show that $\operatorname{rg}(P) = \ker(\operatorname{id} - P)$. Indeed, $x \in \ker(\operatorname{id} - P)$ implies $x = Px \in \operatorname{rg}(P)$ and $y \in \operatorname{rg}(P)$ implies $(P - \operatorname{id})y = Py - y = y - y = 0$, hence $y \in \ker(\operatorname{id} - P)$.

(iii) Obviously $x \mapsto ((\operatorname{id} - P)x,\ Px) \in \ker(P) \oplus \operatorname{rg}(P)$ is well defined, linear, bijective and continuous because $\operatorname{id} - P$ and P are continuous. By the inverse mapping theorem then also the inverse operator is continuous which shows that X and $\ker(P) \oplus \operatorname{rg}(P)$ are isomorphic. □

Theorem 3.40. *Let X be a normed space, $U \subseteq X$ a finite dimensional subspace. Then there exists a linear continuous projection P of X to U with $\|P\| \leq \dim U$.*

Proof. From linear algebra we know that there exist bases $(u_1, \dots, u_n)$ of U and $(\varphi_1, \dots, \varphi_n)$ of U' such that $\|u_k\| = \|\varphi_k\| = 1$ and $\varphi_j(u_k) = \delta_{jk},\ j, k = 1, \dots, n$. By the Hahn-Banach theorem the φ_k can be extended to linear functionals ψ_k on X with $\|\varphi_k\| = \|\psi_k\|$. We define

$$P : X \to X, \qquad Px = \sum_{k=1}^{n} \varphi_k(x) u_k.$$

Obviously P is a linear bounded projection on U and

$$\|Px\| \leq \sum_{k=1}^{n} \|\varphi_k\| \, \|x\| \, \|u_k\| = \sum_{k=1}^{n} \|x\| = n\|x\|.$$

□

Theorem 3.41. *Let X be Banach space, $U, V \subseteq X$ closed subspaces such that X and $U \oplus V$ are algebraically isomorphic. Then the following holds:*

(i) *X is isomorphic to $V \oplus U$ with $\|(u, v)\| = \|u\| + \|v\|$.*

(ii) *There exists a continuous linear projection of X on U.*

(iii) *V is isomorphic to X/U.*

Proof. (i) Since U and V are Banach spaces, their sum $U \oplus V$ is a Banach space. The map $U \oplus V \to X,\ (u, v) \mapsto u + v$ is linear, continuous and bijective. Hence by the inverse mapping theorem, also the inverse is continuous.

(ii) $P : X \to U, u + v \mapsto u$ is the desired projection.

(iii) The map $V \mapsto X/V,\ v \mapsto [v]$ is linear, bijective and continuous. Since U is closed, X/U is a Banach space. By the inverse mapping theorem it follows that V and X/U are isomorphic. □

Definition 3.42. let X be a Banach space. A closed subspace U of X is called *complemented* if there exists a continuous linear projection on U.

Remark 3.43. Note that not every closed subspace of a Banach space is complemented in the sense of the theorem above. For example, c_0 is not complemented as subspace of ℓ_∞.

3.6 Weak convergence

Definition 3.44. Let X be a set and $\mathcal{U} = (U_\lambda)_{\lambda\in\Lambda}$ a family of subsets of sets in X. The smallest topology on X such that all U_λ are open is called the topology *generated by* $\mathcal{U}$, denoted by $\tau(\mathcal{U})$.

Obviously $\tau(\mathcal{U})$ exists and is the intersection of topologies containing all U_λ.

Lemma 3.45. *Let X be a set, $\mathcal{U} = (U_\lambda)_{\lambda\in\Lambda}$ a family of subsets of X. Then the topology generated by $\mathcal{U}$ consists of all sets of the form*

$$\bigcup_{\gamma\in\Gamma}\bigcap_{k=1}^{n} U_{\gamma,k}, \tag{3.10}$$

that is, of arbitrary unions of finite intersections of sets in the family $\mathcal{U}$.

Proof. Let $\tau(\mathcal{U})$ be the topology generated by $\mathcal{U}$ and $\sigma(\mathcal{U})$ the system of sets described in (3.10). It is not hard to see that $\sigma(\mathcal{U})$ is a topology containing $\mathcal{U}$, hence containing $\tau(\mathcal{U})$. On the other hand, all sets of the form (3.10) are open in $\tau(\mathcal{U})$, so $\sigma(\mathcal{U}) \subseteq \tau(\mathcal{U})$. □

Definition 3.46. Let X be a set, Λ be an index set and for every $\lambda \in \Lambda$ let $(Y_\lambda, \tau_\lambda)$ be a topological space. Consider a family $\mathcal{F} = (f_\lambda : X \to Y_\lambda)$ of functions. The smallest topology on X such that all f_λ are continuous, is called the *initial topology* on X, denoted by $\sigma(X, \mathcal{F})$.

Note that $\tau(\mathcal{F}) = \tau\Big(\{f_\lambda^{-1}(U_\lambda) \,:\, \lambda \in \Lambda,\ U_\lambda \in \tau_\lambda\}$.

Definition 3.47. Let X be a normed space. The topology $\sigma(X, X')$ is called the *weak topology* on X. The topology $\sigma(X', X)$ is called the *weak$*$ topology* on X' when X is identified with a subset of X'' by the canonical map J_X.

Note that $\sigma(X', X) \subseteq \sigma(X', X'') \subseteq \sigma_{\|\cdot\|}$.

Lemma 3.48. *Let X be a normed space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is weakly convergent to some $x_0 \in X$ (in the sense of Definition 2.38) if and only if it converges in the weak topology $\sigma(X, X')$.*

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent with $x_0 := w\text{-}\lim_{n\to\infty} x_n$ and let U be a $\sigma(X, X')$-open set containing x_0. Then there exist $\varphi_1, \ldots, \varphi_n$ such that

$$x_0 \in \bigcap_{k=1}^{n} \varphi_j^{-1}(V_j) \subseteq U$$

with V_j open subsets in $\mathbb{R}$ containing $\varphi_j(x_0)$. Since $\lim_{n\to\infty} \varphi(x_n) = \varphi(x_0)$ for all $\varphi \in X'$, we can choose an $N \in \mathbb{N}$ such that $\varphi_j(x_n) \in U_j$ for all $n \geq N$ and all $j = 1, \ldots, n$. Hence $x_n \in \bigcap_{k=1}^{n}\{\varphi_j^{-1}(V_j)\} \subseteq U$ for all $n \geq N$.

Now assume that $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to x_0 in the weak topology. Since by definition of $\sigma(X, X')$ all functionals $\varphi \in X'$ are continuous, it follows that $(\varphi(x_n))_{n\in\mathbb{N}}$ converges to $\varphi(x_0)$ for every $\varphi \in X'$. □

Lemma 3.49. *Let X be a normed space, $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(\varphi_n)_{n\in\mathbb{N}} \subseteq X'$.*

(i) $x_0 = w\text{-}\lim\limits_{n \to \infty} x_n \quad \Longrightarrow \quad \|x_0\| \le \liminf\limits_{n\to\infty} x_n.$

(ii) $\varphi_0 = w*\text{-}\lim\limits_{n \to \infty} \varphi_n \quad \Longrightarrow \quad \|\varphi_0\| \le \liminf\limits_{n\to\infty} \varphi_n.$

Proof. (i) For $x_0 = 0$ the assertion is clear. By the Hahn-Banach theorem there exists an $\varphi \in X'$ such that $\varphi(x_0) = \|x_0\|$ and $\|\varphi\| = 1$. Hence

$$\|x_0\| = \|\lim_{n\to\infty} \varphi(x_n)\| \le \liminf_{n\to\infty} \|\varphi\|\,\|x_n\| = \liminf_{n\to\infty} \|x_n\|.$$

(ii) Let $\varepsilon > 0$. Then there exists an $x \in X$ with $\|x\| = 1$ such that $\|\varphi_0\| - \varepsilon < \|\varphi_0(x)\|$. The statement follows as above:

$$\|\varphi_0\| - \varepsilon < \|\varphi_0(x)\| = \lim_{n\to\infty} \|\varphi_n(x)\| \le \liminf_{n\to\infty} \|\varphi_n\|\,\|x\| = \liminf_{n\to\infty} \|\varphi_n\|.$$ □

Definition 3.50. Let X be a topological space. A function $f : X \to \mathbb{R}$ is called *upper semicontinuous* if $\limsup\limits_{x_n\to x} f(x_n) \le f(x)$. It is called *lower semicontinuous* if $\liminf\limits_{x_n\to x} f(x_n) \ge f(x)$.

Hence the lemma above states that $\|\cdot\|$ is lower semicontinuous in the weak topology.

Definition 3.51. For $\lambda \in \Lambda$ let $(X_\lambda, \tau_\lambda)$ be topological spaces. Define

$$X := \prod_{\lambda\in\Lambda} X_\lambda := \Big\{ f : \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda \ :\ f(\lambda) \in X_\lambda,\ \lambda \in \Lambda \Big\}.$$

The *product topology* on X is the weakest topology such that for every $\lambda \in \Lambda$ the projection

$$\pi_\lambda : X \to X_\lambda, \quad \pi_j(f) = f(j),$$

is continuous.

Lemma 3.52. *Let X as above with the product topology. Let $\mathcal{O} \subseteq \mathbb{P}(X)$ be the family of all sets $U \subseteq X$ such that for every $u \in U$ there exist $\lambda_j \in \Lambda$, $U_j \subseteq X_{\lambda_j}$ open, $j = 1, \dots, n$, such that*

$$u \in \{s \in X \ :\ s(\lambda_j) \in U_j,\ j = 1, \dots, n\} = \bigcap_{j=1}^{n} \underbrace{\pi_{\lambda_j}^{-1}(U_j)}_{\text{open in } \mathcal{O}} \subseteq U.$$

Then $\mathcal{O}$ is the product topology on X.

Proof. This is a special case of Lemma 3.48. □

Theorem 3.53 (Banach-Alaoglu). *Let X be a normed space. Then the closed unit ball $K_1' := \{\varphi \in X' : \|\varphi\| \leq 1\}$ is weak∗-compact.*

Proof. For $x \in X$ define the set $A_x := \{z \in \mathbb{K} : |z| \leq \|x\|\}$ and let $A := \prod_{x\in X} A_x$ together with the product topology. By Tychonoff's theorem A is compact. Note that elements $a \in A$ are maps $X \to \mathbb{K}$ with $|a(x)| \leq \|x\|$, $x \in X$. Hence $K_1' \subseteq A$ because $|\varphi(x)| \leq \|\varphi\|\,\|x\| \leq \|x\|$ for every $\varphi \in K_1'$. The product topology on A is the weakest topology on A such that for every $x \in X$ the map $\pi_x : A \to \mathbb{K},\ a \mapsto a(x)$ is continuous. Hence the topology on K_1' induced by A is exactly the weak∗-topology on K_1'. So it suffices to show that K_1' is closed in A with the product topology.

Let $\varphi \in \overline{K_1'}$ and let $x, y \in X$ and $\varepsilon > 0$. Then

$$U := \{a \in A\,:\, |a(x+y) - \varphi(x+y)| < \varepsilon,\ |a(x) - \varphi(x)| < \varepsilon,\ |a(y) - \varphi(y)| < \varepsilon\}$$

is an open neighbourhood of φ. Hence there exists a $g \in U \cap K_1'$. Since g is linear, it follows that

$$\begin{aligned} |\varphi(x+y) - \varphi(x) - \varphi(y)| &= |\varphi(x+y) - \varphi(x) - \varphi(y) - g(x+y) + g(x) + g(y)| \\ &\leq |\varphi(x+y) - g(x+y)| + |\varphi(x) - g(x)| + |\varphi(y) - g(y)| < 3\varepsilon. \end{aligned}$$

Since ε was arbitrary, this implies $\varphi(x+y) = \varphi(x) + \varphi(y)$. Similarly it can be shown that $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{K}$ and $x \in X$. It follows that φ is linear. Since $\varphi \in A$, it follows that $\|\varphi\| \leq 1$, hence $\varphi \in K_1'$. □

Chapter 4

Hilbert spaces

4.1 Hilbert spaces

Definition 4.1. Let X be a $\mathbb{K}$-vector space. A map

$$\langle \cdot\,, \cdot \rangle : X \times X \to \mathbb{K}$$

is a *sesquilinear form* on X if for all $x, y, z \in X$, $\lambda \in \mathbb{K}$

(i) $\langle \lambda x + y\,, z \rangle = \lambda \langle x\,, z\rangle + \langle y\,, z\rangle$,
(ii) $\langle x\,, \lambda y + z \rangle = \overline{\lambda} \langle x\,, y\rangle + \langle x\,, z\rangle$.

The inner product is called

- *hermitian* $\iff$ $\langle x\,, y\rangle = \overline{\langle y\,, x\rangle}$, $\qquad x, z \in X$,
- *positive semidefinite* $\iff$ $\langle x\,, x\rangle \geq 0$, $\qquad x \in X$,
- *positive (definite)* $\iff$ $\langle x\,, x\rangle > 0$, $\qquad x \in X \setminus \{0\}$.

Definition 4.2. A positive definite hermitian sesquilinear form on a $\mathbb{K}$-vector X is called an *inner product* on X and $(X, \langle \cdot\,, \cdot\rangle)$ is called an *inner product space* (or *pre-Hilbert space*).

Note that $\langle x\,, x\rangle \in \mathbb{R}$, $x \in X$, for a hermitian sesquilinear form X because $\langle x\,, x\rangle = \overline{\langle x\,, x\rangle}$.

Lemma 4.3 (Cauchy-Schwarz inequality). *Let X be a $\mathbb{K}$-vector space with inner product $\langle \cdot\,, \cdot\rangle$. Then for all $x, y \in X$*

$$|\langle x\,, y\rangle|^2 \leq |\langle x\,, x\rangle|\, |\langle y\,, y\rangle|, \tag{4.1}$$

with equality if and only if x and y are linearly dependent.

Proof. For $x = 0$ or $y = 0$ there is nothing to show. Now assume that $y \neq 0$. For all $\lambda \in \mathbb{K}$

$$0 \leq \langle x + \lambda y\,, x + \lambda y\rangle = \langle x\,, x\rangle + \lambda \langle y\,, x\rangle + \overline{\lambda}\langle x\,, y\rangle + |\lambda|^2 \langle y\,, y\rangle.$$

In particular, when we choose $\lambda = -\frac{\langle x\,,y\rangle}{\langle y\,,y\rangle}$ we obtain

$$\begin{aligned} 0 \leq \langle x+\lambda y\,, x+\lambda y\rangle &= \langle x\,,x\rangle - \frac{|\langle y\,,x\rangle|^2}{\langle y\,,y\rangle} - \frac{|\langle x\,,y\rangle|^2}{\langle y\,,y\rangle} + \frac{|\langle x\,,y\rangle|^2}{\langle y\,,y\rangle} \\ &= \langle x\,,x\rangle - \frac{|\langle x\,,y\rangle|^2}{\langle y\,,y\rangle} \end{aligned}$$

which proves (4.1). If there exist $\alpha, \beta \in K$ such that $\alpha x + \beta y = 0$, then obviously equality holds in (4.1). On the other hand, if equality holds, then $\langle x+\lambda y\,, x+\lambda y\rangle = 0$ with λ chosen as above, so x and y are linearly dependent. □

Note that (4.1) is true also in a space X with a semidefinite hermitian sesquilinear form but equality in (4.1) does not imply that x and y are linearly dependent.

Lemma 4.4. *An inner product space $(X, \langle\cdot\,,\cdot\rangle)$ becomes a normed space by setting $\|x\| := \langle x\,,x\rangle^{\frac{1}{2}}$, $x \in X$.*

Proof. The only property of a norm that does not follow immediately from the definition of $\|\cdot\|$ is the triangle inequality. To prove the triangle inequality, choose $x, y \in X$. Using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} \|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x\,,y\rangle + \|y\|^2 &\leq \|x\|^2 + 2|\langle x\,,y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \end{aligned}$$ □

In the following, we will always consider inner product spaces endowed with the topology induced by the norm.

Definition 4.5. A complete inner product space is called a *Hilbert space*.

Lemma 4.6. *Note that the scalar product on a inner product space X is a continuous map $X \times X \to \mathbb{K}$ when $X \times X$ is equipped with the norm $\|(x,y)\| = \|x\|_X + \|y\|_X$.*

Proof. The statement follows from

$$\begin{aligned} |\langle x_1\,,x_2\rangle - \langle y_1\,,y_2\rangle| &= |\langle x_1\,,x_2 - y_2\rangle - \langle y_1 - x_1\,,y_2\rangle| \\ &\leq \|x_1\|\,\|x_2 - y_2\| - \|y_1 - x_1\|\,\|y_2\|. \end{aligned}$$ □

The polarisation formula allows to express the inner product of two elements of X in terms of their norms.

Theorem 4.7 (Polarisation formula). *Let X be an inner product space over $\mathbb{K}$ and $x, y \in X$. Then*

$$\begin{aligned} \langle x\,,y\rangle &= \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big), && \text{if } \mathbb{K} = \mathbb{R}, \\ \langle x\,,y\rangle &= \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2\Big), && \text{if } \mathbb{K} = \mathbb{C}. \end{aligned}$$

Proof. Straightforward calculation. □

A necessary and sufficient criterion for a normed space to be an inner product space is the following.

Theorem 4.8 (Parallelogram identity). *Let X be normed space. Then the norm on X is generated by an inner product if and only if for all $x, y \in X$ the parallelogram identity is satisfied:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

In this case, the inner product is given by the polarisation formula.

Proof. Assume that the norm is generated by the inner product $\langle \cdot\,, \cdot \rangle$ and let $\|x\| = \langle x\,, x\rangle^{\frac{1}{2}}$. Then for all $x, y \in X$ parallelogram identity holds:

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x\,, y\rangle + \|x\|^2 + \|y\|^2 - 2\operatorname{Re}\langle x\,, y\rangle \\ &= 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

Now assume that the norm on X is such that the parallelogram identity holds and for $x, y \in X$ define $\langle x\,, y\rangle$ by the polarisation formula. We prove that $\langle \cdot\,, \cdot \rangle$ is an inner product on X in the case $\mathbb{K} = \mathbb{C}$. The case $\mathbb{K} = \mathbb{R}$ can be proved analogously.

- Positivity.

$$\begin{aligned} 4\langle x\,, x\rangle &= \|x+x\|^2 - \|x-x\|^2 + \mathrm{i}\,\|x+\mathrm{i}x\|^2 - \mathrm{i}\,\|x-\mathrm{i}x\|^2 \\ &= 4\|x\|^2 + \mathrm{i}\,\|x+\mathrm{i}x\|^2 - \mathrm{i}\,\|\mathrm{i}x+x\|^2 = 4\|x\|^2 \geq 0. \end{aligned}$$

- Hermiticity.

$$\begin{aligned} 4\langle x\,, y\rangle &= \|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\,\|x+\mathrm{i}y\|^2 - \mathrm{i}\,\|x-\mathrm{i}y\|^2 \\ &= \|y+x\|^2 - \|y-x\|^2 + \mathrm{i}\,\|-\mathrm{i}x+y\|^2 - \mathrm{i}\,\|\mathrm{i}x+y\|^2 = 4\overline{\langle y\,, x\rangle}. \end{aligned}$$

- Additivity.

$$
\begin{aligned}
4(\langle x\,,y\rangle + \langle x\,,z\rangle)&\\
&= \|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\,\|x+\mathrm{i}y\|^2 - \mathrm{i}\,\|x-\mathrm{i}y\|^2\\
&\quad + \|x+z\|^2 - \|x-z\|^2 + \mathrm{i}\,\|x+\mathrm{i}z\|^2 - \mathrm{i}\,\|x-\mathrm{i}z\|^2\\
&= \Big\|x+\frac{y+z}{2}+\frac{y-z}{2}\Big\|^2 - \Big\|x-\frac{y+z}{2}-\frac{y-z}{2}\Big\|^2\\
&\quad + \Big\|x+\frac{y+z}{2}-\frac{y-z}{2}\Big\|^2 - \Big\|x-\frac{y+z}{2}+\frac{y-z}{2}\Big\|^2\\
&\quad + \mathrm{i}\,\Big\|x+\mathrm{i}\,\frac{y+z}{2}+\mathrm{i}\,\frac{y-z}{2}\Big\|^2 - \mathrm{i}\,\Big\|x-\mathrm{i}\,\frac{y+z}{2}-\mathrm{i}\,\frac{y-z}{2}\Big\|^2\\
&\quad + \mathrm{i}\,\Big\|x+\mathrm{i}\,\frac{y+z}{2}-\mathrm{i}\,\frac{y-z}{2}\Big\|^2 - \mathrm{i}\,\Big\|x-\mathrm{i}\,\frac{y+z}{2}+\mathrm{i}\,\frac{y-z}{2}\Big\|^2\\
&= 2\Big\|x+\frac{y+z}{2}\Big\|^2 + 2\Big\|\frac{y-z}{2}\Big\|^2 - 2\Big\|x-\frac{y+z}{2}\Big\|^2 - 2\Big\|\frac{y-z}{2}\Big\|^2\\
&\quad + 2\mathrm{i}\,\Big\|x+\mathrm{i}\,\frac{y+z}{2}\Big\|^2 + 2\mathrm{i}\,\Big\|\frac{y-z}{2}\Big\|^2 - 2\mathrm{i}\,\Big\|x-\mathrm{i}\,\frac{y+z}{2}\Big\|^2 - 2\mathrm{i}\,\Big\|\frac{y-z}{2}\Big\|^2\\
&= 2\Big\|x+\frac{y+z}{2}\Big\|^2 - 2\Big\|x-\frac{y+z}{2}\Big\|^2 + 2\mathrm{i}\,\Big\|x+\mathrm{i}\,\frac{y+z}{2}\Big\|^2 - 2\mathrm{i}\,\Big\|x-\mathrm{i}\,\frac{y+z}{2}\Big\|^2\\
&= 2\cdot 4\langle x\,,\frac{y+z}{2}\rangle.
\end{aligned}
$$

  If we choose $z=0$ we find $\langle x\,,y\rangle = 2\langle x\,,\frac{y}{2}\rangle$, hence

$$\langle x\,,y\rangle + \langle x\,,z\rangle = 2\Big\langle x\,,\frac{y+z}{2}\Big\rangle = \langle x\,,y+z\rangle.$$

- Homogeneity. From the additivity we obtain $\langle \lambda x\,,y\rangle = \lambda\langle x\,,y\rangle$ for all $\lambda\in\mathbb{Q}$. Note that $\langle \mathrm{i}x\,,y\rangle = \mathrm{i}\langle x\,,y\rangle$, hence homogeneity is proved for $\lambda\in\mathbb{Q}+\mathrm{i}\mathbb{Q}$. Hence for fixed $x,y\in\mathbb{C}$ the two continuous functions $\mathbb{C}\to\mathbb{C}$, $\lambda\mapsto\lambda\langle x\,,y\rangle$ and $\mathbb{C}\to\mathbb{C}$, $\lambda\mapsto\langle\lambda x\,,y\rangle$ must be equal because they are equal on the dense subset $\mathbb{Q}+\mathrm{i}\mathbb{Q}$ of $\mathbb{C}$. □

Theorem 4.9. *The completion of an inner product space is an inner product space.*

Proof. By continuity of the norm, the parallelogram identity holds on the completion $\overline{X}$ of an inner product space X. So $\overline{X}$ is an inner product space. □

Examples 4.10. (i) $\mathbb{R}^n$ and $\mathbb{C}^n$ with the Euclidean inner product

$$\langle x\,,y\rangle = \sum_{k=1}^{n} x_k\overline{y_k}, \qquad x=(x_k)_{k=1}^n,\ y=(y_k)_{k=1}^n,$$

are inner product spaces.

(ii) $\ell_2(\mathbb{N})$ with

$$\langle x\,,y\rangle = \sum_{k=1}^{\infty} x_k\overline{y_k}, \qquad x=(x_k)_{k\in\mathbb{N}},\ y=(y_k)_{k\in\mathbb{N}},$$

is an inner product space.

(iii) Let $\mathcal{R}([0,1])$ be the vector space of the Riemann integrable functions on the interval $[0,1]$. Then

$$\langle f\,, g\rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t, \qquad f,g \in \mathcal{R}([0,1]),$$

defines a sesquilinear form on $\mathcal{R}([0,1])$ which is not positive definite, since, for example, $\chi_{\{0\}} \neq 0$, but $\langle \chi_{\{0\}}\,, \chi_{\{0\}}\rangle = 0$.

The restriction of $\langle \cdot\,, \cdot\rangle$ to the space of the continuous functions $C([0,1])$ is an inner product which is not complete (its closure is the space $\mathscr{L}_2([0,1])$).

4.2 Orthogonality

Definition 4.11. Let X be an inner product space.

(i) Elements $x, y \in X$ are called *orthogonal*, denoted by $x \perp y$, if and only if $\langle x\,, y\rangle = 0$

(ii) Subsets $A, B \subseteq X$ are called *orthogonal*, denoted by $A \perp B$, if and only if $\langle a\,, b\rangle = 0$ for all $a \in A$, $b \in B$.

(iii) The *orthogonal complement* of a set $M \subseteq X$ is

$$M^\perp := \{x \in X : x \perp m,\ m \in M\}.$$

Remarks 4.12. (i) Pythagoras' theorem holds: $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$.

(ii) For every set $M \subseteq X$ its orthogonal complement $M^\perp$ is a closed subspace of X.

(iii) $A \subseteq (A^\perp)^\perp$ for every subset $A \subseteq X$.

(iv) $A^\perp = (\operatorname{span} A)^\perp$ for every subset $A \subseteq X$.

Theorem 4.13 (Projection theorem). *Let H be a Hilbert space, $M \subseteq H$ a nonempty closed and convex subset and $x_0 \in H$. Then there exists exactly one $y_0 \in M$ such that $\|x_0 - y_0\| = \operatorname{dist}(x_0, M)$.*

Proof. Recall that $\operatorname{dist}(x_0, M) := \inf\{\|x_0 - y\| : y \in M\}$. If $x_0 \in M$ then the assertion is clear (choose $y_0 = x_0$).

Now assume that $x_0 \notin M$. Without restriction we may assume $x_0 = 0$.

Existence of y_0. Let $d := \operatorname{dist}(x_0, M) = \inf\{\|y\| : y \in M\}$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq M$ such that $\lim\limits_{n\to\infty} \|y_n\| = d$. We will show that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Note that $\|\frac{y_n+y_m}{2}\|^2 \geq d^2$ because $\frac{y_n+y_m}{2} \in M$ by the convexity of M. Hence the parallelogram identity (Theorem 4.8) yields

$$\begin{aligned}\left\|\frac{y_n - y_m}{2}\right\|^2 &\leq \left\|\frac{y_n - y_m}{2}\right\|^2 + \left\|\frac{y_n + y_m}{2}\right\|^2 - d^2\\ &= \frac{1}{2}(\|y_n\|^2 + \|y_m\|^2) - d^2 \longrightarrow 0, \qquad n, m \to \infty.\end{aligned}$$

Since X is a Banach space, $(y_n)_{n\in\mathbb{N}}$ converges to some $y_0 \in X$, and since M is closed, $y_0 \in M$.

Uniqueness of y_0. Assume that there are y_0, $\tilde{y}_0 \in M$ such that $\|y_0\| = \|\tilde{y}_0\| = d = \operatorname{dist}(x_0, M)$. The parallelogram identity yields

$$d^2 \le \left\| \frac{y_0 + \tilde{y}_0}{2} \right\|^2 \le \left\| \frac{y_0 + \tilde{y}_0}{2} \right\|^2 + \left\| \frac{y_0 - \tilde{y}_0}{2} \right\|^2 = \frac{1}{2}(\|y_0\|^2 + \|\tilde{y}_0\|^2) = d^2.$$

It follows that $\|y_0 - \tilde{y}_0\| = 0$, so $y_0 = \tilde{y}_0$. □

Lemma 4.14. *Let M be a closed and convex subset of a Hilbert space H and fix $x_0 \in H$. For $y_0 \in M$ the following are equivalent:*

(i) $\|x_0 - y_0\| = \operatorname{dist}(x_0, M)$,
(ii) $\operatorname{Re}\langle x_0 - y_0 \,, y - y_0 \rangle \le 0, \qquad y \in M.$

Proof. (i) $\Longrightarrow$ (ii) For $t \in [0,1]$ and $y \in M$ let $y_t := y_0 + t(y - y_0)$. Then $y_t \in M$ by the convexity of M and by assumption on y_0

$$\begin{aligned} \|x_0 - y_0\|^2 \le \|x_0 - y_t\|^2 &= \|x_0 - y_0 - t(y - y_0)\|^2 \\ &= \|x_0 - y_0\|^2 - 2t \operatorname{Re}\langle x_0 - y_0 \,,\, y - y_0 \rangle + t^2 \|y - y_0\|^2. \end{aligned}$$

So for all $t \in (0, 1]$

$$2 \operatorname{Re}\langle x_0 - y_0 \,,\, y - y_0 \rangle \le t \|y - y_0\|^2$$

which implies $\operatorname{Re}\langle x_0 - y_0 \,,\, y - y_0 \rangle \le 0$.
(ii) $\Longrightarrow$ (i) Let $y \in M$. By assumption

$$\begin{aligned} \|x_0 - y\|^2 &= \|(x_0 - y_0) + (y_0 - y)\|^2 \\ &= \|x_0 - y_0\|^2 + \|y_0 - y\|^2 + 2 \operatorname{Re}\langle x_0 - y_0 \,, y_0 - y \rangle \ge \|x_0 - y_0\|^2. \end{aligned}$$ □

Lemma 4.15. *Let U be a closed subspace of a Hilbert space H and fix $x_0 \in H$. For $y_0 \in U$ the following are equivalent:*

(i) $\|x_0 - y_0\| = \operatorname{dist}(x_0, U)$,
(ii) $x_0 - y_0 \perp U$.

Proof. (i) $\Longrightarrow$ (ii) Let $y \in U$. If $y = 0$, then obviously $\langle x_0 - y_0 \,, y \rangle = 0$. If $\|y\| = 1$, let $\lambda = \|y\|^{-1} \langle x_0 - y_0 \,, y \rangle$. By assumption

$$\begin{aligned} \|x_0 - y_0\|^2 &\le \|x_0 - y_0 - \lambda y\|^2 \\ &= \|x_0 - y_0\|^2 - \overline{\lambda} \langle x_0 - y_0 \,, y \rangle - \lambda \langle y \,, x_0 - y_0 \rangle + |\lambda|^2 \|y\|^2 \\ &= \|x_0 - y_0\|^2 + (1 - 2\|y\|^{-2}) |\langle x_0 - y_0 \,, y \rangle|^2 \\ &= \|x_0 - y_0\|^2 - |\langle x_0 - y_0 \,, y \rangle|^2 \end{aligned}$$

so $\langle x_0 - y_0 \,, y \rangle = 0$. By linearity of U then $x_0 - y_0 \perp y$ for all $y \in U$.
(ii) $\Longrightarrow$ (i) Let $y \in U$. By assumption

$$\|x_0 - y\|^2 = \|(x_0 - y_0) + (y_0 - y)\|^2 = \|x_0 - y_0\|^2 + \|y_0 - y\|^2 \ge \|x_0 - y_0\|^2.$$ □

Recall that a linear operator $P : X \to X$ on a Banach space X is called a projection if and only if $P^2 = P$ (see Definition 3.38).

Theorem 4.16. *Let H be a Hilbert space, $U \subseteq H$ a closed subspace with $U \neq \{0\}$. Then there exists a projection $P_U \in L(H)$ on U such that $\|P_U\| = 1$ and $\ker(P_U) = U^\perp$. Also $\operatorname{id} - P_U$ is continuous projection with $\|\operatorname{id} - P_U\| = 0$ if $U = H$ and $\|\operatorname{id} - P_U\| = 1$ if $U \neq H$. If $U \oplus U^\perp$ is equipped with the norm $\|(u,v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$, then $H = U \oplus U^\perp$.*

Definition 4.17. P_U as in the theorem is called the *orthogonal projection on U*.

Proof of Theorem 4.16. Fix $x_0 \in H$ and let $P_U(x_0) := y_0$ the unique element $y_0 \in U$ such that $\|x_0 - y_0\| = \operatorname{dist}(x_0, U)$. Then $\operatorname{rg}(P_U) = U$ and $P_U^2 = P_U$, hence P_U is a projection on U.
By Lemma 4.15, $P_U(x_0)$ is the unique element in U such that $x_0 - P_U(x_0) \in U^\perp$.

$$\operatorname{Re}\langle x_0 - P_U(x_0)\,,\, y - P_U(x_0)\rangle \leq 0, \qquad y \in U.$$

We will show that P_U is linear. Let $x_1, x_2 \in H$ and $\lambda \in \mathbb{K}$. Since $U^\perp$ is a subspace, we obtain

$$\lambda x_1 - x_2 - (\lambda P_U(x_1) - P_U(x_2)) = \lambda(x_1 - P_U(x_1)) - (x_2 - P_U(x_2)) \in U^\perp.$$

Hence, by definition of P_U,

$$P_U(\lambda x_1 - x_2) = \lambda P_U(x_1) - P_U(x_2).$$

We already know that $\operatorname{rg}(P_U) = U$. $\ker(P_U) = U^\perp$ because

$$P_U(x) = 0 \iff x_0 \in U^\perp.$$

Therefore $\operatorname{id} - P_U$ is a projection with $\operatorname{rg}(\operatorname{id} - P_U) = U^\perp$ and $\ker(\operatorname{id} - U) = U$. By Pythagoras' theorem we obtain

$$\|x_0\|^2 = \|P_U(x_0) + (\operatorname{id} - P_U)(x_0)\|^2 = \|P_U(x_0)\|^2 + \|(\operatorname{id} - P_U)(x_0)\|^2.$$

In particular, $H = U \oplus U^\perp$ with norm as in the statement, and $\|P_U\| \leq 1$ and $\|\operatorname{id} - P_U\| \leq 1$. Lemma 3.39 implies $\|P_U\| = 1$, $\|\operatorname{id} - P_U\| = 1$ if $U \neq H$ and $\|\operatorname{id} - P_U\| = 0$ if $U = H$. □

Lemma 4.18. *Let U be a subspace of a Hilbert space H. Then $\overline{U} = U^{\perp\perp}$.*

Proof. By the projection theorem (Theorem 4.16), for every closed subspace V

$$P_V = \operatorname{id} - P_{V^\perp} = \operatorname{id} - (\operatorname{id} - P_{V^{\perp\perp}}) = P_{V^{\perp\perp}},$$

hence $V = V^{\perp\perp}$. Application to $V = \overline{U}$ shows the statement. □

Definition 4.19. Let X, Y be vector spaces. A map $X \to Y$ is called *antilinear* or *conjugate linear* if $f(\lambda x + y) = \overline{\lambda} f(x) + f(y)$ for all $\lambda \in \mathbb{K}$ and $x, y \in X$.

Theorem 4.20 (Fréchet-Riesz representation theorem). *Let H be a Hilbert space. Then the map*

$$\Phi : H \to H', \quad y \mapsto \langle \cdot \,, y\rangle$$

is an isometric antilinear bijection.

Proof. Obviously $\Phi(0) = 0 \in H'$. The Cauchy-Schwarz inequality yields

$$\|\Phi(y)(x)\| = |\langle x \,, y\rangle| \leq \|x\|\,\|y\|, \qquad x,\, y \in H,$$

hence $\|\Phi(y)\| \leq \|y\|$ If $y \neq 0$, then set $x = \|y\|^{-1}y$. Note that $\|x\| = 1$ and $\|\Phi(y)x\| = \|y\|$, implying that $\|\Phi(y)\| = \|y\|$. So we have shown that Φ is well-defined and an isometry. In particular, Φ is injective.
To show that Φ is surjective, fix an $\varphi \in H'$. If $\varphi = 0$, then $\varphi = \Phi(0)$. Otherwise we can assume that $\|\varphi\| = 1$. Since $\ker\{\varphi\}$ is closed, there exists a decomposition $H = \ker\varphi \oplus (\ker\varphi)^{\perp}$. Note that $\operatorname{rg}(\varphi) = \mathbb{K}$, hence $\dim(\ker\varphi)^{\perp} = 1$. Choose $y_0 \in (\ker\varphi)^{\perp}$ with $\varphi(y_0) = 1$. Then $(\ker\varphi)^{\perp} = \operatorname{span}\{y_0\}$. For $x = u + \lambda y_0 \in \ker\varphi \oplus (\ker\varphi)^{\perp}$,

$$\langle x \,, \|y_0\|^{-2}y_0\rangle = \lambda = \lambda\varphi(y) + \varphi(u) = \varphi(x),$$

hence $\varphi = \langle \cdot \,, \|y_0\|^{-1}y_0\rangle$. Since Φ is an isometry, it follows that $1 = \|\varphi\| = \left\| \frac{\|y_0\|}{\|y_0\|^2} \right\| = \frac{1}{\|y_0\|}$, so $\|y_0\| = 1$. □

Corollary 4.21. (i) *Every Hilbert space is reflexive.*

(ii) *The dual H' of a Hilbert space H is an inner product space by*

$$\langle \Phi(x) \,, \Phi(y)\rangle_{H'} = \langle y \,, x\rangle_H$$

with $\Phi : H \to H'$ as in Theorem 4.20.

Proof. (ii) is clear. Let $\Psi : H' \to H''$ as in Theorem 4.20. Then it is easy to check that $\Psi \circ \Phi = J_H$, so J_H is surjective, implying that H is reflexive. □

Corollary 4.22. *Let H be a Hilbert space.*

(i) *A sequence $(x_n)_{n\in\mathbb{N}} \subseteq H$ converges weakly to $x_0 \in H$ if and only if*

$$\langle x_n - x_0 \,, y\rangle \to 0, \qquad y \in H.$$

(ii) *Every bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq H$ contains a weakly convergent subsequence.*

Proof. (i) follows from the Riesz-Fréchet theorem, and (ii) follows with Theorem 2.41. □

4.3 Orthonormal systems

Definition 4.23. Let H be a Hilbert space. A family $S = (x_\lambda)_{\lambda\in\Lambda}$ of vectors in H is called an *orthonormal system* if $\langle x_\lambda\,, x_{\lambda'}\rangle = \delta_{\lambda\lambda'}$. A orthonormal system S is an *orthonormal basis* (or a *complete orthonormal system*) if and only if for every orthonormal system T

$$S \subseteq T \implies S = T.$$

Examples 4.24. (i) The unit vectors $(\mathrm{e}_n)_{n\in\mathbb{N}}$ in $\ell_2(\mathbb{N})$ are a orthonormal system.

(ii) Let $H = L_2(-\pi,\pi)$. An orthonormal system in H is

$$S = \Big\{\frac{1}{\sqrt{2\pi}}\Big\} \cup \Big\{\frac{1}{\sqrt{\pi}}\sin(n\,\cdot\,) : n\in\mathbb{N}\Big\} \cup \Big\{\frac{1}{\sqrt{\pi}}\cos(n\,\cdot\,) : n\in\mathbb{N}\Big\}.$$

Lemma 4.25 (Gram-Schmidt). *Let H be a Hilbert space and $(x_n)_{n\in\mathbb{N}}$ a family of linearly independent vectors. Then there exists a orthonormal system $S = (s_n)_{n\in\mathbb{N}}$ such that $\overline{\operatorname{span} S} = \overline{\operatorname{span}\{x_n : n\in\mathbb{N}\}}$.*

Proof. Let $s_1 := \|x_1\|^{-1}x_1$. Next set $y_2 := x_2 - \langle x_1\,, s_1\rangle s_1$. Note that $y_2 \neq 0$ because x_2 and x_1 are linearly independent. Let $s_2 := \|y_2\|^{-1}y_2$. Then $s_1 \perp s_2$ and $\|s_1\| = \|s_2\| = 1$. Now for $k \geq 1$ let

$$y_{n+1} := x_{n+1} - \sum_{k=1}^{n}\langle x_k\,, s_k\rangle s_k, \qquad s_{n+1} := \|y_{n+1}\|^{-1}y_{n+1}.$$

Since $x_1,\dots,x_{n+1}$ are linearly independent, s_{n+1} is well-defined. By construction, $s_{n+1} \perp s_j$ for $j = 1,\dots n$. Note that for every $n\in\mathbb{N}$, $s_n \in \operatorname{span}\{x_1,\dots,x_n\}$ and $x_n \in \operatorname{span} S$, hence $\overline{\operatorname{span} S} = \overline{\{x_n : n\in\mathbb{N}\}}$. ☐

Example. Let $H = L_2((0,1))$ and $x_n \in H$ defined by $x_n(t) = t^n$. Application of the Gram-Schmidt orthogonalisation yields polynomials $s_n(t) = \sqrt{n+\frac{1}{2}}P_n(t)$ where $P_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n$ is the nth Legendre polynomial.

Theorem 4.26 (Bessel inequality). *Let H be a Hilbert space, $\{s_n : n\in\mathbb{N}\}$ a orthonormal system in H. Then*

$$\sum_{n=1}^{\infty} |\langle x\,, s_n\rangle|^2 \leq \|x\|^2, \qquad x\in H.$$

Proof. For $N\in\mathbb{N}$ let $x_N := x - \sum_{n=1}^{N}\langle x\,, s_n\rangle s_n$. Since $x_N \perp s_n$ for $n = 1,\dots,N$, Pythagoras' theorem yields

$$\|x\|^2 = \|x_N\|^2 + \Big\|\sum_{n=1}^{N}\langle x\,, s_n\rangle s_n\Big\|^2 = \|x_N\|^2 + \sum_{n=1}^{N}|\langle x\,, s_n\rangle|^2 \geq \sum_{n=1}^{N}|\langle x\,, s_n\rangle|^2.$$ ☐

Lemma 4.27. *Let H be a Hilbert space, $S = (s_\lambda)_{\lambda\in\Lambda}$ a orthonormal system in H. Then for every $x \in H$ the set*

$$S_x := \{\lambda \in \Lambda : \langle x\,, s_\lambda\rangle \neq 0\}$$

is at most countable.

Proof. By the Bessel inequality, for every $n \in \mathbb{N}$ the set

$$S_{x,n} := \Big\{\lambda \in \Lambda : |\langle x\,, s_\lambda\rangle| \geq \frac{1}{n}\Big\}$$

is finite. Hence $S_x = \bigcup_{n=1}^\infty S_{x,n}$ is at most countable. □

Definition 4.28. Let X be a normed space, $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$. Then $\sum_{\lambda\in\Lambda} x_\lambda$ *converges unconditionally* to $x \in H$ if and only if $\Lambda_0 := \{\lambda \in \Lambda : x_\lambda \neq 0\}$ is at most countable and $\sum_{n=1}^\infty x_{\lambda_n} = x$ for every enumeration $\Lambda_0 = \{\lambda_n : n \in \mathbb{N}\}$.

Recall that in finite dimensional Banach spaces unconditional convergence is equivalent to absolute convergence. In every infinite dimensional Banach space, however, there exists a unconditionally convergent series that does not converge absolutely (Dvoretzky-Rogers theorem).

Corollary 4.29 (Bessel inequality). *Let H be a Hilbert space and $S \subseteq H$ a orthonormal system. Then*

$$\sum_{s\in S} |\langle x\,, s\rangle| \leq \|x\|^2, \qquad x \in H.$$

Proof. For fixed $x \in H$, the set $S_x = \{s \in S : \langle x\,, s\rangle \neq 0\}$ is at most countable (Lemma 4.27), so the claim follows from the Bessel inequality for countable orthonormal systems. □

Theorem 4.30. *Let H be a Hilbert space and $S \subseteq H$ a orthonormal system. Then*

$$P : H \to H, \qquad Px = \sum_{s\in S} \langle x\,, s\rangle s$$

is an orthogonal projection on $\overline{\operatorname{span} S}$ and the series is unconditionally convergent.

Proof. First we proof that the series in the definition of P is unconditionally convergent (this proves then well-definedness of P). Fix $x \in H$. For fixed $x \in H$, the set $S_x = \{s \in S : \langle x\,, s\rangle \neq 0\}$ is at most countable (Lemma 4.27). Let $S_x = \{s_n : n \in \mathbb{N}\}$ be an enumeration of S_x. Then $\big(\sum_{k=1}^n \langle x\,, s_k\rangle s_k\big)_{n\in\mathbb{N}}$ is a Cauchy sequence because

$$\Big\|\sum_{k=N}^{M} \langle x\,, s_k\rangle s_k\Big\|^2 = \sum_{k=N}^{M} |\langle x\,, s_k\rangle|^2 \longrightarrow 0, \qquad M, K \to \infty$$

by Bessel's inequality. Since H is complete, $y := \sum_{k=1}^\infty \langle x\,, s_k\rangle s_k$ exists. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation. Then also $y_\pi := \sum_{k=1}^\infty \langle x\,, s_\pi(k)\rangle s_{\pi(k)}$ exists. We have to show that $y = y_\pi$. For all

$z \in H$

$$\langle y\,,z\rangle = \sum_{n=1}^{\infty}\langle y\,,s_n\rangle\langle s_n\,,z\rangle = \sum_{n=1}^{\infty}\langle y\,,s_\pi\rangle\langle s_\pi\,,z\rangle = \langle y_\pi\,,z\rangle.$$

We have used that $\sum_{n=1}^{\infty}\langle y\,,s_n\rangle\langle s_n\,,z\rangle$ is absolute convergent and can therefore be rearranged, because, by Hölder's inequality and Bessel's inequality

$$\Big(\sum_{n=1}^{\infty}|\langle y\,,s_n\rangle\langle s_n\,,z\rangle|\Big)^2 \le \Big(\sum_{n=1}^{\infty}|\langle y\,,s_n\rangle|^2\Big)\Big(\sum_{n=1}^{\infty}|\langle s_n\,,z\rangle|^2\Big) \le \|y\|^2\|z\|^2 < \infty.$$

Since $y - y_\pi \perp z$, $z \in H$, it follows that $y = y_\pi$. Therefore the series in the definition of P is unconditionally convergent and P is well-defined.

It is clear that P is a linear and $\|P\| \le 1$ follows from Corollary 4.29. Let $x \in H$. We have to show that $x - Px \in \overline{\operatorname{span} S}^{\perp}$ (Theorem 4.16). This is clear because

$$\Big\langle x - \sum_{s\in S}\langle x\,,s\rangle\,,s_0\Big\rangle = \Big\langle x - \sum_{s\in S_x}\langle x\,,s\rangle\,,s_0\Big\rangle = 0, \qquad s_0 \in S. \qquad \square$$

Theorem 4.31. *Let H be a Hilbert space and $S \subseteq H$ a orthonormal system. Then the following is equivalent.*

(i) *S is a complete orthonormal system.*

(ii) $x \perp S \implies x = 0, \quad x \in H.$

(iii) $H = \overline{\operatorname{span} S}.$

(iv) $x = \sum_{s\in S}\langle x\,,s\rangle s, \qquad x \in H.$

(v) $\langle x\,,y\rangle = \sum_{s\in S}\langle x\,,s\rangle\langle s\,,y\rangle, \qquad x, y \in H.$

(vi) Parseval's equality *holds:* $\|x\|^2 = \sum_{s\in S}|\langle x\,,s\rangle|^2, \qquad x \in H.$

Proof. (i) $\implies$ (ii) If there exists an $x \in H$ such that $x \in S^\perp \setminus \{0\}$, then $S' := S \cup \{\|x\|^{-1}x\}$ is a orthonormal system with $S \subsetneq S'$, contradicting the maximality of S.

(ii) $\implies$ (iii) follows from Lemma 4.18.

(iii) $\implies$ (iv) By theorem 4.30, $x \mapsto \sum_{s\in S}\langle x\,,s\rangle s$ is the orthogonal projection on $\overline{\operatorname{span} S} = H$.

(iv) $\implies$ (v) straightforward.

(v) $\implies$ (vi) Choose $x = y$.

(vi) $\implies$ (i) Assume there exists an orthonormal system $S' \supsetneq S$. Then for every $s' \in S' \setminus S$ we get the contradiction

$$1 = \|s'\|^2 = \sum_{s\in S}|\langle s'\,,s\rangle|^2 = 0. \qquad \square$$

Now we show that the orthonormal systems in Example 4.24 are complete.

Examples 4.32. (i) The set of the unit vectors $\{e_n : n \in \mathbb{N}\}$ in $\ell_2(\mathbb{N})$ are a complete orthonormal system in $\ell_2(\mathbb{N})$ because $\overline{\{e_n : n \in \mathbb{N}\}} = \ell_2(\mathbb{N})$.

(ii) Let Γ be a set and define

$$\ell_2(\Gamma) := \Big\{ f : \Gamma \to \mathbb{K} \;:\; f(\gamma) \neq 0 \text{ for at most countably many } \gamma \in \Gamma \text{ and } \sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty \Big\}.$$

Then $\langle f\,, g\rangle = \sum_{\gamma\in\Gamma} f(\gamma)\overline{g(\gamma)}$ is a well-defined inner product (note that only countably many terms are $\neq 0$ and the sum is absolutely convergent by Hölder's inequality). As in the case $\Gamma = \mathbb{N}$ it can be shown that $\ell_2(\Gamma)$ is a Hilbert space and $(f_\lambda)_{\lambda\in\Gamma}$ where $f_\lambda(\gamma) = \delta_{\lambda\gamma}$ (Kronecker delta) is a complete orthonormal system in $\ell_2(\Gamma)$.

(iii) Let $H = L_2(0, 1)$ and

$$S = \Big\{ \frac{1}{\sqrt{2\pi}} \Big\} \cup \Big\{ \frac{1}{\sqrt{\pi}} \sin(n\,\cdot\,) : n \in \mathbb{N} \Big\} \cup \Big\{ \frac{1}{\sqrt{\pi}} \cos(n\,\cdot\,) : n \in \mathbb{N} \Big\}.$$

Note that $\operatorname{span} S$ is the set of all trigonometric polynomials. Without restriction we can assume that $\mathbb{K} = \mathbb{R}$. By the theorem of Fejér, the trigonometric polynomials are dense in $C_{2\pi} := \{f \in C([-\pi, \pi]) : f(-\pi) = f(\pi)\}$ with respect to $\|\cdot\|_\infty$, hence also with respect to $\|\cdot\|_2$. Since $C_{2\pi}$ is $\|\cdot\|_2$-dense in $L_2([-\pi, \pi])$, S is a total subset of $L_2([-\pi, \pi])$.

Lemma 4.33. *Let H be an infinite dimensional Hilbert space. Then the following is equivalent.*

(i) *H is separable.*
(ii) *Every complete orthonormal system in H is countable.*
(iii) *There exists an countable complete orthonormal system in H.*

Proof. (i) $\Longrightarrow$ (ii) Assume $S \subseteq H$ is an uncountable complete orthonormal system in H. Let $\varepsilon \in (0, 2^{-\frac{1}{2}})$ and $s \neq s' \in S$. Then $B_\varepsilon(s) \cap B_\varepsilon(s') = \emptyset$ because by Pythagoras $\|s - s'\| = \sqrt{\|s\|^2 + \|s'\|^2} = \sqrt{2}$. Let A be a dense subset of H. For every $s \in S$ there exists an $a_s \in A$ such that $a_s \in B_\varepsilon(s)$. In particular, $a_s \neq a_{s'}$ if $s \neq s'$, so A cannot be countable, thus H is not separable.

(ii) $\Longrightarrow$ (iii) The existence of a complete orthonormal system in H follows from Zorn's lemma. By assumption, it must be complete.

(iii) $\Longrightarrow$ (i) Let S be a countable orthonormal system in H. Then $\overline{\operatorname{span} S} = H$ by Theorem 4.31 and H is separable by Theorem 1.25. □

Lemma 4.34. *Let H be Hilbert space and S and T be complete orthonormal system in H. Then $|S| = |T|$.*

Proof. The statement is proved in linear algebra if $|S| < \infty$. Now assume that S is not finite. For $x \in S$ the set $T_x := \{y \in T : \langle x\,, y\rangle \neq 0\}$ is at most countable by Lemma 4.27. By Theorem 4.31 (ii) $T \subseteq \bigcup_{x\in S} T_x$, hence $|T| \leq |S||\mathbb{N}| = |S|$. Analogously, $|S| \leq |T||\mathbb{N}| = |T|$. By the Schröder-Bernstein theorem then $|S| = |T|$. □

Theorem 4.35. *Let H be a Hilbert space and S an orthonormal basis of H. Then $H \cong \ell_2(S)$ (see Example 4.32 (ii)).*

Proof. Define $T : H \to \ell_2(S)$ by $Tx(s) = \langle x\,, s\rangle$, $x \in H$, $s \in S$. T is well-defined by Bessel's inequality. Then $T : H \to \ell_2(S)$ is linear and isometric by Parseval's equality. To show that T is surjective, let $y \in \ell_2(S)$ and define $x := \sum_{s\in S} y(s)s$. Then $x \in H$ (Theorem 4.30) and $Tx = y$. □

Note that by construction $\langle Tx\,, Ty\rangle = \langle x\,, y\rangle$, $x, y \in H$.

Corollary 4.36. *If H is a separable Hilbert space, then $H \cong \ell_2(\mathbb{N})$.*

Corollary 4.37 (Fischer-Riesz theorem). $L_2[0, 1] \cong \ell_2(\mathbb{N})$.

4.4 Linear operators in Hilbert spaces

Definition 4.38. Let H_1, H_2 be Hilbert spaces and $\Phi_j : H_j \to H_j'$ the canonical isomorphism in the Fréchet-Riesz representation theorem (Theorem 4.20). Let $T \in L(H_1, H_2)$. Its (*Hilbert space*) *adjoint operator* is $T^* := \Phi_1^{-1} T' \Phi_2 \in L(H_2, H_1)$ where T' is the Banach space adjoint of T (see Definition 2.26).

Hence T^* is characterised by

$$\langle Tx\,, y\rangle = \langle x\,, T^*y\rangle, \qquad x \in H_1, y \in H_2.$$

Theorem 4.39. *Let H_1, H_2, H_3 be Hilbert spaces, $S, T \in L(H_1, H_2)$, $R \in L(H_2, H_3)$ and $\lambda \in \mathbb{K}$.*

(i) $(\lambda S + T)^* = \overline{\lambda} S^* + T^*$.
(ii) $(RT)^* = T^* R^*$.
(iii) $T^* \in L(H_2, H_1)$ *and* $\|T^*\| = \|T\|$.
(iv) $T^{**} = T$.
(v) $\|TT^*\| = \|T^*T\| = \|T\|^2$.
(vi) $\ker T = (\mathrm{rg}(T^*))^\perp$, $\ker T^* = (\mathrm{rg}(T))^\perp$.
(vii) *If T is invertible, then* $(T^{-1})^* = (T^*)^{-1}$.

Proof. (i)–(iv) are clear. For the proof of (v) note that for $\|x\| = 1$

$$\|Tx\|^2 = \langle Tx\,, Tx\rangle = \langle x\,, T^*Tx\rangle \le \|x\|\,\|T^*Tx\| \le \|T^*T\| \le \|T^*\|\,\|T\| = \|T\|^2.$$

Taking the supremum over all $x \in H$ with $\|x\| = 1$ shows the desired equalities.
(vi) $\ker T = (\mathrm{rg}\, T^*)^\perp$ because for $x \in H$

$$\begin{aligned} Tx = 0 \quad &\iff \quad \forall y \in H_2 \quad \langle Tx\,, y\rangle = 0 \quad \iff \quad \forall y \in H_2 \quad \langle x\,, T^*y\rangle = 0 \\ &\iff \quad x \perp \mathrm{rg}(T^*). \end{aligned}$$

Then also $\ker T^* = (\mathrm{rg}(T^{**}))^\perp = (\mathrm{rg}(T))^\perp$. □

Definition 4.40. Let H_1, H_2 be Hilbert spaces, $T \in L(H_1, H_2)$.

(i) T is called *unitary* if T is invertible and $TT^* = \mathrm{id}_{H_2}$ and $T^*T = \mathrm{id}_{H_1}$.

(ii) T is called *normal* if $H_1 = H_2$ and $TT^* = T^*T$.

(iii) T is called *selfadjoint* if $H_1 = H_2$ and $T = T^*$.

Remarks. (i) T selfadjoint $\implies$ T normal.

(ii) $T \in L(H_1, H_2) \implies TT^*$ and T^*T are selfadjoint.

Next we show that a length preserving linear map between Hilbert spaces also preserves angles.

Lemma 4.41. *Let H_1, H_2 be Hilbert spaces and $T \in L(H_1, H_2)$.*

(i) *T is an isometry $\iff$ $\langle Tx\,, Ty\rangle = \langle x\,, y\rangle$, $x, y \in H_1$.*

(ii) *T is unitary $\iff$ T is a surjective isometry.*

Proof. (i) The direction "⇐" is clear; "⇒" follows from the polarisation formula (Theorem 4.7).
(ii) "⇒" Since T is unitary, if follows that $\mathrm{rg}(T) \supseteq \mathrm{rg}(TT^*) = \mathrm{rg}(\mathrm{id}_{H_2}) = H_2$, so T is surjective. T is an isometry because for all $x, y \in H_1$

$$\langle Tx\,, Ty\rangle = \langle T^*Tx\,, y\rangle = \langle x\,, y\rangle,$$

"⇐" Assume that T as a surjective isometry. Since

$$\langle x\,, y - T^*Ty\rangle = \langle x\,, y\rangle - \langle Tx\,, Ty\rangle = 0, \qquad x, y \in H_1,$$

it follows that $T^*Ty = y$, so $T^*T = \mathrm{id}_{H_1}$. In particular T^* is surjective. Now we will show that T^* is an isometry. Let $\xi, \eta \in H_2$. Then there exist $x, y \in H_1$ such that $Tx = \xi$ and $Ty = \eta$. It follows that

$$\langle T^*\xi\,, T^*\eta\rangle = \langle T^*Tx\,, T^*Ty\rangle = \langle x\,, y\rangle = \langle Tx\,, Ty\rangle = \langle \xi\,, \eta\rangle.$$

By the same argument as for T we conclude that $\mathrm{id}_{H_2} = T^{**}T^* = TT^*$. □

Examples 4.42. (i) Let H_1, H_2 be Hilbert spaces with $\dim H_1 = \dim H_2 = n < \infty$. After choice of bases, a linear operator $T : H_1 \to H_2$ has a representation $(a_{ij})_{ij=1}^n \in M_n(\mathbb{C})$. The matrix corresponding to T^* is then $(\overline{a_{ji}})_{ij=1}^n$.

(ii) Let $H = L_2[0, 1]$. For $k \in L_\infty([0, 1] \times [0, 1])$ define

$$T_k : L_2[0, 1] \to L_2[0, 1], \qquad (T_k f)(t) = \int_0^1 k(s, t) f(s)\ \mathrm{d}s.$$

Then $T_k \in L_2[0, 1]$ and

$$T_k^* : L_2[0, 1] \to L_2[0, 1], \qquad (T_k f)(t) = \int_0^1 \overline{k(s, t)} f(s)\ \mathrm{d}s,$$

that is $T_k^* = T_{\overline{k}}$.

Theorem 4.43 (Hellinger-Toeplitz). *Let H be a Hilbert space, $T : H \to H$ a linear operator such that*

$$\langle Tx\,, y\rangle = \langle x\,, Ty\rangle, \qquad x, y \in H.$$

Then T is bounded, hence selfadjoint.

Proof. It suffices to show that T is closed because $\mathcal{D}(T) = H$ is closed. Let $(x_n)_{n\in\mathbb{N}} \subseteq H$ with $x_n \to 0$ and $Tx_n \to y$. Observe that

$$\|y\|^2 = \lim_{n\to\infty} \langle Tx_n\,, y\rangle = \lim_{n\to\infty} \langle x_n\,, Ty\rangle = \langle \lim_{n\to\infty} x_n\,, Ty\rangle = \langle 0\,, Ty\rangle = 0,$$

so $y = 0$. This implies that T is closable, hence closed since $\mathcal{D}(T) = H$. ☐

Theorem 4.44. *Let H be a complex Hilbert space. For $T \in L(H)$ the following is equivalent.*

(i) $\langle Tx\,, x\rangle \in \mathbb{R}$, $x \in H$.

(ii) *T is selfadjoint.*

Proof. (ii) $\implies$ (i) follows from

$$\langle Tx\,, x\rangle = \langle x\,, Tx\rangle = \overline{\langle Tx\,, x\rangle}, \qquad x \in H.$$

(i) $\implies$ (ii) Let $x, y \in H$ and $\lambda \in \mathbb{C}$.

$$A := \langle T(\lambda x + y)\,,\ \lambda x + y\rangle = |\lambda|^2\langle Tx\,, x\rangle + \langle Ty\,, y\rangle + \lambda\langle Tx\,, y\rangle + \overline{\lambda}\langle Ty\,, x\rangle,$$

$$B := \overline{\langle T(\lambda x + y)\,,\ \lambda x + y\rangle} = |\overline{\lambda}|^2\overline{\langle Tx\,, x\rangle} + \overline{\langle Ty\,, y\rangle} + \overline{\lambda}\langle y\,, Tx\rangle + \lambda\langle x\,, Ty\rangle.$$

By assumption, $A = B$, so in the special cases $\lambda = 1$ and $\lambda = \mathrm{i}$ we obtain

$$\langle Tx\,, y\rangle + \langle Ty\,, x\rangle = \langle y\,, Tx\rangle + \langle x\,, Ty\rangle,$$
$$\langle Tx\,, y\rangle - \langle Ty\,, x\rangle = -\langle y\,, Tx\rangle + \langle x\,, Ty\rangle,$$

so finally $\langle Tx\,, y\rangle = \langle x\,, Ty\rangle$. ☐

Theorem 4.45. *Let H be a Hilbert space, $T \in L(H)$ selfadjoint. Then*

$$\|T\| = \sup_{\|x\|\le 1} |\langle Tx\,, x\rangle|.$$

Proof. Let $M := \sup_{\|x\|\le 1} |\langle Tx\,, x\rangle|$. Obviously $M \le \|T\|$ because for $\|x\| \le 1$

$$|\langle Tx\,, x\rangle| \le \|T\|\|x\|^2 \le \|T\|.$$

To show the reverse inequality fix $x, y \in H$. Observe that

$$\begin{aligned}\langle T(x+y)\,, x+y\rangle - \langle T(x-y)\,, x-y\rangle &= 2\langle Tx\,, y\rangle + 2\langle Ty\,, x\rangle \\ &= 2\langle Tx\,, y\rangle + 2\langle y\,, Tx\rangle = 4\,\mathrm{Re}\langle Tx\,, y\rangle.\end{aligned}$$

Hence, by the parallelogram identity (Theorem 4.8), for $\|x\| \le 1$, $\|y\| \le 1$,

$$\begin{aligned}\operatorname{Re}\langle Tx\,,y\rangle &\le \frac{1}{4}\big(|\langle T(x+y)\,,x+y\rangle| + |\langle T(x-y)\,,x-y\rangle|\big)\\ &\le \frac{1}{4}\big(M\|x+y\|^2 + M\|x-y\|^2\big) = \frac{M}{2}(\|x\|^2+\|y\|^2) \le M.\end{aligned}$$

Now choose $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $\lambda\langle Tx\,,y\rangle = |\langle Tx\,,y\rangle|$, so

$$|\langle Tx\,,y\rangle| = \langle T(\lambda x)\,,y\rangle = |\operatorname{Re}\langle T(\lambda x)\,,y\rangle| \le M, \qquad \|x\| \le 1,\ \|y\| \le 1.$$

In particular, $\|\langle\cdot\,,Tx\rangle\| \le M$, so $\|Tx\| \le 1$ for $\|x\| \le 1$. This shows $\|T\| \le M$. □

Corollary 4.46. *Let H be a Hilbert space and $T \in L(H)$ selfadjoint. If $\langle Tx\,,x\rangle = 0$, $x \in H$, then $T = 0$.*

Note that the condition $\langle Tx\,,x\rangle = 0$ automatically implies that T is selfadjoint in the case of a complex Hilbert space. In a real Hilbert spaces H the assumption that T is selfadjoint is necessary for the statement in the corollary. For example, let $T = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation about $90°$. Then $T \neq 0$ but $\langle Tx\,,x\rangle = 0$ for all $x \in \mathbb{R}^2$.

Lemma 4.47. *Let H be a Hilbert space, $T \in L(H)$ a normal operator. Then*

$$\|Tx\| = \|T^*x\|, \qquad x \in H,$$

in particular, $\ker T = \ker T^*$.

Proof. $0 = \langle T^*Tx - TT^*x\,,x\rangle = \|Tx\|^2 - \|T^*x\|^2$. □

Definition 4.48. Let H be a Hilbert space. A bounded selfadjoint operator $T \in L(H)$ is called *non-negative*, denoted by $T \ge 0$, if $\langle Tx\,,x\rangle \ge 0$ for all $x \in H$. It is called *positive*, denoted by $T > 0$, if $\langle Tx\,,x\rangle > 0$ for all $x \in H \setminus \{0\}$. We write $T \le S$ if and only if $S - T \ge 0$. A sequence $(T_n)_{n\in\mathbb{N}} \in L(H)$ is *increasing* if and only if $T_n \le T_{n+1}$, $n \in \mathbb{N}$. A sequence $(T_n)_{n\in\mathbb{N}} \in L(H)$ is *decreasing* if and only if $(-T_n)_{n\in\mathbb{N}} \in L(H)$ is increasing.

Theorem 4.49. *Let H be a Hilbert space. Every monotonic bounded sequence of selfadjoint linear operators on H converges strongly.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a bounded monotonic sequence of selfadjoint operators. Without restriction we assume that it is increasing. Let

$$s_{nm} : H \times H \to \mathbb{K}, \qquad s_{nm}(x,y) = \langle (T_n - T_m)x\,,y\rangle$$

is a positive semidefinite sesquilinear form on H if $n \ge m$. Let M be a bound of $(T_n)_{n\in\mathbb{N}}$. Note that then $\|T_n - T_m\| \le 2M$. Then, using Cauchy-Schwarz inequality, we find for $n \ge m$ and $x \in H$

with $\|x\| = 1$

$$\begin{aligned}\|(T_n - T_m)x\|^2 &= \langle (T_n - T_m)x \,, (T_n - T_m)x\rangle = s_{nm}(x, (T_n - T_m)x)\\ &\le s_{nm}(x,x)^{\frac12}\ s_{nm}((T_n - T_m)x,\ (T_n - T_m)x)^{\frac12}\\ &= \langle (T_n - T_m)x \,, x\rangle^{\frac12} \langle (T_n - T_m)x \,, (T_n - T_m)^2 x\rangle^{\frac12}\\ &\le \langle (T_n - T_m)x \,, x\rangle^{\frac12} \|T_n - T_m\|^{\frac12} \|T_n - T_m\|\\ &\le (2M)^{\frac32} \langle (T_n - T_m)x \,, x\rangle^{\frac12}.\end{aligned}$$

By assumption $(\langle T_n x \,, x\rangle)_{n\in\mathbb{N}}$ is a monotonically increasing bounded sequence in $\mathbb{R}$, hence convergent. It follows that $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence, hence T converges strongly to some $T \in L(H)$ (Proposition 3.13). That T is selfadjoint follows from

$$\langle Tx \,, y\rangle = \lim_{n\to\infty} \langle T_n x \,, y\rangle = \lim_{n\to\infty} \langle x \,, T_n y\rangle = \langle x \,, Ty\rangle, \qquad x, y \in H. \qquad \square$$

4.5 Projections in Hilbert spaces

Proposition 4.50. *Let H be a Hilbert space, $P \in L(H)$ a projection. If $P \neq 0$ then the following is equivalent.*

(i) *P is an orthogonal projection.*
(ii) $\|P\| = 1$.
(iii) *P is selfadjoint.*
(iv) *P is normal.*
(v) $\langle Px \,, x\rangle \ge 0$, $x \in H$.

Proof. (i) $\Longrightarrow$ (ii) follows from Theorem 4.16.
(ii) $\Longrightarrow$ (i) Let $x \in \ker P$ and $y \in \mathrm{rg}(P)$. Then for all $\lambda \in \mathbb{K}$

$$\|\lambda y\|^2 = \|P(x + \lambda y)\|^2 \le \|x + \lambda y\|^2 = \|x\|^2 + |\lambda|^2 \|y\|^2 + 2\,\mathrm{Re}(\lambda \langle x \,, y\rangle).$$

In particular, $0 \le \|x\|^2 + 2\lambda\,\mathrm{Re}\langle x \,, y\rangle$ for all $\lambda \in \mathbb{R}$, and $0 \le \|x\|^2 + 2\mathrm{i}\lambda\,\mathrm{Im}\langle x \,, y\rangle$ for all $\lambda \in \mathrm{i}\mathbb{R}$, hence $\mathrm{Re}\langle x \,, y\rangle = \mathrm{Im}\langle x \,, y\rangle = 0$.
(i) $\Longrightarrow$ (iii) Observe that $\langle Px \,, y\rangle = \langle x \,, Py\rangle$ for all $x, y \in H$ because

$$\begin{aligned}\langle Px \,, y\rangle &= \langle Px \,, y - Py + Py\rangle = \langle Px \,, Py\rangle,\\ \langle x \,, Py\rangle &= \langle x - Px + Px \,, Py\rangle = \langle Px \,, Py\rangle.\end{aligned}$$

(iii) $\Longrightarrow$ (iv) is clear.
(iv) $\Longrightarrow$ (i) By Lemma 4.47, $\ker P = \ker P^* = (\mathrm{rg}\,P)^\perp$.
(i) $\Longrightarrow$ (v) For all $x \in H$: $\langle Px \,, x\rangle = \langle Px \,, x - Px + Px\rangle = \langle Px \,, Px\rangle \ge 0$.
(v) $\Longrightarrow$ (i) Let $x \in \ker P$, $y \in \mathrm{rg}\,P$. Since for all $\lambda \in \mathbb{R}$

$$0 \le \langle P(x + \lambda y) \,, x + \lambda y\rangle = \langle \lambda y \,, x + \lambda y\rangle = \lambda^2 \|y\|^2 + \lambda \langle y \,, x\rangle,$$

it follows that $\langle x \,, y\rangle = 0$. $\square$

Lemma 4.51. *Let H Hilbert space H. A linear operator $P : H \to H$ is an orthogonal projection if and only if $P^2 = P$ and $\langle x\,, Py\rangle = \langle y\,, Px\rangle$ for all $x, y \in H$.*

Proof. Assume that P is an orthogonal projection. Then $P^2 = P$ and by Proposition 4.50 P is selfadjoint.
If $P^2 = P$ and $\langle x\,, Py\rangle = \langle y\,, Px\rangle$ for all $x, y \in H$, then P is a projection. By the theorem of Hellinger-Toeplitz (Theorem 4.43) P is selfadjoint, hence P is an orthogonal projection by Proposition 4.50. □

Lemma 4.52. *Let H be a Hilbert space, $U_1, U_2 \subseteq H$ closed subspaces and P_1, P_2 the corresponding orthogonal projections. Then the following is equivalent:*

(i) $P_1P_2 = P_2P_1 = 0$.
(ii) $U_1 \perp U_2$.
(iii) *$P_1 + P_2$ is an orthogonal projection.*

If one of the equivalent conditions above hold, then $\operatorname{rg}(P_1 + P_2) = U_1 \oplus U_2$.

Proof. (i) $\Longrightarrow$ (ii) By assumption, $U_2 = \operatorname{rg} P_2 \subseteq \ker P_1 = (\operatorname{rg} P_1)^\perp = U_1^\perp$, hence $U_1 \perp U_2$.
(ii) $\Longrightarrow$ (i) By assumption, $\operatorname{rg} P_2 = U_2 \subseteq U_1^\perp = \ker P_1$, hence $P_1P_2 = 0$. Since (ii) is symmetric in U_1 and U_2, it follows also that $P_2P_1 = 0$.
(i),(ii) $\Longrightarrow$ (iii) Observe that $P_1P_2 = P_2P_1 = 0$, so $P_1 + P_2$ is a projection because

$$(P_1 + P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_2.$$

Since the sum of two selfadjoint operators is selfadjoint, $P_1 + P_2$ is selfadjoint, hence, by Proposition 4.50 an orthogonal projection.
(iii) $\Longrightarrow$ (i) Since $P_1 + P_2$ is an orthogonal projection, it follows that

$$P_1P_2 + P_2P_1 = (P_1 + P_2)^2 - (P_1 + P_2) = 0.$$

In particular $0 = (P_1P_2 + P_2P_1)P_2x = (\operatorname{id} + P_2)P_1P_2x$. Note that for $y \in H \setminus \{0\}$ the vectors $(\operatorname{id} - P_2)y$ and P_2y are linearly independent, hence $(\operatorname{id} + P_2)y = (\operatorname{id} - P_2)y + 2P_2y$ is zero if and only if $(\operatorname{id} - P_2)y = 0$ and $P_2y = 0$, hence $y = 0$. Therefore $\operatorname{rg} P_1P_2 \subseteq \ker(\operatorname{id} + P_2) = \{0\}$. □

Lemma 4.53. *Let H be a Hilbert space and P_1 and P_2 orthogonal projections on subspaces U_1 and U_2.*

(i) *P_1P_2 is an orthogonal projection if and only if $P_1P_2 = P_2P_1$. In this case, P_1P_2 is an projection on $U_1 \cap U_2$.*
(ii) *$P_1 - P_2$ is an orthogonal projection if and only if $P_1P_2 = P_2P_1 = P_2$.*

Proof. (i) If P_1P_2 is an orthonormal projection, then, by Proposition 4.50, P_1P_2 is selfadjoint, that is $P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$. On the other hand, if P_1 and P_2 commute, then it is easy to verify that $(P_1P_2)^2 = P_1P_2$ and $(P_1P_2)^* = P_1P_2$, hence P_1P_2 is an orthogonal projection. In this case, $\operatorname{rg}(P_1P_2) = \operatorname{rg}(P_2P_1)$, so $\operatorname{rg}(P_1P_2) \subseteq U_1 \cap U_2$. On the other hand, $P_1P_2x = x$ for every $x \in U_1 \cap U_2$, so also $\operatorname{rg}(P_1P_2) \supseteq U_1 \cap U_2$ holds.

(ii) Using Lemma 4.52 we obtain

$$\begin{aligned} P_1 - P_2 \text{ orthonormal projection} &\iff 1-(P_1-P_2) \text{ orthonormal projection}\\ &\iff (1-P_1)+P_2 \text{ orthonormal projection}\\ &\iff P_2(1-P_1)=(1-P_1)P_2=0\\ &\iff P_2P_1=P_1P_2=P_2. \quad \square \end{aligned}$$

Lemma 4.54. *Let H be a Hilbert space and P_1, P_2 orthogonal projections on H_0, $H_1 \subseteq H$. Then the following is equivalent.*

(i) $H_0 \subseteq H_1$,

(ii) $\|P_0x\| \le \|P_1x\|, \quad x \in H$.

(iii) $\langle P_0x\,, x\rangle \le \langle P_1x\,, x\rangle, \quad x \in H$.

(iv) $P_0P_1 = P_0$.

Proof. (ii) $\iff$ (iii) Let $x \in H$ and P an orthogonal projection. Then $\langle Px\,, x\rangle = \langle P^2x\,, x\rangle = \langle Px\,, Px\rangle = \|Px\|^2$.
(i) $\iff$ (iv)

$$\begin{aligned} P_0P_1 = P_0 &\iff P_0(\mathrm{id}-P_1)=0 &&\iff \mathrm{rg}(\mathrm{id}-P_1)\subseteq \ker P_0\\ &\iff (\mathrm{rg}\,P_1)^\perp \subseteq (\mathrm{rg}\,P_0)^\perp &&\iff H_1^\perp \subseteq H_0^\perp\\ &\iff H_0\subseteq H_1. \end{aligned}$$

(iv) $\implies$ (ii) For all $x \in H$: $\|P_0x\| = \|P_0P_1x\| \le \|P_0\|\|P_1x\| \le \|P_1x\|$.
(iii) $\implies$ (i) Let $x \in H_1^\perp = \ker P_1$. Then $0 = \langle P_1x\,, x\rangle \ge \langle P_0x\,, x\rangle = \|P_0\|^2$, hence $H_1^\perp \subseteq \ker P_0 = H_0^\perp$. □

Lemma 4.55. *Let H be a Hilbert space and $(P_n)_{n\in\mathbb{N}}$ a sequence of orthogonal projections with $\langle P_mx\,, x\rangle \le \langle P_nx\,, x\rangle$ for all $x \in X$ and $m < n$. Then $(P_n)_{n\in\mathbb{N}}$ converges strongly to an orthogonal projection.*

Proof. By Theorem 4.49 we already know that $s\text{-}\lim P_n =: P$ exists and is a selfadjoint operator. It remains to be shown that P is a projection, that is, that $P^2 = P$. For $x \in H$ and $n \in \mathbb{N}$

$$P^2x = (P-P_n+P_n)(P-P_n+P_n)x = (P-P_n)Px + P_n(P-P_n)x + P_n^2x.$$

Note that $(P-P_n)Px \to 0$, $n\to\infty$, and also $P_n(P-P_n)x$ because $\|P_n\| = 1$, $n \in \mathbb{N}$. Since $P_n^2x = P_nx \to Px$, it follows that $P^2 = P$. □

4.6 The adjoint of an unbounded operator

In sections 2.4 and section 4.4 we have defined the adjoint of bounded linear operators between Banach or Hilbert spaces. Now we define the adjoint of an unbounded linear operator. Recall that $T(X \to Y)$ denotes a possibly unbounded linear operators defined on a subspace $\mathcal{D}(T) \subseteq X$.

Definition 4.56. Let X, Y be Banach spaces and $\mathcal{D}(T) \subseteq X$ a dense subspace. For a linear map $T : X \supseteq \mathcal{D}(T) \to Y$ we define

$$\mathcal{D}(T') := \{\varphi \in Y' \ : \ x \mapsto \varphi(Tx) \text{ is a bounded linear functional on } \mathcal{D}(T)\},$$

Since $\mathcal{D}(T)$ is dense in X, the map $\mathcal{D}(T) \to \mathbb{K},\ x \mapsto \varphi(Tx)$ has a unique continuous extension $T'\varphi \in X'$ for $\varphi \in \mathcal{D}(T')$. Hence the *Banach space adjoint* T'

$$T' : Y' \supseteq \mathcal{D}(T') \to X', \qquad (T'\varphi)(x) = \varphi(Tx), \quad x \in \mathcal{D}(T),\ \varphi \in \mathcal{D}(T').$$

is well-defined.

Theorem 4.57. *Let X, Y be Banach spaces, $\mathcal{D}(T) \subseteq X$ a dense subspace and $T : X \supseteq \mathcal{D}(T) \to Y$ be a linear operator. Then T' is closed.*

Proof. Let $G(T') = \{(y', T'y') : \varphi \in \mathcal{D}(T')\} \subseteq Y' \times X'$ be the graph of T'.
Note that $(y', x') \in G(T')$ if and only if $x'x = y'(Tx)$ for all $x \in \mathcal{D}(T)$. Now let $((y'_n, x'_n))_{n\in\mathbb{N}} \subseteq G(T')$ a convergent sequence with $\lim_{n\to\infty}(y'_n, x'_n) = (y'_0, x'_0)$. For all $x \in \mathcal{D}(T)$ it follows that

$$x'_0 x = \lim_{n\to\infty} x'_n x = \lim_{n\to\infty} y'_n(Tx) = \lim_{n\to\infty} y'_0(Tx),$$

thus $(y'_0, x'_0) \in G(T')$ which implies that T' is closed. □

Definition 4.58. Let X, Y be Banach spaces. For linear operators S, T from X to Y we write $S \subseteq T$ if T is an extension of S, that is, if $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $T|_{\mathcal{D}(S)} = S$.

Theorem 4.59. *Let X, Y, Z be Banach spaces.*

(i) *Let $(S, \mathcal{D}(S))$ and $(T, \mathcal{D}(T))$ be densely defined linear operators $X \to Y$. If $S \subseteq T$ then $T' \subseteq S'$.*

(ii) *Assume $S(X \to Y)$ and $T(Y \to Z)$ are densely defined such that also TS is densely defined. Then $S'T' \subseteq (TS)'$.*

(iii) *Assume $S(X \to Y)$ and $T(X \to Y)$ are densely defined such that also $T+S$ is densely defined. Then $(S' + T') \subseteq (S + T)'$.*

Proof. (i) is clear from the definition of the adjoint operator.
(ii) Let $z' \in \mathcal{D}(S'T')$. Then $T'z' \in \mathcal{D}(S')$ and the map

$$\mathcal{D}(S) \to \mathbb{K}, \quad x \mapsto (T'z')(Sx)$$

is continuous. Then also its restriction

$$\mathcal{D}(TS) \to \mathbb{K}, \quad x \mapsto (T'z')(Sx) = z'(TSx)$$

is continuous. Note that by assumption $\mathcal{D}(TS)$ is dense in X, hence $z' \in \mathcal{D}((TS)')$ and $(TS)'z' = S'T'z'$.

(iii) Let $y' \in \mathcal{D}(T' + S') = \mathcal{D}(T') \cap \mathcal{D}(S')$. Then the map

$$\mathcal{D}(T+S) \to \mathbb{K}, \quad x \mapsto y'(Tx) + y'(Sx) = y'((T+S)x)$$

is continuous. Since by assumption $\mathcal{D}(T+S)$ is dense in X, $y' \in \mathcal{D}((T+S)')$ and $(T+S)'y' = (T'+S')y'$. □

If S and T are bounded, then "=" holds in (ii) and (iii) (Theorem 2.27). Note that for unbounded linear operators $T' + S' = (T+S)'$ is not necessarily true. For example, if $T(X \to Y)$ is a densely defined unbounded linear operator such that also T' is densely defined with $\mathcal{D}(T') \neq Y'$. Then $\mathcal{D}(T' - T') \neq Y' = \mathcal{D}(T-T)'$.

Corollary 4.60. *Let X be a Banach space, T a densely defined linear operator in X with bounded inverse $T^{-1} \in L(X)$. Then T' is invertible and*

$$(T')^{-1} = (T^{-1})'.$$

Proof. By Theorem 4.59 (ii) it follows that $(T^{-1})'T' \subseteq (TT^{-1})' = \mathrm{id}'_X = \mathrm{id}_X$, hence $(T^{-1})'T' = \mathrm{id}_{\mathcal{D}(T')}$.
Again by Theorem 4.59 (ii) we find $T'(T^{-1})' \subseteq (T^{-1}T)' = \mathrm{id}'_{\mathcal{D}(T)} = \mathrm{id}_X$, so it suffices to show $\mathcal{D}(T'(T^{-1})') = \mathcal{D}(T')$. Let $\varphi \in \mathcal{D}(T')$ and $\eta = (T^{-1})'\varphi$. For every $x \in \mathcal{D}(T)$ it follows that $\eta(Tx) = ((T^{-1})'\varphi)(Tx) = \varphi(T^{-1'}Tx) = \varphi(x)$, which implies $\eta \in \mathcal{D}(T')$, hence $\mathcal{D}(T'(T^{-1})') = \mathcal{D}(T')$. □

Note that above Corollary follows also from Theorem 5.20.
More general is Theorem 4.65 due to Phillips.

Definition 4.61. Let X be a Banach space. For subspaces $A \subseteq X$ and $B \subseteq X'$ we define the annihilators

$$A^\circ := \{\varphi \in X' : \varphi(x) = 0,\ x \in A\} \subseteq X',$$
$${}^\circ B := \{x \in X : \varphi(x) = 0,\ \varphi \in B\} \subseteq X.$$

Remark 4.62. The sets A° and ${}^\circ B$ are closed subspaces and ${}^\circ(A^\circ) = \overline{A}$. If X is reflexive, then also $({}^\circ B)^\circ = \overline{B}$.

Proof. Obviously, A° and ${}^\circ B$ are subspaces. Let $(x'_n)_{n\in\mathbb{N}} \subseteq A^\circ$ be a convergent sequence. Then $x'_0 := \lim_{n\to\infty} x'_n \in A^\circ$ because $x'_0 x = \lim_{n\to\infty} x'_n x = 0$ for all $x \in A$. Let $(x_n)_{n\in\mathbb{N}} \subseteq {}^\circ B$ be a convergent sequence. Then $x_0 := \lim_{n\to\infty} x_n \in {}^\circ B$ because $\varphi x_0 = \lim_{n\to\infty} \varphi x_n = 0$ for all $\varphi \in B$.

Now we show that ${}^\circ(A^\circ) = \overline{A}$. Since obviously $A \subseteq {}^\circ(A^\circ)$, also $\overline{A} \subseteq {}^\circ(A^\circ)$. Assume that there exists an $a \in {}^\circ(A^\circ) \setminus \overline{A}$. By a corollary to the Hahn-Banach theorem (Corollary 2.20) there exists a $\varphi \in X'$ such that $\varphi|_{\overline{A}} = 0$ and $\varphi(a) \neq 0$. Therefore $\varphi \in A^\circ$, so by definition of ${}^\circ(A^\circ)$, also $\varphi(a) = 0$.

$({}^\circ B)^\circ = \overline{B}$ follows if we identify X with X'' using the canonical map J_X. □

Lemma 4.63. *Let X, Y be Banach space, $Y \neq \{0\}$ and $T(X \to Y)$ a densely defined closed linear operator and $y_0 \in Y \setminus \{0\}$. Then there exists a $\varphi \in \mathcal{D}(T')$ such that $\varphi(y_0) \neq 0$, in particular, $\mathcal{D}(T') \neq \{0\}$.*

Proof. By assumption, the graph $G(T)$ of T is closed and $(0, y_0) \neq G(T)$. Hence, by a corollary to the Hahn-Banach theorem (Corollary 2.20) there exists $\psi \in (X \times Y)'$ such that $\psi|_{G(T)} = 0$ and $\psi((0, y_0)) \neq 0$. Let $\varphi : Y \to \mathbb{K}$, $\varphi(y) = \psi((0, y))$. Obviously $\varphi \in Y'$ and $\varphi(y_0) \neq 0$. Moreover, $\varphi \in \mathcal{D}(T')$ because for all $x \in \mathcal{D}(T)$

$$\begin{aligned} \varphi(Tx) &= \psi((0, Tx)) = \psi((x, Tx) - (x, 0)) = \psi((x, Tx)) - \psi((x, 0)) \\ &= -\psi((x, 0)). \end{aligned}$$ □

Theorem 4.64. *Let X and Y be Banach spaces. For a densely defined closed linear operator $T(X \to Y)$ the following holds:*

(i) $\operatorname{rg}(T)^\circ = \overline{\operatorname{rg}(T)}^\circ = \ker T'$.
(ii) $\overline{\operatorname{rg} T} = {}^\circ(\ker T')$.
(iii) $\overline{\operatorname{rg} T} = Y \iff T'$ *is injective.*
(iv) ${}^\circ(\operatorname{rg} T') \cap \mathcal{D}(T) = \ker T$.
(v) $\overline{\operatorname{rg} T'} \subseteq (\ker T)^\circ$.

Proof. (i) The first equality is clear. The second equality follows from

$$\begin{aligned} \varphi \in \operatorname{rg}(T)^\circ \quad &\iff \quad \forall\, y \in \operatorname{rg}(T) \quad \varphi(y) = 0 \\ &\iff \quad \forall\, x \in \mathcal{D}(T) \quad \varphi(Tx) = 0 \\ &\iff \quad \varphi \in \mathcal{D}(T'),\ T'\varphi = 0 \\ &\iff \quad \varphi \in \ker(T'). \end{aligned}$$

(ii) $\overline{\operatorname{rg} T} = {}^\circ((\operatorname{rg} T)^\circ) = {}^\circ(\ker T')$ by (i) and Remark 4.62.

(iii) By (ii), $\overline{\operatorname{rg} T} = Y$ if and only if ${}^\circ(\ker T') = Y$. This is the case if and only if $\varphi(y) = 0$ for all $\varphi \in \ker T'$ and $y \in Y$, that is, if and only if $\ker T' = \{0\}$.

(iv) Let $x \in \ker(T)$ and $x' \in \operatorname{rg} T'$. Choose $y' \in \mathcal{D}(T')$ with $T'y' = x'$. Then $x'x = (T'y')x = y'(Tx) = y'(0) = 0$, hence $x \in {}^\circ(\operatorname{rg} T')$.
Now let $x \in {}^\circ(\operatorname{rg} T') \cap \mathcal{D}(T)$. Then $y'(Tx) = (T'y')x = 0$ for all $y' \in Y'$. Since T is closed, it follows by Lemma 4.63 that $Tx = 0$, hence $x \in \ker T$.

(v) Let $x' \in \operatorname{rg}(T')$ and $x \in \ker T$. Choose $y' \in \mathcal{D}(T')$ such that $T'y' = x'$. Then $x'x = (T'y')x = y'(Tx) = y'(0) = 0$. It follows that $\operatorname{rg}(T') \subseteq (\ker T)^\circ$, and since $(\ker T)^\circ$ is closed, the statement is proved. □

Theorem 4.65 (Phillips). *Let X, Y be a Banach spaces, $T(X \to Y)$ a densely defined injective linear operator with $\overline{\operatorname{rg}(T)} = Y$. Then*

$$(T')^{-1} = (T^{-1})'. \tag{4.2}$$

Moreover, $\operatorname{rg}(T) = Y$ *and T^{-1} is bounded if and only if T is closed and $(T')^{-1}$ is bounded on X'.* (T^{-1} denotes the inverse of $T : \mathcal{D}(T) \to \operatorname{rg}(T)$, similar for $(T')^{-1}$.)

Proof. By assumption, $\mathcal{D}(T^{-1}) = \operatorname{rg}(T)$ is dense in Y, hence Y, $(T^{-1})'$ exists. Moreover, T' is injective by Theorem 4.64 (iii), hence $(T')^{-1} : \operatorname{rg}(T') \to Y'$ exists.

Let us prove (4.2). To show that $(T')^{-1} \subseteq (T^{-1})'$, let $\varphi \in \operatorname{rg}(T') = \mathcal{D}((T')^{-1})$. Note that for every $y \in \operatorname{rg}(T) = \mathcal{D}(T^{-1})$, we have that

$$\varphi(T^{-1}y) = (T'(T')^{-1}\varphi)(T^{-1}y) = ((T')^{-1}\varphi)(TT^{-1}y) = ((T')^{-1}\varphi)y$$

hence $\varphi \in \mathcal{D}((T^{-1})')$ and hence $(T^{-1})'\varphi = (T')^{-1}\varphi$.
To show that $(T')^{-1} \supseteq (T^{-1})'$, let $\psi \in \operatorname{rg}(T') = \mathcal{D}((T^{-1})')$. Note that $(T^{-1})'\varphi \in \mathcal{D}(T')$ because for all $x \in \mathcal{D}(T)$ we have that

$$((T^{-1})'\varphi)(Tx) = \varphi((T^{-1})Tx) = \varphi((x)$$

and hence $\varphi = T'((T^{-1})'\varphi \in \operatorname{rg}(T') = \mathcal{D}((T')^{-1}$.

If $\operatorname{rg}(T) = Y$ and T^{-1} is bounded, then T is closed and $T^{-1} \in L(Y, X)$ which implies that also $(T^{-1})' \in L(X', Y')$.

If on the other hand T is closed and $(T')^{-1} \in L(X', Y')$, then also T^{-1} is closed. For all $\varphi \in X'$ and $y \in \mathcal{D}(T^{-1})$ with $\|y\| = 1$, we have that

$$|\varphi(T^{-1}y)| = |T'(T')^{-1}\varphi(T^{-1}y)| = |(T')^{-1}\varphi(y)| \leq \|(T')^{-1}\|\,\|\varphi\|.$$

Hence $\{T^{-1}y : y \in \mathcal{D}(T^{-1}), \|y\| = 1\}$ is bounded by Corollary 3.8, the$\overline{\text{refore}}$ T^{-1} is bounded its domain is closed by the closed graph theorem, hence $\mathcal{D}(T^{-1}) = \operatorname{rg} T = \overline{\operatorname{rg}(T)} = Y$. Since T^{-1} is closed by □

Theorem 4.66 (Closed range theorem). *Let X, Y be reflexive Banach spaces and $T : X \supseteq \mathcal{D}(T) \to Y$ a closed densely defined linear operator. The following is equivalent:*

(i) $\operatorname{rg}(T)$ *is closed.*

(ii) $\operatorname{rg}(T')$ *is closed.*

(iii) $T : X \supseteq \mathcal{D}(T) \to \overline{\operatorname{rg}(T)}$ *is open.*

(iv) $T' : Y' \supseteq \mathcal{D}(T') \to \overline{\operatorname{rg}(T')}$ *is open.*

(v) $\operatorname{rg}(T) = {}^\circ(\ker T')$.

(vi) $\operatorname{rg}(T') = (\ker T)^\circ$.

Proof. (i) $\iff$ (iii) Since T is closed, $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space and

$$\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \to \operatorname{rg} T, \quad \widetilde{T}x = Tx$$

is continuous (Lemma 3.32). Observe that also $i : (\mathcal{D}(T), \|\cdot\|_T) \to X,\ x \mapsto x$ is continuous and that $T = \widetilde{T} \circ i^{-1} : X \supseteq \mathcal{D}(T) \to Y$. Note that $\overline{\operatorname{rg} T}$ is a Banach space.

If $\operatorname{rg} T$ is closed, then $\widetilde{T} : (\mathcal{D}, \|\cdot\|_T) \to \operatorname{rg} T$ is open by the open mapping theorem (Theorem 3.22), then also $T = \widetilde{T} \circ i^{-1} : X \supseteq \mathcal{D}(T) \to \operatorname{rg} T$ is open as composition of open maps. If $T : \mathcal{D}(T) \to \overline{\operatorname{rg} T}$ is open, then it is surjective, hence $\operatorname{rg} T$ is closed.

Note that T' is closed (Theorem 4.57), hence (ii) $\iff$ (iv) is proved analogously.

(i) $\iff$ (v) follows from theorem 4.64 (ii).

(ii) $\Longleftrightarrow$ (vi) follows from theorem 4.64 (ii)

$$\overline{\operatorname{rg}(T')} = {}^{\circ}(\ker T'') = (\ker T)^{\circ}.$$

(iii) $\Longleftrightarrow$ (iv) Recall that T is open if and only if there exists an , $r > 0$ such that the image of the open ball in X with centre 0 and radius r contains the open unit ball in Y. That is, there exists a $r > 0$ such that $T(B_X(0, r)) \supseteq B_Y(0, 1)$. Assume that T is open and let r as above.
To show that T' is open, we have to show that for every $x_0' \in \operatorname{rg}(T')$ with $\|x_0'\| < 1$, there exists a $y_0' \in \mathcal{D}(T')$ with $T'y_0' = x_0'$ and $\|y_0'\| < r$. Define a linear functional φ on $\operatorname{rg}(T)$ as follows: for $y \in \operatorname{rg} T$ with $\|y\| < 1$ choose $x \in \mathcal{D}(T)$ such that $\|x\| < r$ and $Tx = y$. Set $\varphi(y) = x_0'x$ and extend φ linearly to $\operatorname{rg} T$. Note that $|\varphi(y)| = |x_0'x| \le \|x_0'\|\|x\| \le r\|y\|$, φ is bounded, so by the theorem of Hahn-Banach it can be extended to a functional $y_0' \in Y'$ with $\|y_0'\| \le r$. Note that

$$\mathcal{D}(T) \to \mathbb{K}, \qquad \mapsto y_0'(Tx) = \varphi(Tx) = x_0'x$$

is continuous, so $y_0' \in \mathcal{D}(T)$.

(iv) $\Longleftrightarrow$ (iii) Follows analogously if we note that $T'' = T$ by the reflexivity of X and Y. □

Definition 4.67. Let H_1, H_2 be Hilbert spaces and $\mathcal{D}(T) \subseteq H_1$ a dense subspace. For a linear map $T : H_1 \supseteq \mathcal{D}(T) \to H_2$ its *Hilbert space adjoint* T^* is defined by

$$\mathcal{D}(T^*) := \{y \in H_2 \ : \ x \mapsto \langle Tx\,, y\rangle \ \text{ is a bounded on } \ \mathcal{D}(T)\},$$
$$T^* : H_2 \supseteq \mathcal{D}(T^*) \to H_1, \qquad T^*y = y^*,$$

where $y^* \in H_1$ such that $\langle Tx\,, y\rangle = \langle x\,, y^*\rangle$ for all $x \in \mathcal{D}(T)$.
Note that for $y \in \mathcal{D}(T^*)$ the map $x \mapsto \langle Tx\,, y\rangle$ is continuous and densely defined and can therefore be extended uniquely to an element $\varphi_y \in H_1'$. By the Riesz representation theorem (Theorem 4.20) there exists exactly one $y^* \in H_1$ as desired.

Definition 4.68. Let H_1, H_2 be Hilbert spaces and $\mathcal{D}(T) \subseteq H_1$, $\mathcal{D}(S) \subseteq H_2$ subspaces. The linear maps $T : H_1 \supseteq \mathcal{D}(T) \to H_2$ and $S : H_2 \supseteq \mathcal{D}(S) \to H_1$ are called *formally adjoint* if

$$\langle Tx\,, y\rangle_{H_2} = \langle x\,, Sy\rangle_{H_1}, \qquad x \in \mathcal{D}(T),\, y \in \mathcal{D}(S).$$

Note that the formal adjoint of a non-densely defined linear operator is not unique; in particular, the operator trivial operator with $\mathcal{D} = \{0\}$ is formally adjoint to every linear operator.
If T is densely defined, then its adjoint T^* is its maximal formally adjoint operator.

Lemma 4.69. *Let H_1 and H_2 be Hilbert spaces and define*

$$U : H_1 \times H_2 \to H_2 \times H_1, \qquad (x, y) \mapsto (y, -x).$$

If $T(H_1 \to H_2)$ is a densely defined linear operator, then

$$G(T^*) = U(G(T)^{\perp}) = [U(G(T))]^{\perp}. \tag{4.3}$$

Proof. Observe that U is unitary, hence $U(G(T)^\perp) = [U(G(T))]^\perp$. The first equality in (4.3) follows from

$$\begin{aligned}(y_0, x_0) \in G(T^*) \quad &\iff \quad \langle Tx\,, y_0\rangle_Y = \langle x\,, x_0\rangle_X, \quad x \in \mathcal{D}(T)\\ &\iff \quad \langle Tx\,, y_0\rangle - \langle x\,, x_0\rangle = 0, \quad x \in \mathcal{D}(T)\\ &\iff \quad \langle (Tx, -x)\,,\ (y_0, x_0)\rangle_{H_2\times H_1} = 0, \quad x \in \mathcal{D}(T)\\ &\iff \quad \langle U(x, Tx)\,,\ (y_0, x_0)\rangle_{H_2\times H_1} = 0, \quad x \in \mathcal{D}(T)\\ &\iff \quad (y_0, x_0) \in [U(G(T))]^\perp. \end{aligned}$$

□

Theorem 4.70. *Let H_1 and H_2 be Hilbert spaces. For a densely defined linear operator $T(X \to Y)$ the following holds:*

(i) *T^* is closed.*

(ii) *If T is closable, then T^* is densely defined and $T^{**} = \overline{T}$.*

Proof. (i) follows immediately from (4.3).

(ii) Let $y_0 \in \mathcal{D}(T^*)^\perp$. Then $\langle y_0\,, y\rangle = 0$ for all $y \in \mathcal{D}(T)$. This implies

$$0 = \langle (0, y_0)\,,\ (-z, y)\rangle_{H_1\times H_2} = \langle (0, y_0)\,,\ U(y, z)\rangle_{H_1\times H_2}, \qquad (y, z) \in G(T^*).$$

Hence by Lemma 4.69,

$$(0, y_0) \in [U^{-1}(G(T^*))]^\perp = G(T)^{\perp\perp} = \overline{G(T)} = G(\overline{T}).$$

It follows that $y_0 = \overline{T}0 = 0$, so $\overline{\mathcal{D}(T^*)} = Y$. Let

$$V : H_2 \times H_1 \to H_1 \times H_2, \qquad V(y, x) = (x, -y).$$

Obviously $VU = -\operatorname{id}_{H_1\times H_2}$ and application of Lemma 4.69 to T^* yields

$$\begin{aligned} G(T^{**}) &= [V(G(T^*))]^\perp = [VU(G(T)^\perp)]^\perp = [-(G(T)^\perp)]^\perp = G(T)^{\perp\perp} = \overline{G(T)}\\ &= G(\overline{T}). \end{aligned}$$

hence $T^{**} = \overline{T}$. □

Theorem 4.71. *Let H_1, H_2, H_3 be Hilbert spaces.*

(i) *Let $T(H_1 \to H_2)$ and $S(H_1 \to H_2)$ be densely defined linear operators. If $S \subseteq T$ then $T^* \subseteq S^*$.*

(ii) *Assume $S(H_1 \to H_2)$ and $T(H_2 \to H_3)$ are densely defined with $\overline{TS} = H_1$. Then $S^*T^* \subseteq (TS)^*$.*

(iii) *Assume $S(H_1 \to H_2)$ and $T(H_1 \to H_2)$ are densely defined with $\overline{T + S} = H_1$. Then $(T^* + S^*) \subseteq (S + T)^*$.*

If S and T are bounded, then "=" holds in (ii) and (iii).

Proof. As is in the Banach space case. □

Corollary 4.72. *Let H be a Hilbert space, T a densely defined linear operator in H with bounded inverse $T^{-1} \in L(H)$. Then T^* is invertible and*

$$(T^*)^{-1} = (T^{-1})^* =: T^{-*}.$$

Proof. By Theorem 4.71 (ii) it follows that $(T^{-1})^*T^* \subseteq (TT^{-1})^* = \mathrm{id}_{H^*} = \mathrm{id}_H$, hence $(T^{-1})^*T^* = \mathrm{id}_{\mathcal{D}(T^*)}$.
Again by Theorem 4.71 (ii) we find $T^*(T^{-1})^* \subseteq (T^{-1}T)^* = \mathrm{id}^*_{\mathcal{D}(T)} = \mathrm{id}_H$, so it suffices to show $\mathcal{D}(T^*(T^{-1})^*) = \mathcal{D}(T^*)$. Let $y \in \mathcal{D}(T^*)$ and $z = (T^{-1})^*y$. For every $x \in \mathcal{D}(T)$ it follows that $\langle Tx\,, z\rangle = \langle Tx\,, (T^{-1})^*y\rangle = \langle T^{-1}Tx\,, y\rangle = \langle x\,, y\rangle$, so $z \in \mathcal{D}(T^*)$ which implies $\mathcal{D}(T^*(T^{-1})^*) = \mathcal{D}(T^*)$. □

Theorem 4.73. *Let H_1, H_2 be Hilbert spaces, $T(H_1 \to H_2)$ a densely defined closed linear operator. Then the following holds.*

(i) $\mathrm{rg}(T)^\perp = \overline{\mathrm{rg}(T)}^\perp = \ker T^*$.

(ii) $\overline{\mathrm{rg}(T)} = (\ker T^*)^\perp$.

(iii) $\mathrm{rg}(T^*)^\perp = \ker T$.

(iv) $\overline{\mathrm{rg}(T^*)} = (\ker T)^\perp$.

Proof. (i) Note that $y \in \mathrm{rg}(T)^\perp$ if and only if $\langle Tx\,, y\rangle$ for all $x \in \mathcal{D}(T)$. This is equivalent to $y \in \mathcal{D}(T^*)$ and $T^*y = 0$.

(ii) By (i) $\overline{\mathrm{rg}(T)} = \overline{\mathrm{rg}(T)}^{\perp\perp} = (\ker T^*)^\perp$.

(iii) By Theorem 4.70 T^* is closed and densely defined and $T^{**} = T$. Application of (i) to T^* shows $\mathrm{rg}(T^*)^\perp = \ker T$.

(iv) Application of (ii) to T^* shows $\overline{\mathrm{rg}(T^*)} = (\ker T)^\perp$. □

Example 4.74. Let $H = L_2[0,1]$. Let

$$\begin{aligned}
\mathcal{D}(T_1) &:= W_2^1(0,1) = \{x \in\ L_2[0,1] : x \text{ absolutely continuous},\ x' \in L_2[0,1]\},\\
\mathcal{D}(T_2) &:= \mathcal{D}(T_1) \cap \{x \in L_2[0,1] : x(0) = x(1)\}\\
\mathcal{D}(T_3) &:= \mathcal{D}(T_1) \cap \{x \in L_2[0,1] : x(0) = x(1) = 0\}.
\end{aligned}$$

For $k = 1, 2, 3$ let

$$T_k : H \supseteq \mathcal{D}(T_k) \to H, \qquad T_k x = \mathrm{i}x'.$$

Obviously, the T_k are well-defined and $\mathcal{D}(T_k)$ is dense in H (Theorem A.27). We will show: $T_1^* = T_3$, $T_3^* = T_1$, $T_2^* = T_2$, in particular all T_k are closed.

Proof. Let $x, y \in \mathcal{D}(T_1)$. Then, using integration by parts,

$$\begin{aligned}\langle T_1 x\,, y\rangle &= \int_0^1 \mathrm{i}x'(t)\overline{y(t)}\;\mathrm{d}t = \mathrm{i}x(t)\overline{y(t)}\Big|_0^1 - \int_0^1 \mathrm{i}x(t)\overline{y'(t)}\;\mathrm{d}t \\ &= \mathrm{i}x(1)\overline{y(1)} - \mathrm{i}x(0)\overline{y(0)} + \langle x\,, T_1 y\rangle.\end{aligned}$$

In particular we obtain

$$\begin{aligned}\langle Tx\,, y\rangle &= \langle x\,, Ty\rangle, \qquad x \in \mathcal{D}(T_1),\ y \in \mathcal{D}(T_3), \\ \langle Tx\,, y\rangle &= \langle x\,, Ty\rangle, \qquad x, y \in \mathcal{D}(T_2).\end{aligned}$$

This shows that

$$\mathcal{D}(T_3) \subseteq \mathcal{D}(T_1^*), \quad \mathcal{D}(T_2) \subseteq \mathcal{D}(T_2^*) \quad \text{and} \quad \mathcal{D}(T_1) \subseteq \mathcal{D}(T_3^*)$$

and $T_1^*|_{\mathcal{D}(T_3)} = T_3$, $T_3^*|_{\mathcal{D}(T_3)} = T_1$ and $T_2^*|_{\mathcal{D}(T_3)} = T_2$.
To prove the inclusion $\mathcal{D}(T_1^*) \subseteq \mathcal{D}(T_3)$ let $g \in \mathcal{D}(T_1^*)$ and $\varphi = T_1^* g$. Define $\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s$. Then Φ is absolutely continuous and $\Phi' = \varphi$. For $x \in \mathcal{D}(T_1)$

$$\begin{aligned}\int_0^1 \mathrm{i}x'(t)\overline{g(t)}\;\mathrm{d}t = \langle T_1 x\,, g\rangle = \langle x\,, \varphi\rangle &= \int_0^1 \mathrm{i}x(t)\overline{\varphi(t)}\;\mathrm{d}t \\ &= x(t)\overline{\Phi(t)}\Big|_0^1 - \int_0^1 \mathrm{i}x'(t)\overline{\Phi(t)}\;\mathrm{d}t \\ &= x(1)\overline{\Phi(1)} - \int_0^1 \mathrm{i}x'(t)\overline{\Phi(t)}\;\mathrm{d}t.\end{aligned}$$

Note that $\Phi(1) = 0$ as can be seen if x is chosen to be a constant function. Hence

$$\int_0^1 \mathrm{i}x'(t)\overline{(g(t)\mathrm{i}\Phi(x))}\;\mathrm{d}t = 0, \qquad x \in \mathcal{D}(T_1),$$

implying that $g + \mathrm{i}\Phi \in \operatorname{rg}(T_1)^\perp = \{0\}$. It follows that g is absolutely continuous and $g(0) = \mathrm{i}\varphi(0) = 0$, $g(1) = \mathrm{i}\varphi(1) = 0$, so $g \in \mathcal{D}(T_3)$.
Analogously, $T_2^* = T_2$ and $T_3^* = T_1$ can be shown. □

Definition 4.75. Let H be a Hilbert spaces, $\mathcal{D}(T) \subseteq H$ a dense subspace and $T : H \supseteq \mathcal{D}(T) \to H$ a linear map.

(i) T is called *symmetric* if $T \subseteq T^*$.
(ii) T is called *selfadjoint* if $T = T^*$.
(iii) T is called *essentially selfadjoint* if $\overline{T} = T^*$.

The operator T_2 in the example above is selfadjoint, the operator T_3 is symmetric.

Chapter 5

Spectrum of linear operators

If not stated explicitely otherwise, all Hilbert and Banach spaces in this chapter are assumed to be complex vector spaces.

5.1 The spectrum of a linear operator

Definition 5.1. Let X be a Banach space and $T(X \to X)$ a densely defined linear operator.

$$\begin{aligned} \rho(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is bijective}\} && \textit{resolvent set} \text{ of } T, \\ \sigma(T) &:= \mathbb{C} \setminus \rho(T) && \textit{spectrum} \text{ of } T. \end{aligned}$$

The spectrum of T is further divided in *point spectrum* $\sigma_{\mathrm{p}}(T)$, *continuous spectrum* $\sigma_{\mathrm{c}}(T)$ and *residual spectrum* $\sigma_{\mathrm{r}}(T)$:

$$\begin{aligned} \sigma_{\mathrm{p}}(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is not injective}\}, \\ \sigma_{\mathrm{c}}(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is injective}, \ \operatorname{rg}(T - \lambda \operatorname{id}) \neq X, \ \overline{\operatorname{rg}(T - \lambda \operatorname{id})} = X\}, \\ \sigma_{\mathrm{r}}(T) &:= \{\lambda \in \mathbb{C} : \lambda \operatorname{id} - T \text{ is injective}, \ \overline{\operatorname{rg}(T - \lambda \operatorname{id})} \neq X\}. \end{aligned}$$

It follows immediately from the definition that

$$\sigma(T) = \sigma_{\mathrm{p}}(T) \,\dot\cup\, \sigma_{\mathrm{c}}(T) \,\dot\cup\, \sigma_{\mathrm{r}}(T).$$

In the following, we often write $\lambda - T$ instead of $\lambda \operatorname{id} - T$.

Definition 5.2. (i) Elements $\lambda \in \sigma_{\mathrm{p}}(T)$ are called *eigenvalues* of T.

(ii) For $\lambda \in \sigma_{\mathrm{p}}(T)$ we define the *geometric eigenspace* of T in λ, $N_\lambda(T)$, and the *algebraic eigenspace* of T in λ, $A_\lambda(T)$, by

$$\begin{aligned} N_\lambda(T) &:= \ker(T - \lambda), \\ A_\lambda(T) &:= \{x \in X : (T - \lambda)^n x = 0 \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$

(iii) For $\lambda \in \rho(T)$ the *resolvent* of T in λ is $(\lambda \operatorname{id} - T)^{-1} := R(\lambda, T)$. The map

$$\rho(T) \to L(X), \quad \lambda \mapsto R(\lambda, T)$$

is the *resolvent map*.

Remark 5.3. If T is closed, then $(T-\lambda)^{-1}$ is closed if it exists. Therefore, by the closed graph theorem,

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is injective and } (T-\lambda)^{-1} \in L(X)\}.$$

Remark 5.4. Often the resolvent set of a linear operator is defined slightly differently: Let $T(X \to X)$ is a densely defined linear operator. Then $\lambda \in \rho(T)$ if and only if $\lambda - T$ is bijective and $(\lambda - T)^{-1} \in L(X)$. With this definition it follows that $\rho(T) = \emptyset$ for every non-closed $T(X \to X)$ because one of the following cases holds:

(i) $\lambda - T$ is not bijective $\implies \lambda \notin \rho(T)$;
(ii) $\lambda - T$ is bijective, then $(\lambda - T)^{-1}$ is defined everywhere and not closed, so it cannot be bounded, which implies $\lambda \notin \rho(T)$.

Remark 5.5. If $\dim X < \infty$, then $\sigma_{\mathrm{c}}(T) = \sigma_{\mathrm{r}}(T) = \emptyset$ and $\sigma_{\mathrm{p}}(T)$ is the set of all eigenvalues of T.

Theorem 5.6 (Spectral mapping theorem for polynomials). *Let X be a Banach space, $T \in L(X)$ and $P \in \mathbb{C}[X]$ a polynomial. Then*

$$\sigma(P(T)) = P(\sigma(T)).$$

Proof. Let $\lambda \in \mathbb{C}$. Then there exists a polynomial Q such that $P(X) - P(\lambda) = (X - \lambda)Q(X)$. In particular, $P(T) - P(\lambda) = (T - \lambda)Q(T) = Q(T)(T - \lambda)$. Hence, if $\lambda \in \sigma(T)$, then $(T - \lambda)$ is not bijective, so $P(T) - P(\lambda)$ is not bijective which implies $P(\sigma(T)) \subseteq \sigma(P(T))$.
Now assume $\mu \in \sigma(P(T))$. There exist $a, \lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that $P(X) - \mu = a(X - \lambda_1)\cdots(X - \lambda_n)$. Since $P(T) - \mu$ is not invertible, at least one of the terms $\lambda_j - T$ cannot be invertible, that is at least one λ_j must belong to the spectrum of T and $\mu = P(\lambda_j) \in P(\sigma(T))$. □

5.2 The resolvent

In this section we will study the resolvent map $\rho(T) \to L(X)$, $\lambda \mapsto R(\lambda, T) = (\lambda - T)^{-1}$. We will show that its domain is open and that it is analytic.

Lemma 5.7. *Let X be a Banach space and $T(X \to X)$ a closed linear operator.*

(i) $\|R(\lambda_0, T)\| \geq \dfrac{1}{\operatorname{dist}(\lambda_0, \sigma(T))}$ *for all $\lambda_0 \in \rho(T)$.*
(ii) *For $\lambda_0 \in \rho(T)$ and $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \|R(\lambda_0, T)\|^{-1}$*

$$R(\lambda, T) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (R(\lambda_0, T))^{n+1}.$$

Note that (ii) shows that locally around a $\lambda_0 \in \rho(T)$ the resolvent has a power series expansion with coefficients depending only on λ_0 and T.

Proof of Lemma 5.7. Recall that for a bounded linear operator $S \in L(X)$ with $\|S\| < 1$ the operator $(\mathrm{id} - S)^{-1} \in L(X)$ and it is given explicitly by the Neumann series (Theorem 2.10)

$$(\mathrm{id} - S)^{-1} = \sum_{n=0}^{\infty} S^n.$$

Let $\lambda_0 \in \rho(T)$. For $\lambda \in \mathbb{C}$ we find

$$\lambda - T = \lambda_0 - T - (\lambda_0 - \lambda) = \big[\mathrm{id} - (\lambda_0 - \lambda)(\lambda_0 - T)^{-1}\big](\lambda_0 - T).$$

If $|\lambda_0 - \lambda| < \|(\lambda_0 - T)^{-1}\|^{-1}$, then the term in brackets is invertible, hence so is $\lambda - T$ and we obtain

$$\begin{aligned}(\lambda - T)^{-1} &= (\lambda_0 - T)^{-1}\big[\mathrm{id} - (\lambda_0 - \lambda)(\lambda_0 - T)^{-1}\big]^{-1} \\ &= (\lambda_0 - T)^{-1}\Big(\sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 - T)^{-n}\Big) \\ &= \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n(\lambda_0 - T)^{-(n+1)}\end{aligned}$$

which proves (ii). If $\mu \in \mathbb{C}$ with $|\mu| < \|(T - \lambda_0)^{-1}\|^{-1}$, then $\lambda_0 + \mu \in \rho(T)$, hence $\mathrm{dist}(\lambda_0, \sigma(T)) \geq \|(T - \lambda_0)^{-1}\|^{-1}$, so also (i) is proved. □

As a corollary we obtain the following theorem.

Theorem 5.8. *Let X be a Banach space and $T(X \to X)$ a closed linear operator.*

(i) *$\sigma(T)$ is closed.*

(ii) *If $T \in L(X)$, then $\sigma(T)$ is compact.*

Proof. (i) $\mathbb{C} \setminus \sigma(T) = \rho(T)$ is open by Lemma 5.7.
(ii) Let $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$. Then $\lambda - T = \lambda(\mathrm{id} - \lambda^{-1}T)$ is invertible since $\|\lambda^{-1}T\| < 1$ (Neumann series, Theorem 2.10), hence $\lambda \in \rho(T)$ It follows that $\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \supseteq \rho(T)$. Since σ is closed and bounded, it is compact. □

Next we prove the so-called resolvent identities.

Theorem 5.9. *Let X be a Banach space and $T(X \to X)$, $S(X \to X)$ a linear operators with $\mathcal{D}(S) = \mathcal{D}(T)$.*

(i) *1st resolvent identity:*

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T), \qquad \lambda, \mu \in \rho(T).$$

In particular, the resolvents commute.

(ii) *2nd resolvent identity:*

$$R(\lambda, T) - R(\lambda, S) = R(\lambda, T)(T - S)R(\lambda, S), \qquad \lambda \in \rho(T) \cap \rho(S).$$

Proof. (i) follows from a straightforward calculation:

$$\begin{aligned} R(\lambda, T) - R(\mu, T) &= (\lambda - T)^{-1} - (\mu - T)^{-1} \\ &= (\lambda - T)^{-1}\big[\mu - T - (\lambda - T)\big](\mu - T)^{-1} \\ &= (\mu - \lambda)R(\lambda, T)R(\mu, T). \end{aligned}$$

(ii) is shown similarly:

$$\begin{aligned} R(\lambda, T) - R(\lambda, S) &= (\lambda - T)^{-1} - (\lambda - S)^{-1} \\ &= (\lambda - T)^{-1}\big[\lambda - S - (\lambda - T)\big](\lambda - S)^{-1} \\ &= R(\lambda, T)(T - S)R(\lambda, S), \end{aligned}$$ □

Next we study properties of the resolvent map $\rho(T) \to L(X)$, $\lambda \mapsto R(\lambda, T)$. By Lemma 5.7 we already now that its domain is open and that it is analytic, that is, locally it has a power series representation.

Definition 5.10. Let $\Omega \in \mathbb{C}$ be an open set, X a Banach space and $f : \Omega \to X$.

(i) f is called *holomorphic in* $z_0 \in \Omega$ if and only if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in the norm topology. f is called *holomorphic* if and only if it is holomorphic in every $z_0 \in \Omega$.

(ii) f is called *weakly holomorphic in* $z_0 \in \Omega$ if and only if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in the weak topology. f is called *weakly holomorphic* if and only if it is weakly holomorphic in every $z_0 \in \Omega$. Hence, for every $\varphi \in X'$ the map $\Omega \to \mathbb{C}$, $z \mapsto \varphi(f(z))$ is holomorphic in the usual sense.

Lemma 5.11. *Let X be a Banach space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence if and only if the sequence $(\varphi(x_n))_{n\in\mathbb{N}} \subseteq X$ is uniformly Cauchy for $\varphi \in X'$ with $\|\varphi\| \le 1$ (that is, for every $\varepsilon > 0$ exists a $N \in \mathbb{N}$ such that $|\varphi(x_n) - \varphi(x_m)| < \varepsilon$ for all $m, n \ge N$ and all $\varphi \in X'$ with $\|\varphi\| \le 1$).*

Proof. Assume that $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence and let $\varepsilon > 0$. Then there exists a $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for $m, n \ge N$. It follows that $\|\varphi(x_n) - \varphi(x_m)\| \le \|\varphi\|\|x_n - x_m\| < \varepsilon$ for all $m, n \ge N$ and all $\varphi \in X'$ with $\|\varphi\| \le 1$.

Now let $\varepsilon > 0$ and assume that there exists an $N \in \mathbb{N}$ such that $|\varphi(x_n) - \varphi(x_m)| < \varepsilon$ for all $m, n \geq N$ and all $\varphi \in X'$ with $\|\varphi\| \leq 1$. Recall that the map $J_X : X \to X''$ is an isometry. It follows for $m, n \geq N$

$$\begin{aligned}\|x_n - x_m\| &= \|J_X x_n - J_X x_m\| = \sup\{|(J_X x_n - J_X x_m)\varphi| \,:\, \varphi \in X',\ \|\varphi\| \leq 1\} \\ &= \sup\{|\varphi(x_n) - \varphi(x_m)| \,:\, \varphi \in X',\ \|\varphi\| \leq 1\} < \varepsilon.\end{aligned}$$ □

Recall the following fundamental theorem of complex analysis.

Theorem 5.12 (Cauchy's integral formula). *Let $\Omega \in \mathbb{C}$ open and let $f : \Omega \to \mathbb{C}$ holomorphic. Let $z_0 \in \Omega$ and $r > 0$ such that $K_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\} \subseteq \Omega$. Then*

$$f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_r(z_0)} \frac{f(z)}{z - a}\, \mathrm{d}z, \qquad a \in B_r(z_0) \tag{5.1}$$

where $\Gamma_r(z_0)$ is the positively oriented boundary of $K_r(z_0)$. More generally, for $n \in \mathbb{N}_0$,

$$f^{(n)}(a) = \frac{n!}{2\pi \mathrm{i}} \int_{\Gamma_r(z_0)} \frac{f(z)}{(z - a)^{n+1}}\, \mathrm{d}z, \qquad a \in B_r(z_0). \tag{5.2}$$

Theorem 5.13 (Dunford). *Let X be a Banach space and let $\Omega \in \mathbb{C}$ open. A map $f : \Omega \to X$ is holomorphic if and only if it is weakly holomorphic.*

Proof. Clearly, holomorphy of f implies weak holomorphy. Now assume that f is weakly holomorphic. Let $z_0 \in \Omega$. Choose $r > 0$ such that $K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\} \in \Omega$. and let $\Gamma_r(z_0)$ be the positively oriented boundary of $K_r(z_0)$. For every $\varphi \in X'$ Cauchy's integral formula (5.1) yields

$$\varphi(f(a)) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_r(z_0)} \frac{\varphi(f(z))}{z - a}\, \mathrm{d}z, \qquad a \in B_r(z_0).$$

For $a \in B_r(z_0)$ and $0 < |h| < r - |z_0 - a|$ it follows that $a + h \in K_r(z_0)$, hence with Cauchy's integral formula we obtain

$$\begin{aligned}&\frac{1}{h}\Big(\varphi(f(a+h)) - \varphi(f(a))\Big) - (\varphi \circ f)'(a) \\ &\qquad = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_r(z_0)} \frac{1}{h}\Big[\frac{1}{z - a - h} - \frac{1}{z - a} - \frac{h}{(z - a)^2}\Big]\varphi(f(z))\, \mathrm{d}z \\ &\qquad = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_r(z_0)} \Big[\frac{1}{(z - a)(z - a - h)} - \frac{1}{(z - a)^2}\Big]\varphi(f(z))\, \mathrm{d}z \\ &\qquad = \frac{h}{2\pi \mathrm{i}} \int_{\Gamma_r(z_0)} \frac{\varphi(f(z))}{(z - a)^2 (z - a - h)}\, \mathrm{d}z.\end{aligned}$$

Since $z \mapsto \varphi(f(z))$ is holomorphic in a neighbourhood of $\Gamma_r(z_0)$, it is in particular continuous. Hence there exists C_φ such that

$$|\varphi(f(z))| < C_\varphi, \qquad z \in \Gamma_r(z_0).$$

By a corollary to the theorem of Banach-Steinhaus (Corollary 3.8), there exists $C > 0$ such that

$$\|f(z)\| < C, \qquad z \in \Gamma_r(z_0).$$

Hence we obtain

$$\Big| \frac{1}{h}\Big(\varphi(f(a+h)) - \varphi(f(a))\Big) - \frac{\mathrm{d}}{\mathrm{d}z}(\varphi \circ f)(a) \Big| \leq h\|\varphi\|C'.$$

This implies that

$$\lim_{h\to 0} \varphi\Big(\frac{1}{h}\big(f(a+h) - f(a)\big)\Big) = \lim_{h\to 0} \frac{1}{h}\Big(\varphi(f(a+h)) - \varphi(f(a))\Big) = (\varphi \circ f)'(a),$$

uniformly for $\varphi \in X'$, $\|\varphi\| \leq 1$. Therefore, by Lemma 5.11, $\lim\limits_{h\to 0} \frac{1}{h}\big(f(a+h) - f(a)\big)$ exists. ☐

Theorem 5.14 (Dunford). *Let X be a Banach space, $\Omega \subseteq \mathbb{C}$ open and $T : \Omega \to L(X)$. Then the following is equivalent:*

(i) *T is holomorphic in the operator norm.*

(ii) *T is strongly holomorphic.*

(iii) *T is weakly holomorphic.*

Proof. (i) $\Longrightarrow$ (ii) follows from the definition. (ii) $\Longleftrightarrow$ (iii) follows form Theorem 5.13. It remains to prove (iii) $\Longrightarrow$ (i). As in the proof of Theorem 5.13 we obtain for $x \in X$ and $\varphi \in X'$

$$\frac{1}{h}\Big(\varphi(T(a+h)x - T(a)x)\Big) - \frac{\mathrm{d}}{\mathrm{d}z}|_{z=a}(\varphi T(z)x) = \frac{h}{2\pi\mathrm{i}} \int_{\Gamma_r(z_0)} \frac{\varphi(T(z)x)}{(z-a)^2(z-a-h)}\,\mathrm{d}z.$$

Since $z \mapsto \varphi(T(z)x)$ is holomorphic in a neighbourhood of $\Gamma_r(z_0)$, it is continuous, so there exists $C_{x,\varphi}$ such that

$$|\varphi(T(z)x)| < C_{x,\varphi}, \qquad z \in \Gamma_r(z_0).$$

By a corollary to the theorem of Banach-Steinhaus (Corollary 3.8), there exists $C_x > 0$ such that

$$\|T(z)x\| < C_x, \qquad z \in \Gamma_r(z_0),$$

and by the theorem of Banach-Steinhaus (Theorem 3.7), there exists $C > 0$ such that

$$\|T(z)\| < C, \qquad z \in \Gamma_r(z_0).$$

This implies that

$$\lim_{h\to 0} \frac{1}{h}\Big(\varphi(T(a+h)x - T(a)x)\Big) = \varphi\Big(\lim_{h\to 0}\frac{1}{h}(T(a+h)x - T(a)x)\Big)$$

exists, uniformly for $\varphi \in X'$, $\|\varphi\| \leq 1$. Therefore, by Lemma 5.11,

$$\lim_{h\to 0} \frac{1}{h}(T(a+h)x - T(a)x)$$

exists and convergence is uniform for $x \in X$ with $\|x\| = 1$. Analogously as in the proof of Lemma 5.11 it follows the existence of

$$\lim_{h\to 0} \frac{1}{h}(T(a+h) - T(a)).$$ ☐

Theorem 5.15. *Let X be a Banach space, $T(X \to X)$ a densely defined closed linear operator. Then the resolvent map*

$$\rho(T) \to L(X), \quad \lambda \mapsto R(\lambda, T) = (\lambda - T)^{-1}$$

is holomorphic.

Proof. Let $\lambda_0 \in \rho(T)$ and $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \|R(\lambda_0, T)\|$. For fixed $x \in X$ and $\varphi \in X'$ we have by Lemma 5.7

$$\begin{aligned}\varphi(R(\lambda, T)x) &= \varphi\Big(\Big(\sum_{n=0}^{\infty}(\lambda - \lambda_0)^n (R(\lambda_0, T))^{n+1}\Big)x\Big) \\ &= \sum_{n=0}^{\infty}(\lambda - \lambda_0)^n \varphi\big((R(\lambda_0, T))^{n+1}x\big)\end{aligned}$$

where we used that the operator series converges and φ is continuous. Since the last sum is absolutely convergent, it follows that $\lambda \to \varphi(R(\lambda, T)x)$ is analytic locally at λ_0, hence holomorphic. Since weak holomorphy is equivalent to holomorphy in the operator norm (Theorem 5.14), the theorem is proved. □

The preceding theorem allows us to apply theorems of complex analysis to the resolvent map.

Theorem 5.16. *Let X be a Banach space and $T \in L(X)$. Then $\sigma(T) \neq \emptyset$.*

Proof. Assume $\sigma(T) = \emptyset$. Observe that this implies $X \neq \{0\}$ and $T^{-1} \in L(X)$. Let $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$. Then $\lambda \in \rho(T)$ and using the Neumann series

$$\|R(\lambda, T)\| = \Big\| \sum_{n=0}^{\infty} \lambda^n T^{-(n+1)} \Big\| \leq \sum_{n=0}^{\infty} |\lambda^n| \|T\|^{-(n+1)} = \frac{1}{\|T\| - |\lambda|}.$$

In particular, $\|R(\lambda, T)\| \to 0$ for $|\lambda| \to \infty$. Hence for every $x \in X$ and $\varphi \in X'$ the map $\lambda \to \varphi(R(\lambda, T)x)$ is holomorphic and bounded in $\mathbb{C}$, so constant by the Liouville theorem. Since $\varphi(R(\lambda, T)x) \to 0$ for $|\lambda| \to \infty$, it follows that $\varphi(R(\lambda, T)x) = 0$ for all $\lambda \in \mathbb{C}$, $x \in X$ and $\varphi \in X'$. By a corollary to the Hahn-Banach theorem (Corollary 2.17) it follows that $R(\lambda, T)x = 0$ for all $x \in X$ and $\lambda \in \mathbb{C}$, hence $R(\lambda, T) = 0$, $\lambda \in \mathbb{C}$. This contradicts the fact that $1 = \|TT^{-1}\| \leq \|T\|\|T^{-1}\| = 0$. □

The following example shows that for unbounded linear operators the cases $\sigma(T) = \emptyset$ and $\sigma(T) = \mathbb{C}$ are possible.

Examples 5.17. (i) Let $X = C([0, 1])$ and

$$T : X \supseteq C^1([0, 1]) \to X, \quad Tx = x'.$$

Then T is unbounded and closed and $\sigma(T) = \sigma_{\mathrm{p}}(T) = \mathbb{C}$.

(ii) Let $X = \{x \in C([0,1]) : x(0) = 0\}$, $\mathcal{D}(T) = \{x \in X \cap C^1([0,1]) : x' \in X\}$ and

$$T : X \supseteq \mathcal{D}(T) \to X, \quad Tx = x'.$$

Then T is unbounded and closed and $\sigma(T) = \emptyset$.

Proof. (i) Obviously, T is unbounded and densely defined. If $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$ such that $x_n \to x$ and $Tx_n \to y \in X$, then, by a theorem of Analysis 1, x is differentiable, hence in $\mathcal{D}(T)$ and $Tx = x' = y$ which implies that T is closed.
For every $\lambda \in \mathbb{C}$ the differential equation $x' - \lambda x = 0$ has the solution $x_\lambda(t) = \mathrm{e}^{\lambda t}$. Note that $x_\lambda \in \mathcal{D}(T)$ and $(T - \lambda)x_\lambda = 0$, so $\lambda \in \sigma_{\mathrm{p}}(T)$.

(ii) Obviously, T is unbounded and densely defined. If $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$ such that $x_n \to x$ and $Tx_n \to y \in X$, then, by a theorem of Analysis 1, x is differentiable and $x' = y$. Moreover, $x(0) = \lim_{n\to\infty} x_n(0) = 0$, so in $\mathcal{D}(T)$ and $Tx = x' = y$ which implies that T is closed.
For every $\lambda \in \mathbb{C}$ and every $y \in X$ the initial value problem $x' - \lambda x = y$, $x(0)$ has exactly one solution x_λ given by

$$x_\lambda(t) = \mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{-\lambda s}\, y(s)\ \mathrm{d}s.$$

Obviously $x_\lambda \in C^1[0,1]$, $x_\lambda(0) = 0$ and $x'_\lambda(0) = \lambda x_\lambda(0) + y(0) = 0$. Hence $T - \lambda$ is bijective, in particular $\lambda \in \rho(T)$. □

Note that in the last example the continuity of $(T - \lambda)$ can be seen immediately:

$$\begin{aligned}\|(T-\lambda)^{-1}y\|_\infty = \|x_\lambda\|_\infty &= \sup\left\{\left|\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda s}\, y(s)\ \mathrm{d}s\right| : t \in [0,1]\right\}\\ &\le \|y\|_\infty \max\{1,\ \mathrm{e}^{\lambda}\} \int_0^1 \mathrm{e}^{-\lambda s}\ \mathrm{d}s.\end{aligned}$$

Definition 5.18. Let X be a Banach space The *spectral radius* of $T \in L(X)$ is

$$r(T) := \limsup \|T^n\|^{\frac{1}{n}}.$$

Theorem 5.19. *Let X be a Banach space, $T \in L(X)$ and $r(T)$ its spectral radius.*

(i) $r(T) \le \|T^m\|^{1/m} \le \|T\|$ *for all $m \in \mathbb{N}$, in particular* $r(T) = \lim_{m\to\infty} \|T^m\|^{1/m}$.

(ii) $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le r(T)\}$.

(iii) *If X is a complex Banach space, then there exists a $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$, in particular*

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

(iv) *If X is Hilbert space and T is normal, then $r(T) = \|T\|$.*

(v) *If X is a complex Hilbert space and T is normal with $r(T) = 0$, then $T = 0$.*

Proof. (i) Let $m \in \mathbb{N}$ arbitrary. For every $n \in \mathbb{N}$ there exist $p_n, q_n \in \mathbb{N}_0$ with $q_n < m$ and $n = p_n m + q_n$. Let $M := \max\{1, \|T\|, \dots, \|T^{m-1}\|\}$. Then

$$\|T^n\| = \|T^{p_n m + q_n}\| \leq \|T^{p_n m}\| \, \|T^{q_n}\| \leq M \|T^m\|^{p_n}.$$

This implies $r(T) = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} M^{\frac{1}{2}} \|T^m\|^{\frac{1}{m} - \frac{q_m}{nm}} = \|T^m\|^{\frac{1}{m}}$.

(ii) By the formula of Hadamard, the radius of convergence of $\sum_{n=0}^{\infty} z^{n+1}\|T^n\|$ is $(\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}})^{-1} = r(T)^{-1}$. Hence for all $\lambda \in \mathbb{C}$, $|\lambda| > r(T)$, the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n =: A$ converges in norm. By Theorem 2.10 (Neumann series), A is the inverse of $\lambda - T$. Because T is closed, it follows that $\{\lambda \in \mathbb{C} : |\lambda| > r(T)\} \subseteq \rho(T)$, or equivalently $\{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\} \subseteq \sigma(T)$.

(iii) Let $r_0 := \max\{|\lambda| : \lambda \in \sigma(T)\}$. It follows from (ii) that $r_0 \leq r(T)$. Now choose any $\mu \in \mathbb{C}$ with $|\mu| > r_0$. We have to show that $|\mu| > r(T)$. Observe that by definition of $R(T)$ and by the formula of Hadamard

$$(\lambda - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n, \qquad |\lambda| > r(T), \tag{5.3}$$

where the series on the right hand side converges in norm. In particular, for every $\varphi \in L(X)'$

$$\varphi(\lambda - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} \varphi(T^n), \qquad |\lambda| > r(T).$$

Hence $\lambda \mapsto \varphi(T - \lambda)^{-1}$ defines an analytic function for $|\lambda| > r(T)$. It follows from complex analysis that then the equality in (5.3) holds for all λ in the largest open ring where $\lambda \mapsto \varphi(\lambda - T)$ is analytic, that is for all $\lambda > r(T)$. In particular, $\sum_{n=0}^{\infty} \mu^{-(n+1)} \varphi(T^n)$ converges for every $\varphi \in L(X)'$, hence it is weakly convergent, and therefore $(\mu^{-(n+1)} \varphi(T^n))_{n\in\mathbb{N}}$ converges to 0. It follows that $(\mu^{-(n+1)} T^n)_{n\in\mathbb{N}}$ is weakly convergent to 0, hence it is bounded (Corollary 3.9). Let $M \in \mathbb{R}$ such that $\|\mu^{-(n+1)}\|T^n\| < M$, $n \in \mathbb{N}$. Then $\|\|T^n\|^{\frac{1}{n}} < M^{\frac{1}{n}} \mu^{1+\frac{1}{n}}$ for all $n \in \mathbb{N}$, in particular $r(T) = \lim_{n\to\infty} \|\|T^n\|^{\frac{1}{n}} \leq \mu$.

(iv) Recall that $\|TT^*\| = \|T\|^2$ for a normal operator T (Theorem 4.39). Hence

$$\|T^2\|^2 = \|T^2(T^*)^2\| = \|(TT^*)^2\| = \|(TT^*)\|^2 = \|T\|^4,$$

hence $\|T^2\| = \|T\|^2$. By induction, it can be shown that hence $\|T^{2^n}\| = \|T\|^{2^n}$ for all $n \in \mathbb{N}$, implying that

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|T^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|T\| = \|T\|.$$

(v) follows directly from (iv). □

Note that in general $r(T) < \|T\|$, for example $r(T) = 0$ for every nilpotent linear operator.

5.3 The spectrum of the adjoint operator

Theorem 5.20. (i) *Let X be a Banach space and $T(X \to X)$ a densely defined closed linear operator. Then $\sigma(T') = \sigma(T)$ and $R(\lambda, T)' = R(\lambda, T')$ for $\lambda \in \rho(T)$.*

(ii) *Let H be a Hilbert space and $T(H \to H)$ a densely defined closed linear operator. Then $\sigma(T^*) = \overline{\sigma(T)} = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma(T)\}$ and $R(\lambda, T)^* = R(\lambda^*, T^*)$ for $\lambda \in \rho(T)$.*

Proof. If $\lambda \in \rho(T)$, then $\lambda \in \rho(T')$ and $((T-\lambda)^{-1})' = ((T-\lambda)')^{-1}$ by Theorem 4.65. (This follows also from Corollary .)
If on the other hand $\lambda \in \rho(T')$, then

$$\begin{aligned} \overline{\operatorname{rg}(T-\lambda)} &= {}^\circ(\ker(T'-\lambda)) = {}^\circ\{0\} = Y, \\ \ker(T-\lambda) &= {}^\circ(\operatorname{rg}(T'-\lambda)) \cap \mathcal{D}(T) = {}^\circ(Y') \cap \mathcal{D}(T) = \{0\}. \end{aligned}$$

Hence $T - \lambda$ is injective and has dense range. Therefore Theorem 4.65 shows that $\operatorname{rg}(T-\lambda) = Y$, hence $\lambda \in \rho(T)$.

□

Lemma 5.21. *Let X be a Banach space and $T(X \to X)$ densely defined and closed.*

(i) $\lambda \in \sigma_p(T) \implies \lambda \in \sigma_p(T') \cup \sigma_r(T')$.

(ii) $\lambda \in \sigma_r(T) \implies \lambda \in \sigma_p(T')$.

Proof. (i) If $\lambda \in \sigma_{\mathrm{p}}(T)$, then $\ker(\lambda - T) \supsetneq \{0\}$, $\overline{\operatorname{rg}(\lambda - T')} \subseteq \ker(T)^\circ \neq X$. It follows that $\lambda \in \sigma_{\mathrm{p}}(T')$ or $\lambda \in \sigma_{\mathrm{r}}(T')$.
(i) If $\lambda \in \sigma_{\mathrm{r}}(T)$, then $\overline{\operatorname{rg}(\lambda - T)} \neq X$. By Theorem 4.64 $\overline{\operatorname{rg}(\lambda - T)} = X$ if and only if $(\lambda - T)' = \lambda - T'$ is not injective, hence $\lambda \in \sigma_{\mathrm{p}}(T')$. □

Theorem 5.22. *Let H be a complex Hilbert space, $T(H \to H)$ a symmetric operator and $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

(i) $\|(\lambda - T)x\| \geq |\operatorname{Im}(\lambda)|\,\|x\|$ *for all $x \in \mathcal{D}(T)$.*
In particular $T - \lambda : \mathcal{D}(T) \to \operatorname{rg}(T-\lambda)$ is invertible with continuous inverse and the point spectrum of T is real.

(ii) *If T is closed, then $\operatorname{rg}(\lambda - T)$ is closed.*

Proof. (i) For all $x \in \mathcal{D}(T)$

$$\begin{aligned} \|(\lambda - T)x\|\,\|x\| &\geq \big|\langle(\lambda - T)x\,, x\rangle\big| = \big|\langle(\operatorname{Re}\lambda - T)x\,, x\rangle + \mathrm{i}\langle \operatorname{Im}\lambda x\,, x\rangle\big| \\ &\geq |\operatorname{Im}\lambda|\|x\|. \end{aligned}$$

In particular, $\lambda - T$ is injective, which implies that $\lambda \notin \sigma_{\mathrm{p}}(T)$.
(i) If $(\lambda - T)$ is continuous and closed, to its domain $\operatorname{rg}(\lambda - T)$ is closed. □

Theorem 5.23. *Let H be a complex Hilbert space and $T(H \to H)$ a symmetric operator. Then the following is equivalent.*

(i) *T is selfadjoint.*

(ii) $\operatorname{rg}(\lambda - T) = H$ *for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

(iii) $\operatorname{rg}(\pm \mathrm{i} - T) = H$.

(iv) *There exist* $z_\pm \in \mathbb{C}$ *with* $\operatorname{Im} z_+ > 0$ *and* $\operatorname{Im} z_- < 0$ *such that* $\operatorname{rg}(z_\pm - T) = H$.

(v) $\sigma(T) \subseteq \mathbb{R}$.

(vi) *T is closed and* $\ker(\pm\mathrm{i} - T^*) = H$.

Proof. (i) $\Longrightarrow$ (ii) Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $\operatorname{rg}(\lambda - T) \neq H$ is closed by Theorem 5.22 and $\lambda^* \notin \sigma_{\mathrm{p}}(T)$. It follows by Theorem 4.73 that

$$\operatorname{rg}(\lambda - T) = \operatorname{rg}(\lambda - T)^{\perp\perp} = \ker(\lambda^* - T^*)^\perp = \ker(\lambda^* - T)^\perp = \{0\}^\perp = H.$$

(ii) $\Longrightarrow$ (i) By assumption, T is symmetric, hence $T \subseteq T^*$, so it suffices to show that $\mathcal{D}(T^*) \subseteq \mathcal{D}(T)$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $\lambda - T$ and $\overline{\lambda} - T$ are bijective. For $x \in \mathcal{D}(T^*)$ there exists a $y \in \mathcal{D}(T)$ such that $(\lambda - T^*)x = (\lambda - T)y$. Since $T \subseteq T^*$, it follows that $Ty = T^*$, hence $x - y \in \ker(\lambda - T^*) = \{0\}$ which implies $x = y \in \mathcal{D}(T)$.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (v) Let $z_\pm \in \mathbb{C}$ with $\operatorname{Im} z_+ > 0$ and $\operatorname{Im} z_- < 0$ such that $\operatorname{rg}(z_\pm - T) = H$. By Theorem 5.22, it follows that $z_\pm - T$ is injective and its inverse is bounded by $|\Im z_\pm|$. Hence, by Lemma 5.7, every $\lambda \in \mathbb{C}$ with $|\lambda - z_\pm| < |\Im z_\pm|$ belongs to $\rho(T)$. Given any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, repeating the argument above finitely many times shows that $\lambda \in \rho(T)$.

(v) $\Longrightarrow$ (ii) is obvious.

(vi) $\Longrightarrow$ (iii) Since T is closed, the range of $\pm\mathrm{i} - T$ is closed by Theorem 5.22. Therefore $\operatorname{rg}(\pm\mathrm{i} - T) = \operatorname{rg}(\pm\mathrm{i} - T)^{\perp\perp} = \ker(\mp\mathrm{i} - T^*)^\perp = \{0\}^\perp = H$.

(i) $\Longrightarrow$ (vi) Since $T = T^*$, it is closed and $\mathbb{C} \setminus \mathbb{R} \subseteq \rho(T)$, in particular $\ker(\pm\mathrm{i} - T) = \{0\}$. ☐

Analogously, we find a characterisation of essentially selfadjoint operators.

Theorem 5.24. *Let H be a complex Hilbert space and $T(H \to H)$ a symmetric operator. Then the following is equivalent.*

(i) *T is essentially selfadjoint.*

(ii) $\overline{\operatorname{rg}(\lambda - T)} = H$ *for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

(iii) $\overline{\operatorname{rg}(\pm \mathrm{i} - T)} = H$.

(iv) *There exist* $z_\pm \in \mathbb{C}$ *with* $\operatorname{Im} z_+ > 0$ *and* $\operatorname{Im} z_- < 0$ *such that* $\overline{\operatorname{rg}(z_\pm - T)} = H$.

(v) $\sigma(\overline{T}) \subseteq \mathbb{R}$.

(vi) $\ker(\pm\mathrm{i} - T^*) = H$.

Definition 5.25. Let X be a Banach space and $T(X \to X)$ densely defined and closed. $\lambda \in \mathbb{C}$ is called *approximate eigenvalue* if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} (T - \lambda)x_n = 0$. The set of all approximate eigenvalues is denoted by $\sigma_{\mathrm{ap}}(T)$.

Proposition 5.26. (i) *Every approximate eigenvalue belongs to* $\sigma(T)$.

(ii) *Every boundary point of* $\sigma(T) \subseteq \mathbb{C}$ *is an approximate eigenvalue of* T.

(iii) *If* X *is a Hilbert space and if* T *is selfadjoint, then every* $\lambda \in \sigma(T)$ *is an approximate eigenvalue of* T.

Proof. (i) Let λ be an approximate eigenvalue of T. Choose a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $(\lambda - T)x_n \to 0$. Assume that $\lambda \in \rho(T)$. Then $R(\lambda, T) = (\lambda - T)^{-1}$ is bounded, therefore

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} R(\lambda - T)(\lambda - T)x_n = R(\lambda - T) \lim_{n\to\infty} (\lambda - T)x_n = 0,$$

in contradiction to $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

(ii) Let λ be a boundary point of $\sigma(T)$. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \rho(T)$ which converges to λ. For every $n \in \mathbb{N}$ choose $x_n \in X$ such that $\|x_n\| = 1$ and $\|R(\lambda_n, T)x_n\| \geq \frac{1}{2}\|R(\lambda_n, T)\|$. From Lemma 5.7 we know that $\|R(\lambda_n, T)\| \geq \frac{1}{\operatorname{dist}(\lambda_n, \sigma(T))}$. Set $y_n := \|R(\lambda_n, T)\|^{-1} R(\lambda_n, T)x_n$. Then $y_n \in \mathcal{D}(T)$ and $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Moreover

$$\begin{aligned} \|(\lambda - T)y_n\| &\leq \|(\lambda - \lambda_n)y_n\| + \|(\lambda_n - T)y_n\| \\ &= |\lambda - \lambda_n| + \|R(\lambda_n - T)x_n\|^{-1} \\ &\leq |\lambda - \lambda_n| + 2\|R(\lambda_n - T)\|^{-1} \longrightarrow 0, \qquad n \to \infty. \end{aligned}$$

Hence $\lambda \in \sigma_{\mathrm{ap}}(T)$.

(iii) By Theorem 5.23 the spectrum of a selfadjoint operator is real, so $\sigma(T) = \partial\sigma(T) \subseteq \sigma_{\mathrm{ap}}(T) \subseteq \sigma(T)$. □

Lemma 5.27. *Let* H *be Hilbert space and* $T \in L(H)$ *selfadjoint. Then* $\sigma(T) \subseteq [m, M]$ *where* $m := \inf\{\langle Tx\,, x\rangle : \|x\| = 1\}$ *and* $M := \sup\{\langle Tx\,, x\rangle : \|x\| = 1\}$. *Moreover,* $m, M \in \sigma(T)$.

Proof. Let $\lambda \in \mathbb{R}$, $\lambda < m$. Then $\lambda - T$ is injective because for all $x \in X$

$$\|(\lambda - T)x\|\|x\| \geq \langle(\lambda - T)x\,, x\rangle \geq (\lambda - m)\|x\|^2. \tag{5.4}$$

In particular, $\operatorname{rg}(\lambda - T) = \mathcal{D}((\lambda - T)^{-1})$ is closed because $(\lambda - T)^{-1} : \operatorname{rg}(\lambda - T) \to H$ is closed and continuous by (5.4). Hence $\operatorname{rg}(\lambda - T) = \overline{\operatorname{rg}(\lambda - T)} = \ker(\lambda - T)^{\perp} = H$. It follows that $(-\infty, m) \in \rho(T)$. Analogously $(M, \infty) \in \rho(T)$ is shown.

Now we show that $m \in \sigma(T)$. By Proposition 5.26 it suffices to show that $m \in \sigma_{\mathrm{ap}}(T)$. By definition of m there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\langle Tx_n\,, x_n\rangle \searrow m$. Since $s(x, y) := \langle(T - m)x\,, y\rangle$ defines a positive semidefinite sesquilinear form, Cauchy-Schwarz inequality implies

$$\begin{aligned} \|(T - m)x_n\|^2 &= |s(x_n, (T - m)x_n)| \leq s(x_n, x_n)^{\frac{1}{2}} s((T - m)x_n)^{\frac{1}{2}} \\ &= \langle(T - m)x_n\,, x_n\rangle^{\frac{1}{2}} \langle(T - m)^2 x_n\,, (T - m)x_n\rangle^{\frac{1}{2}}. \end{aligned}$$

Since the first term in the product tends to 0 for $n \to \infty$ and the second term is bounded by $(\|T\| - m)^{\frac{3}{2}} < \infty$, it follows that $\|(T - m)x_n\|$ tends to 0 for $n \to \infty$. This shows that $m \in \sigma_{\mathrm{ap}}(T)$. The proof of $M \in \sigma(T)$ is analogous. □

5.4 Compact operators

Recall that a metric space M is *compact* if and only if every open cover of M contains a finite cover. M is called *totally bounded* if and only if every for every $\varepsilon > 0$ there exists a covering of M with finitely many open balls of radius ε. M is called *precompact* (or *precompact*) if and only if $\overline{M}$ is compact. It can be shown that a totally bounded metric M is compact if and only if M is complete. In particular, a subset of a complete metric space is totally bounded if and only if its closure is compact. A subset of a metric space is called *relatively compact* if and only if its closure is compact.

Definition 5.28. Let X, Y be normed spaces. An operator $T \in L(X, Y)$ is called *compact* if for every bounded set $A \subseteq X$ the set $T(A)$ is relatively compact. The set of all compact operators from X to Y is denoted by $K(X, Y)$.

Remark 5.29. Sometimes compact operators are called *completely continuous*.

Remarks 5.30. (i) Every compact linear operator is bounded.

(ii) $T \in L(X, Y)$ is compact if and only if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a convergent subsequence.

(iii) $T \in L(X, Y)$ is compact if and only if $T(B_X(0,1))$ is relatively compact.

(iv) Let $T \in L(X, Y)$ with finite dimensional $\operatorname{rg}(T)$. The T is compact.

(v) The identity map $\operatorname{id} \in L(X)$ is compact if and only if X is finite-dimensional.

Theorem 5.31. *Let X, Y be Banach spaces. Then $K(X, Y)$ is a closed subspace of $L(X, Y)$.*

Proof. Obviously, $0 \in K(X, Y)$ and Remark 5.30 (ii) implies that the linear combination of compact operators is compact. Now let $(T_n)_{n\in\mathbb{N}} \subseteq K(X, Y)$ a Cauchy sequence. Since $L(X, Y)$ is complete, there exists a $T \in L(X, Y)$ such that $T_n \to T$. We have to show $T \in K(X, Y)$. Take an arbitrary bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and choose $M \in \mathbb{R}$ such that $\|x_n\| \leq M$, $n \in \mathbb{N}$. Since T_1 is compact, there exists a subsequence $(x_n^{(1)})$ such that $(T_1 x_n^{(1)})_{n\in\mathbb{N}}$ converges. Continuing like this, for every $k \geq 2$ we find a subsequence $(x_n^{(k)})$ of $(x_n^{(k-1)})$ such that $(T_k x_n^{(k)})_{n\in\mathbb{N}}$ converges. Let $(y_n)_{n\in\mathbb{N}} = (x_n^{(n)})_{n\in\mathbb{N}}$ the diagonal sequence. Then, for every $k \in \mathbb{N}$, the sequence $(T_k y_n)_{n\in\mathbb{N}}$ converges. Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $\|T - T_k\| < \frac{\varepsilon}{3M}$ and $N \in \mathbb{N}$ such that $\|T_k x_n - T_k x_m\| \leq \frac{\varepsilon}{3}$ for $m, n \geq N$. Then, for all $m, n \geq N$,

$$\begin{aligned} \|Ty_n - Ty_m\| &\leq \|Ty_n - T_k y_n\| + \|T_k y_n - T_k y_m\| + \|T_k y_m - Ty_m\| \\ &\leq \frac{M\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{M\varepsilon}{3M} = \varepsilon. \end{aligned}$$

Hence $(Ty_n)_{n\in\mathbb{N}}$ is Cauchy sequence in the Banach space Y, hence convergent. □

Lemma 5.32. *Let X, Y, Z be Banach spaces, $S \in L(X, Y)$ and $T \in L(Y, Z)$. Then TS is compact if at least one of the operators S or T is compact.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. If S is compact, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Sx_{n_k})_{k\in\mathbb{N}}$ converges. By continuity of T, also $(TSx_{n_k})_{k\in\mathbb{N}}$ converges.

Now assume that T is compact. Since S is bounded, $(Sx_n)_{k\in\mathbb{N}}$ is bounded, hence there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(TSx_{n_k})_{k\in\mathbb{N}}$ converges. □

Theorem 5.33 (Schauder). *Let X, Y be Banach space and $T \in L(X,Y)$. Then T is compact if and only if T' is compact.*

For the proof we use the Ascoli-Arzelá theorem.

Theorem 5.34 (Arzelá-Ascoli). *Let (M,d) be a compact metric space and $A \subseteq C(M)$ a family of real or complex valued continuous functions on M such that*

(i) *A is bounded,*
(ii) *A is closed,*
(iii) *A is equicontinuous, that is,*

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, f \in A \qquad d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Then A is compact.

Proof. See, e. g., [Rud91] or [Yos95]. □

Proof of Theorem 5.33. First assume that T is compact. Let $K_X(0,1) := \{x \in X : \|x\| \le 1\}$ be the closed unit ball in X. By assumption $K := \overline{T(K_X(0,1))}$ is compact in Y and bounded by $\|T\|$. Now let $(\varphi_n)_{n\in\mathbb{N}} \subseteq Y'$ be a bounded sequence and $C \in \mathbb{R}$ such that $\|\varphi_n\| \le C$, $n \in \mathbb{N}$. We define the functions

$$f_n : K \to \mathbb{K}, \quad f_n(y) := \varphi_n(y).$$

Then $(f_n)_{n\in\mathbb{N}}$ is bounded by C and equicontinuous because $|f(y_1) - f(y_2)| \le C\|y_1 - y_2\|$ for all $y_1, y_2 \in K$. By the Ascoli-Arzelá, $(f_n)_{n\in\mathbb{N}}$ is compact, so there exists a convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$. Then also $(T'\varphi_{n_k})_{k\in\mathbb{N}}$ converges because

$$\begin{aligned}\|T'\varphi_{n_k} - T'\varphi_{n_m}\| &= \sup\{\|\varphi_{n_k}(Tx) - \varphi_{n_m}(Tx)\| : x \in K_X(0,1)\}\\ &= \sup\{\|\varphi_{n_k}(y) - \varphi_{n_m}(y)\| : y \in K\} = \|f_{n_k} - f_{n_m}\|.\end{aligned}$$

Now assume that T' is compact. Then $T'' \in L(X'', Y'')$ is compact. By Lemma 5.32 $T'' \circ J_X$ is compact. Recall that $J_Y \circ T = T \circ J_X$ (Lemma 2.34), so $J_Y \circ T : X \to Y''$ is compact. Since Y is closed in Y'', $T : X \to Y$ is compact. □

Example 5.35. Let $k \in C([0,1]^2)$ and

$$T_k : C([0,1]) \to C([0,1]), \qquad (T_k x)(t) = \int_0^1 k(s,t)x(s)\,\mathrm{d}s.$$

Then T_k is compact.

Proof. Obviously T_k is well-defined and bounded. Let $(x_n)_{n\in\mathbb{N}} \subseteq C([0,1])$ a bounded sequence with bound C. Hence $(T_k x_n)_{n\in\mathbb{N}}$ is bounded. To show that it is equicontinuous fix $\varepsilon > 0$. Since k is uniformly continuous, there exists a $\delta > 0$ such that $|k(s,t) - k(s',t')| < \varepsilon$ if $\|(s,t) - (s',t')\| < \delta$. Now for $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$ and $n \in \mathbb{N}$ we obtain

$$|T_k x_n(t_1) - T_k x_n(t_2)| \le \int_0^1 |k(s,t_1) - k(s,t_2)||x_n(s)|\, \mathrm{d}s < \varepsilon \|x_n\|_\infty \le C\varepsilon.$$

By the Ascoli-Arzelá theorem it follows that $(T_k x_n)_{n\in\mathbb{N}}$ is relatively compact, hence it contains a convergent subsequence. □

Let X be vector space and $T : X \to X$ a linear operator. Then obviously

$$\begin{aligned} &\{0\} \subseteq \ker T \subseteq \ker T^2 \subseteq \ker T^3 \subseteq \dots, \\ &X \supseteq \operatorname{rg} T \supseteq \operatorname{rg} T^2 \supseteq \operatorname{rg} T^3 \supseteq \dots. \end{aligned}$$

Lemma 5.36. *Let X a vector space and $T : X \to X$ a linear operator.*

(i) *Assume that $\ker T^{k+1} = \ker T^k$ for some $k \in \mathbb{N}_0$. Then $\ker T^n = \ker T^k$ for all integer $n \ge k$.*

(ii) *Assume that $\operatorname{rg} T^{k+1} = \operatorname{rg} T^k$ for some $k \in \mathbb{N}_0$. Then $\operatorname{rg} T^n = \operatorname{rg} T^k$ for all integer $n \ge k$.*

Proof. We prove the lemma by induction. The case when $n = k$ is clear by assumption.
(i) Assume that $n > k$ and $\ker T^n = \ker T^k$. Then

$$\ker T^{n+1} = \{x \in X : T^{n+1}x = 0\} = \{x \in X : Tx \in \ker T^k\} = \ker T^{k+1} = \ker T^k.$$

(i) Assume that $n > k$ and $\operatorname{rg} T^n = \operatorname{rg} T^k$. Then

$$\operatorname{rg} T^{n+1} = T(\operatorname{rg} T^n) = T(\operatorname{rg} T^k) = \operatorname{rg} T^{k+1} = \operatorname{rg} T^k.$$

□

Definition 5.37. Let X be a vector space and $T : X \to X$ a linear operator. We define

$$\textit{ascent of } T := \alpha(T) := \begin{cases} \min\{k \in \mathbb{N}_0 : \ker T^k = \ker T^{k+1}\}, & \text{if the minimum exists,} \\ \infty & \text{else} \end{cases}$$

$$\textit{descent of } T := \delta(T) := \begin{cases} \min\{k \in \mathbb{N}_0 : \operatorname{rg} T^k = \operatorname{rg} T^{k+1}\}, & \text{if the minimum exists,} \\ \infty & \text{else.} \end{cases}$$

Lemma 5.38. *Let X be a vector space and $T : X \to X$ a linear operator. If both the ascent $\alpha(T)$ and the descent $\delta(T)$ are finite, then $\alpha(T) = \delta(T) =: p$ and $X = \operatorname{rg}(T^p) \oplus \ker(T^p)$.*

Proof. Let $p := \alpha(T)$ and $q := \delta(T)$. We divide the proof in several steps.
Step 1. $\operatorname{rg}(T^p) \cap \ker(T^n) = \{0\}$ for every $n \in \mathbb{N}_0$.
To see this, choose $x \in \operatorname{rg}(T^p) \cap \ker(T^n)$. Then there exists a $y \in X$ such that $x = T^p y$, so $0 = T^n x = T^{p+n} y$. Hence $y \in \ker T^{p+n} = \ker T^p$ by Lemma 5.36 i. It follows that $x = T^p y = 0$.

Step 2. $X = \operatorname{rg}(T^n) + \ker(T^q)$ for every $n \in \mathbb{N}_0$.

For the proof fix $x \in X$. Then $T^q x \subseteq \operatorname{rg}(T^q) = \operatorname{rg}(T^{q+n})$. Hence there exists $y \in X$ such that $T^q x = T^{q+n} y$. Then $T^q(x - T^n y) = 0$, and therefore $x = T^n y + (x - T^n y) \in \operatorname{rg}(T^n) + \ker(T^q)$.

Step 3. $\alpha(T) \le \delta(T) = q$.

Let $x \in \ker T^{q+1}$. We have to show $x \in \ker T^q$. By step 2, with $n = p$, there exist $x_1 \in \operatorname{rg}(T^p)$ and $x_2 \in \ker(T^q)$ such that $x = x_1 + x_2$. Hence $x_1 = x - x_2 \subseteq \ker(T^{q+1}) \cap \operatorname{rg}(T^p) = \{0\}$ by step 1. Therefore $x = x_2 \in \ker(T^q)$.

Step 4. $\delta(T) \le \alpha(T) = p$.

By step 1 and step 2, we have that $X = \operatorname{rg}(T^p) \oplus \ker(T^q)$. Since $\operatorname{rg}(T^{p+1}) \cap \ker(T^q) \subseteq \operatorname{rg}(T^p) \cap \ker(T^q) = \{0\}$, we also have $X = \operatorname{rg}(T^{q+1}) \oplus \ker(T^q)$, implying $\operatorname{rg} R(T^{p+1}) = \operatorname{rg}(T^p)$, hence $\delta \le p$. □

Theorem 5.39. *Let X be a Banach space, $T \in L(X)$ a compact operator and $\lambda \in \mathbb{C} \setminus \{0\}$.*

(i) $\ker(\lambda - T)^n$ *is finite dimensional for every* $n \in \mathbb{N}_0$.

(ii) *If $U \subseteq X$ is a closed subspace with $U \cap \ker(\lambda - T)^n = \{0\}$, then $(\lambda - T)(U)$ is closed and $\lambda - T : U \to \operatorname{rg}((\lambda - T)|_U$ has a bounded inverse.*

(iii) $\operatorname{rg}(\lambda - T)^n$ *is closed for every* $n \in \mathbb{N}_0$.

Proof. Note that $(\lambda - T)^n = \lambda^n - \sum_{n=1}^n \binom{n}{k} \lambda^{n-k} T^k$ and the operator sum is compact. Hence it suffices to show the assertions for $n = 1$.

(i) Observe that $T|_{\ker(\lambda - T)} = \lambda \operatorname{id}|_{\ker(\lambda - T)}$. Hence $\lambda \operatorname{id}|_{\ker(\lambda - T)}$ is compact. By Remark 5.30 (v) this is case if and only if $\ker(\lambda - T)$ is finite dimensional.

(ii) Since $U \cap \ker(\lambda - T) = \{0\}$, the restriction $(\lambda - T)|_U$ is invertible. We will show that its inverse is bounded. Assume $\big((\lambda - T)|_U\big)^{-1}$ is not bounded. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} (\lambda - T)x_n = 0$. Since T is compact, there exists a convergent subsequence $(T x_{n_k})_{k\in\mathbb{N}}$. Hence

$$\lambda x_{n_k} = T x_{n_k} + \underbrace{(\lambda - T)x_{n_k}}_{\to 0} \longrightarrow \lim_{n\to\infty} T x_{n_k} =: y.$$

Note that $y \in U$ because U is closed. Moreover, $y \in \ker(\lambda - T)$ because

$$(\lambda - T)y = (\lambda - T) \lim_{n\to\infty} x_{n_k} = \lim_{n\to\infty} (\lambda - T)x_{n_k} = 0.$$

Hence $y \in \ker(\lambda - T) \cap U = \{0\}$ in contradiction to $\|y\| = \lim\limits_{n\to\infty} \|\lambda x_n\| = \lambda \ne 0$. Hence $\big((\lambda - T)|_U\big)^{-1} : \operatorname{rg}(\lambda - T)|_U \to U$ is bounded. Since it is also closed, its domain $\operatorname{rg}(\lambda - T)|_U$ must be closed.

(iii) By (i) we already know that $\dim \ker(\lambda - T) < \infty$. Then by the following lemma 5.40 there exists a closed subspace $U \subseteq X$ such that $X = \ker(\lambda - T) \oplus U$. Hence $\operatorname{rg}(\lambda - T) = \operatorname{rg}((\lambda - T)|_U)$ is closed by (ii). □

Lemma 5.40. *Let X be a Banach space and $M \subseteq X$ a finite dimensional subspace. Then there exists a closed subspace U of X such that $X = M \oplus U$.*

Proof. Let $x_1, \dots, x_n$ a basis of M. Then there exist $\varphi_1, \dots, \varphi_n \in M'$ such that $\|\varphi_k\| = 1$ and $\varphi_k(x_j)\delta_{kj}$ for all $j, k = 1, \dots, n$. By the Hahn-Banach theorem the φ_k can be extended to functionals $\psi_k \in X'$ with $\|\psi_k\| = 1$, $k = 1, \dots, n$. Let $P : X \to X, Px = \sum_{j=1}^{n} \varphi_j(x)x$. Obviously $P = P^2$, hence P is a projection. Note that $M = P(X)$. Hence $X = \operatorname{rg}(P) \oplus \ker P = M \oplus \ker P$. □

Theorem 5.41. *Let X be a Banach space, $T \in L(X)$ a compact operator and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\alpha(\lambda - T) = \delta(\lambda - T) = p < \infty$ and $X = \ker(\lambda - T)^p \oplus \operatorname{rg}(\lambda - T)^p$.*

The number $p = \alpha(\lambda - T) = \delta(\lambda - T)$ is called the *Riesz index* of $\lambda - T$.

Proof. By Lemma 5.38 it suffices to show that $\alpha(T)$ and $\delta(T)$ are finite.

Assume that α is not finite. Since in this case $\ker(\lambda - T) \subsetneq \ker(\lambda - T)^2 \subsetneq \dots$ we can find a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that for all $n \in \mathbb{N}$

$$\|x_n\| = 1, \quad x_n \in \ker(\lambda - T)^n, \quad \text{and} \quad \|x_n - z\| \geq \frac{1}{2} \text{ for all } z \in \ker(\lambda - T)^{n-1}.$$

The last condition can be satisfied by the Riesz lemma (Theorem 1.21) because $\ker(\lambda - T)^n$ is closed for all $n \in \mathbb{N}$. Then for all $1 \leq m < n$

$$\|Tx_n - Tx_m\| = \|\lambda x_n \underbrace{-\lambda x_m - (\lambda - T)x_n + (\lambda - T)x_m}_{\in \ker(\lambda - T)^{n-1}}\| \geq \frac{1}{2}.$$

Therefore $(Tx_n)_{n\in\mathbb{N}}$ does not contain a convergent subsequence in contradiction to T being compact.

Assume that δ is not finite. Since in this case $\operatorname{rg}(\lambda - T) \supsetneq \operatorname{rg}(\lambda - T)^2 \supsetneq \dots$ we can choose a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that for all $n \in \mathbb{N}$

$$\|x_n\| = 1, \quad x_n \in \operatorname{rg}(\lambda - T)^n, \quad \text{and} \quad \|x_n - z\| \geq \frac{1}{2} \text{ for all } z \in \operatorname{rg}(\lambda - T)^{n+1}.$$

The last condition can be satisfied by the Riesz lemma because $\operatorname{rg}(\lambda - T)^n$ is closed for all $n \in \mathbb{N}$ by Theorem 5.39. Then for all $1 \leq m < n$

$$\|Tx_n - Tx_m\| = \|\lambda x_n \underbrace{-\lambda x_m - (\lambda - T)x_n + (\lambda - T)x_m}_{\in \operatorname{rg}(\lambda - T)^{n+1}}\| \geq \frac{1}{2}.$$

Therefore $(Tx_n)_{n\in\mathbb{N}}$ does not contain a convergent subsequence in contradiction to T being compact. □

Theorem 5.42 (Spectrum of a compact operator). *Let X be a Banach space. For a compact operator $T \in L(X)$ the following holds.*

(i) *If $\lambda \in \mathbb{C} \setminus \{0\}$, then λ either belongs to $\rho(T)$ or it is an eigenvalue of T, that is $\mathbb{C} \setminus \{0\} \subseteq \rho(T) \cup \sigma_p(T)$.*

(ii) *The spectrum of T is at most countable and 0 is the only possible accumulation point.*

(iii) *If* $\lambda \in \sigma(T) \setminus \{0\}$*, then the dimension of the algebraic eigenspace* $\mathcal{A}_\lambda(T)$ *is finite and* $\mathcal{A}_\lambda(T) = \ker(\lambda - T)^p$ *where* p *is the Riesz index of* $\lambda - T$*.*

(iv) $X = \ker(\lambda - T)^p \oplus \operatorname{rg}(\lambda - T)^p$ *for* $\lambda \in \sigma(T) \setminus \{0\}$ *where* p *is the Riesz index of* $\lambda - T$ *and* $\ker(\lambda - T)^p$ *and* $\operatorname{rg}(\lambda - T)^p$ *are* T*-invariant.*

(v) $\sigma_p(T) \setminus \{0\} = \sigma_p(T') \setminus \{0\}$ *and* $\sigma(T) = \sigma(T')$*. If* H *is a Hilbert space then* $\sigma_p(T) \setminus \{0\} = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma_p(T^*)\} \setminus \{0\} = \overline{\sigma_p(T^*)} \setminus \{0\}$*, where the bar denotes complex conjugation, and* $\sigma(T) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma(T^*)\} = \overline{\sigma(T^*)}$*.*

Proof. (i) Let $\lambda \in \mathbb{C} \setminus \{0\}$. By Theorem 5.41 the Riesz index p of $\lambda - T$ is finite. If $p = 0$, then $X = \operatorname{rg}(\lambda - T)$ by the proof of Lemma 5.38 (step 2), hence $\lambda \in \rho(T)$. If $p \neq 0$, then $\lambda \in \sigma_{\mathrm{p}}(T)$.

(ii) It suffices to show that for every $\varepsilon > 0$ the set $\{\lambda \in \sigma(T) : |\lambda| > \varepsilon\}$ is finite. Assume there exists an $\varepsilon > 0$ such that the set is not finite. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \neq \lambda_m$ for $n \neq m$ and $|\lambda_n| > \varepsilon$, $n \in \mathbb{N}$. Since $\sigma(T) \setminus \{0\}$ consists of eigenvalues, we can choose eigenvectors x_n of T with eigenvalues λ_n. Note that the x_n are linearly independent because $\lambda_n \neq \lambda_m$ for $n \neq m$. Let $U_n := \operatorname{span}\{x_1, \dots, x_n\}$. Note that all U_n are T-invariant, closed and that $U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \dots$. Using the Riesz Lemma, we can choose a sequence $(y_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$\|y_n\| = 1, \quad y_n \in U_n, \quad \text{and} \quad \|y_n - z\| \geq \frac{1}{2} \text{ for all } z \in U_{n-1}.$$

Let $1 \leq m < n$. Note that $Ty_m \in U_m$. Let $y_n = \sum_{j=1}^n \alpha_j x_j$ for some $\alpha_j \in \mathbb{C}$. Then

$$(\lambda_n - T)y_n = \alpha_n(\lambda_n - T)x_n + \sum_{j=1}^n \alpha_j (T - \lambda_n)x_j = \sum_{j=1}^n \alpha_j(\lambda_j - \lambda_n)x_j \in U_{n-1}.$$

Hence

$$\|Ty_n - Ty_m\| = \|\lambda_n y_n \underbrace{-(\lambda_n - T)y_n - Ty_m}_{\in U_{n-1}}\| \geq \frac{1}{2}. \tag{5.5}$$

Therefore $(Tx_n)_{n\in\mathbb{N}}$ does not contain a convergent which contradicts the assumption that T is compact.

(iii) and (iv) follow from Theorem 5.42.

(v) By Schauder's theorem T' is compact (theorem 5.33) Hence for $\lambda \in \mathbb{C}$ it follows that

$$\begin{aligned} \lambda \in \rho(T) \quad &\iff \quad \ker(\lambda - T) = \{0\} \text{ and } \operatorname{rg}(\lambda - T) = X \\ &\iff \quad {}^\circ\operatorname{rg}(\lambda - T') = \{0\} \text{ and } {}^\circ\ker(\lambda - T') = X \\ &\iff \quad \operatorname{rg}(\lambda - T') = X' \text{ and } \ker(\lambda - T') = \{0\} \\ &\iff \quad \lambda \in \rho(T) \end{aligned}$$

□

Theorem 5.43 (Fredholm alternative; Riesz-Schauder theory). *Let* X *be a Banach space,* $T \in L(X)$ *a compact operator and* $\lambda \in \mathbb{C} \setminus \{0\}$*. Then exactly one of the following is true:*

(i) *For every* $y \in X$ *the equation* $(\lambda - T)x = y$ *has exactly one solution* $x \in X$*.*

(ii) $(\lambda - T)x = 0$ *has a non-trivial solution* $x \in X$.

Proof. (i) is equivalent to $\lambda \in \rho(T)$ and (ii) is equivalent to $\lambda \in \sigma_{\mathrm{p}}(T)$. Since $\lambda \neq 0$, the latter is equivalent to $\lambda \in \sigma(T)$. The assertion follows from Theorem 5.42. □

A more precise formulation of the Fredholm alternative is the following.

Theorem 5.44. *Let X be a Banach space, $T \in L(X)$ a compact operator and $\lambda \in \mathbb{C} \setminus \{0\}$. For $x, y, \in X$ and $\varphi, \eta \in X'$ consider the equations*

$$\begin{aligned} &(\mathrm{A}) \quad (\lambda - T)x = y, &&\qquad (\mathrm{C}) \quad (\lambda - T')\varphi = \eta, \\ &(\mathrm{B}) \quad (\lambda - T)x = 0, &&\qquad (\mathrm{D}) \quad (\lambda - T')\varphi = 0. \end{aligned}$$

Then

(i) *For $y \in X$ the following is equivalent:*

  (a) (A) *has a solution* x.

  (b) $\varphi(y) = 0$ *for every solution φ of* (D).

(ii) *For $\eta \in X'$ the following is equivalent:*

  (a) (C) *has a solution* φ.

  (b) $\eta(x) = 0$ *for every solution x of* (B).

(iii) *Fredholm alternative: Exactly one of the following holds:*

  (a) *For all $y \in X$ and $\eta \in X'$ the equations* (A) *and* (C) *have exactly one solution (in particular* (B) *and* (D) *have only the trivial solutions).*

  (b) (B) *and* (D) *have non-trivial solutions. In this case* $\dim(\ker(\lambda - T)) = \dim(\ker(\lambda - T')) > 0$ *and* (A) *and* (C) *have solutions if and only if*

$$\begin{aligned} \varphi(y) &= 0 \quad \text{for all solutions } \varphi \text{ of (D)}, \\ \eta(x) &= 0 \quad \text{for all solutions } x \text{ of (B)}. \end{aligned}$$

Definition 5.45. Let X, Y be Banach spaces. $T \in L(X)$ is called *Fredholm operator* if $\mathrm{rg}(T)$ is closed and $n(T) := \dim(\ker T) < \infty$ and $d(T) := \mathrm{codim}_Y(\mathrm{rg}\, T) := \dim(Y/\mathrm{rg}(T)) < \infty$. In this case, $\chi(T) := n(T) - d(T)$ is called the *Fredholm index*.

Proof of Theorem 5.44. .... □

Now we return to the spectrum of compact operators.

Lemma 5.46. *Let H be Hilbert space, $\neq \{0\}$, and $T \in L(H)$ a selfadjoint compact operator. Then at least one the values $\|T\|$ or $-\|T\|$ is an eigenvalue of T. In particular, if $T \neq 0$, then T has at least one eigenvalue distinct from 0.*

Proof. If $\|T\| = 0$, the assertion is clear. Now assume that $\|T\| \neq 0$. Recall that $\|T\| = \sup\{|\langle Tx\,, x\rangle| : x \in X, \|x\| = 1\}$ (Theorem 4.45).
By Lemma 5.27 the numbers $m = \inf\{\langle Tx\,, x\rangle : x \in X, \|x\| = 1\}$ and $M = \inf\{\langle Tx\,, x\rangle : x \in X, \|x\| = 1\}$ belong to the spectrum of T. Since T is compact and $\|T\| \neq 0$, it follows that $\emptyset \neq \{\pm\|T\|\} \cap \sigma(T) = \{\pm\|T\|\} \cap \sigma_{\mathrm{p}}(T)$. □

Theorem 5.47 (Spectral theorem for compact selfadjoint operators). *Let H be a Hilbert space and $T \in L(H)$ a compact selfadjoint operator.*

(i) *There exists an orthonormal system $(\mathrm{e}_n)_{n=1}^N$ of eigenvectors of T with eigenvalues $(\lambda_n)_{n=1}^N$ where $N \in \mathbb{N} \cup \{\infty\}$ such that*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x\,, \mathrm{e}_n\rangle\, \mathrm{e}_n, \qquad x \in H. \tag{5.6}$$

The λ_n can be chosen such that $|\lambda_1| \geq |\lambda_2| \geq \cdots > 0$. The only possible accumulation point of the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is 0.

(ii) *If P_0 is the orthogonal projection on $\ker T$, then*

$$x = P_0 x + \sum_{n=1}^{N} \langle x\,, \mathrm{e}_n\rangle\, \mathrm{e}_n, \qquad x \in H. \tag{5.7}$$

(iii) *If $\lambda \in \rho(T)$, $\lambda \neq 0$*

$$(\lambda - T)^{-1}x = \lambda^{-1}P_0 x + \sum_{n=1}^{N} \frac{\langle x\,, \mathrm{e}_n\rangle}{\lambda_n - \lambda}\, \mathrm{e}_n, \qquad x \in H.$$

Proof. (i) Let $X_1 = X$ and $T_1 = T$. If $T \neq 0$, then there exists a $\lambda_1 \in \sigma_{\mathrm{p}}(T_1)$ such that $|\lambda_1| = \|T_1\| \neq 0$. Let B_1 be an orthonormal basis of $\ker(\lambda_1 - T_1)$. Note that B_1 is finite because T is compact (Theorem 5.42). Let $X_1 := \ker(\lambda_1 - T)^{\perp} = \overline{\mathrm{rg}(\lambda_1 - T)} = \mathrm{rg}(\lambda_1 - T)$. Here we used that T is selfadjoint and consequently $\lambda \in \sigma_{\mathrm{p}}(T) \subseteq \mathbb{R}$. By Theorem 5.42, X_2 is T_1-invariant, hence $T_2 := T_1|_{X_2} \in L(X_2)$. Obviously, T_2 is selfadjoint and compact. If $T_2 \neq 0$, then there exists a $\lambda_2 \in \sigma_{\mathrm{p}}(T_2)$ such that $|\lambda_2| = \|T_2\| \neq 0$. Let B_2 be an orthonormal basis of $\ker(\lambda_2 - T_2)$. Note that B_1 is finite because T is compact (Theorem 5.42). Hence $B_1 \cup B_2$ is an orthonormal basis of $\mathrm{span}\{\ker(\lambda_1 - T), \ker(\lambda_2 - T)\}$. Let $X_3 := \mathrm{span}\{\ker(\lambda_1 - T), \ker(\lambda_2 - T)\}^{\perp}$ and $T_3 := T_2|_{X_3}$. Continuing like this we obtain a sequence of Banach spaces X_n and a sequence of compact selfadjoint operators $T_n \in L(X_n)$. Let $x \in X$. Define

$$x_{n+1} = x - \sum_{\mathrm{e}_n \in B_1 \cup \ldots B_n} \langle x\,, \mathrm{e}_n\rangle\, \mathrm{e}_n \in X_{n+1}.$$

It follows that

$$\|Tx - T \sum_{\mathrm{e}_n \in B_1 \cup \ldots B_n} \langle x\,, \mathrm{e}_n\rangle\, \mathrm{e}_n\,\| = \|T_{n+1}x_{n+1}\| \leq |\lambda_{n+1}|\|x\| \longrightarrow 0, \qquad n \to \infty.$$

This implies that

$$Tx = \sum_{n=1}^{N} \langle x\,, \mathrm{e}_n\rangle T\,\mathrm{e}_n = \sum_{n=1}^{N} \lambda_n \langle x\,, \mathrm{e}_n\rangle\,\mathrm{e}_n\,.$$

(ii) Note that ...

(iii)

□

Corollary 5.48. *Let H be a Hilbert space and $T \in L(H)$ a compact selfadjoint operator. There exists a sequence $(P_n)_{n=1}^N$ of pairwise orthogonal projections with $N \in \mathbb{N} \cup \{\infty\}$ and a sequence $|\lambda_1| \geq |\lambda_2| \geq \dots$ such that*

$$T = \sum_{n=1}^{N} \lambda_n P_n \tag{5.8}$$

where the series converges to T in the operator norm. If $(\lambda_n)_n$ is an infinite sequence, then $\lim\limits_{n\to\infty} \lambda_n = 0$. The representation (5.8) is unique if the λ_n are pairwise distinct.

Proof. If the series is a finite sum, the assertion is clear. Now assume that the series is an infinite. Note that for every $k \in \mathbb{N}$ the operator $\sum_{n=k}^{\infty} \lambda_n P_n$ is normal and that the norm of a normal operator is equal to maximum of the moduli of the elements of its spectrum (Theorem 5.20). Since $|\lambda_{k+1}| \to 0$ for $k \to \infty$ the claim follows from

$$\Big\| T - \sum_{n=1}^{k} \lambda_n P_n \Big\| = \sup\{|\lambda_n| : n \geq k+1\} = |\lambda_{k+1}|.$$ □

The representation (5.8) allows us to define the root of a positive compact selfadjoint operator.

Theorem 5.49. *Let H be a Hilbert space and $K \in L(H)$ a compact operator.*

(i) *T is positive $\iff$ all eigenvalues of T are positive.*
T is strictly positive $\iff$ all eigenvalues of T are strictly positive.

(ii) *If T is positive and $k \in \mathbb{N}$ then there exists exactly one positive compact selfadjoint operator R such that $R^k = T$.*

Note that the theorem does not imply that there cannot be non-compact operators $A \in L(H)$ such that $A^2 = T$. In Corollary 5.60 we will show that every bounded positive selfadjoint operator has a unique positive root.

Proof of Theorem 5.49. Recall that a linear operator T is positive if and only if $\langle Tx\,, x\rangle \geq 0$ for all $x \in H$. Let P_0, λ_n and e_n as in (5.7). Then (i) follows from

$$\langle Tx\,, x\rangle = \Big\langle \sum_n \lambda_n \langle x\,, \mathrm{e}_n\rangle\,\mathrm{e}_n\,,\ P_0 x + \sum_n \lambda_n \langle x, \mathrm{e}_n\,, \mathrm{e}\rangle_n \Big\rangle = \sum_n \lambda_n |\langle x\,, \mathrm{e}_n\rangle|^2 \geq 0.$$

For the proof of (ii) define $R = \sum_n \lambda_n^{1/k} \langle \cdot\,, e_n \rangle\, e_n$. Obviously $R^k = T$. To show uniqueness, assume that there exists a compact selfadjoint positive linear operator S such that $S^k = T$. Since S is compact, it has a representation $S = \sum_n \mu_n Q_n$ with pairwise orthogonal projections Q_n. By assumption

$$T = S^k = \sum_n \mu_n^k Q_n.$$

Hence the μ_n are the kth roots the eigenvalues λ_n of T, so $S = R$. □

Definition 5.50. Let H be a Hilbert space and $T \in L(H)$ a positive selfadjoint compact operator. Then $|T| := (T^*T)^{\frac{1}{2}}$. The non-zero eigenvalues s_n of $|T|$ are the *singular values* of T.

Obviously $|T|$ and $|T^*|$ are positive selfadjoint compact operators.

Lemma 5.51. (i) *$\|\,|T|x\,\| = \|Tx\|$ and $\|\,|T^*|y\,\| = \|T^*y\|$ and for $x \in H_1$ and $y \in H_2$.*

(ii) *s is a singular value of T if and only if s^2 is an eigenvalue of T^*T and TT^*.*

Proof. (i) For all $x \in H_1$

$$\|\,|T|x\,\|^2 = \langle |T|x\,, |T|x \rangle = \langle |T|^2 x\,, x \rangle = \langle T^*Tx\,, x \rangle = \|Tx\|^2.$$

An analogous calculation shows $\|\,|T^*|y\,\| = \|T^*y\|$ and for $y \in H_2$.
(ii) follows from the uniqueness of the representation (5.8). □

Note that $|T|$ can be defined more generally for positive selfadjoint operators on a Hilbert space H, see Definition 5.61.

A representation similar to (5.6) exists for arbitrary compact operators.

Theorem 5.52. *Let H_1, H_2 be Hilbert spaces and $T \in L(H_1, H_2)$ a compact operator.*

(i) *Let $s_1 \geq s_2 \geq \cdots > 0$ be the singular values of T and $(\varphi_n)_{n=1}^N \subseteq H_1$ and $(\psi_n)_{n=1}^N \subseteq H_2$ such that*

$$Tx = \sum_{n=1}^N s_n \langle x\,, \varphi_n \rangle \psi_n, \qquad x \in H_1,$$
$$T^*y = \sum_{n=1}^N s_n \langle y\,, \psi_n \rangle \varphi_n, \qquad y \in H_2.$$

If there are infinitely many s_n, then $\lim_{n\to\infty} s_n = 0$.

(ii) *The non-zero eigenvalues of $|T|$ and $|T^*|$ coincide and are equal to the s_n. The s_n^2 are the eigenvalues of T^*T and TT^*. Moreover, the $\psi_n = \frac{1}{s_n} T\varphi_n$ are eigenvectors of T^*.*

Proof. (i) Let $(\varphi_n)_{n\in\mathbb{N}} \subseteq H_1$ a ONS such that, see Theorem 5.47,

$$|T|x = \sum_{n=1}^N s_n \langle x\,, \varphi_n \rangle \varphi_n, \qquad T^*Tx = \sum_{n=1}^N s_n^2 \langle x\,, \varphi_n \rangle \varphi_n.$$

Let $\psi_n := \frac{1}{x_n} T\varphi_n$. Then $(\psi_n)_{n\in\mathbb{N}}$ is an ONS in H_2 because

$$\langle \psi_n , \psi_m \rangle = \frac{1}{s_n^2} \langle T\varphi_n , T\varphi_m \rangle = \frac{1}{s_n^2} \langle T^*T\varphi_n , \varphi_m \rangle = \frac{1}{s_n^2}\, s_n^2 \delta_{nm} = \delta_{nm}.$$

Moreover

$$TT^*\psi_n = \frac{1}{s_n}\, TT^*T\varphi_n = \frac{s_n^2}{s_n} T\varphi_n = s_n^2 \psi_n.$$

Hence $\sigma_{\mathrm{p}}(T^*T) \setminus \{0\} = \{s_n^2 : 1 \le n \le N\} \subseteq \sigma_{\mathrm{p}}(TT^*) \setminus \{0\}$. Similarly the reverse inclusion can be shown, so that $\sigma_{\mathrm{p}}(T^*T) \setminus \{0\} \subseteq \sigma_{\mathrm{p}}(TT^*) \setminus \{0\}$.

(ii) . . . □

Theorem 5.53 (Min-Max-Principle). *Let H_1, H_2 be Hilbert spaces, $K \in L(H_1, H_2)$ a compact operator with singular values $s_1 \ge s_2 \ge s_3 \ge \dots$. Then $s_1 = \|K\|$ and for $n \ge 2$*

$$s_{n+1} = \inf_{x_1,\dots x_n \in H_1} \sup \Big\{ \|Kx\| \,:\, x \in H_1,\, x \perp \operatorname{span}\{x_1, \dots, x_n\},\ \|x\| = 1 \Big\}.$$

Proof. . . . □

5.5 Hilbert-Schmidt operators

Definition 5.54. Let H_1, H_2 be Hilbert spaces and $K \in L(H_1, H_2)$. K is called a *Hilbert-Schmidt operator* if and only if there exists an ONB $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ of H_1 such that

$$\sum_{\lambda\in\Lambda} \|K\,\mathrm{e}_\lambda\,\|^2 < \infty.$$

The set of all Hilbert-Schmidt operators from H_1 to H_2 is denoted by $\mathrm{H}S(H_1, H_2)$.

Theorem 5.55. *Let H_1, H_2 be Hilbert spaces.*

(i) *A operator $K \in L(H_1, H_2)$ is a Hilbert-Schmidt operator if and only if K^* is a Hilbert-Schmidt operator. In this case:*

$$\sum_{\alpha\in A} \|K\,\mathrm{e}_\alpha\,\|^2 = \sum_{\beta\in B} \|K\,\mathrm{e}_\beta\,\|^2 = \sum_{\lambda\in\Lambda} \|K\,\mathrm{e}_\lambda\,\|^2 < \infty$$

for all ONBes $(\mathrm{e}_\alpha)_{\alpha\in A}$ of H_1 and $(\mathrm{e}_\beta)_{\beta\in B}$ of H_2.

(ii) *Every Hilbert-Schmidt operator is compact.*

(iii) *Let $K \in L(H_1, H_2)$ be a compact operator with singular values $x_1 \ge x_2 \ge s_3 \ge \dots$. Then K is a Hilbert-Schmidt operator if and only if K^* is a Hilbert-Schmidt operator if and only if*

$$\sum_n s_n^2 < \infty.$$

Theorem 5.55 (i) shows that for $K \in \mathrm{H}S(H_1, H_2)$ the Hilbert-Schmidt norm

$$\|K\|_{\mathrm{H}S} := \Big(\sum_{\alpha \in A} \|K \,\mathrm{e}_\alpha\,\|^2 \Big)^{\frac{1}{2}} \quad \text{for an ONB } \ (\mathrm{e}_\alpha)_{\alpha \in A}.$$

is well-defined.

Proof of Theorem 5.55. (i) Let K be a Hilbert-Schmidt operator and $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ an ONB of H_1 such that $\sum_{\lambda\in\Lambda} \|K \,\mathrm{e}_\lambda\,\|^2 < \infty$. For an arbitray ONB $(\psi_\beta)_{\beta\in B}$ of H_2 we find, using Parseval's equality (Theorem 4.31) First we show that K^* is also a Hilbert-Schmidt operator.

$$\begin{aligned}\sum_{\beta\in B} \|K^*\psi_\beta\|^2 &= \sum_{\beta\in B} \Big\| \sum_{\lambda\in\Lambda} \langle K^*\psi_\beta \,, \mathrm{e}_\lambda\rangle \,\mathrm{e}_\lambda \Big\|^2 = \sum_{\lambda\in\Lambda}\sum_{\beta\in B} |\langle K^*\psi_\beta \,, \mathrm{e}_\lambda\rangle|^2 \\ &= \sum_{\lambda\in\Lambda}\sum_{\beta\in B} |\langle \psi_\beta \,, K\,\mathrm{e}_\lambda\rangle|^2 = \sum_{\lambda\in\Lambda} \|K\psi_\lambda\|^2 < \infty.\end{aligned}$$

In particular, the Hilbert-Schmidt norm of K^* does not depend on the chosen ONB of H_2. Applying the same proof to K^*, it follows that the Hilbert-Schmidt norm of $K = K^{**}$ does not depend on the chosen ONB of H_1. For the proof of $\|K\| \le \|K\|_{HS}$ we observe that every $x \in H_1$ with $\|x\| = 1$ can be extended to a ONB of H_1. Hence

$$\|K\|_{HS} \ge \|Kx\| \ge \sup\{\|Ky\| : y \in H_1,\ \|y\| = 1\} = \|K\|.$$

(ii) Let $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ an ONS of H_1 and $(\mathrm{e}_n)_{n\in\mathbb{N}}$ a subset containing all e_λ with $K\,\mathrm{e}_\lambda \neq 0$ (this family is at most countable by Lemma 4.27). For $n \in \mathbb{N}$ let P_n be the orthogonal projection on $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$. Note that all P_n are compact because they have finite-dimensional range. Since K is a Hilbert-Schmidt operator, we find that

$$\|K - KP_n\|^2 = \|K(\mathrm{id} - P_n)\|^2 \le \|K(\mathrm{id} - P_n)\|_{\mathrm{H}S}^2 = \sum_{m=n+1}^{\infty} \|K \,\mathrm{e}_n\,\|^2 \ \longrightarrow\ \infty,$$

in particular K is compact because it is the norm limit of compact operators.

(iii) Assume that K is compact. By Theorem 5.52 we can choose ONSs $(\varphi_n)_{\lambda\in\mathbb{N}}$ of H_1 and $(\psi_n)_{\lambda\in\mathbb{N}}$ of H_2 such that $Kx = \sum_{n\in\mathbb{N}} s_n \langle x \,, \varphi_n\rangle \psi_n$ where $s_1 \ge s_2 \ge \dots \ge 0$ are the singular values of K.

If K is a Hilbert-Schmidt operator, then

$$\sum_{n=1}^{N} s_n^2 = \sum_{n=1}^{N} \|K\varphi_n\|^2 \le \|K\|_{\mathrm{H}S}^2 < \infty.$$

Now assume that $\sum_{n=1}^{N} s_n^2 < \infty$ and choose an arbitrary ONB of H_1 containing $(\varphi_n)_{n\in\mathbb{N}}$. It follows that

$$\sum_{\lambda\in\Lambda} \|K\varphi_\lambda\|^2 = \sum_{n=1}^{N} \|K\varphi_n\|^2 \le \|K\|_{\mathrm{H}S}^2 = \sum_{n=1}^{N} s_n^2 < \infty,$$

implying that K is a Hilbert-Schmidt operator. ∎

Lemma 5.56. *The finite-rank operators are dense in the Hilbert-Schmidt operators.*

Proof. Let H be a Hilbert space and $S \in \mathrm{H}S(H)$. In particular, S is compact and there exist ONBs $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ such that $S = \sum_{n=1}^{N} s_n\langle\cdot\,,\varphi_n\rangle\psi_n$. For $M \in \mathbb{N}$ let us define $S_M = \sum_{n=1}^{M} s_n\langle\cdot\,,\varphi_n\rangle\psi_n$. Then $\|S - S_M\|^2 \le \|S - S_M\|_{\mathrm{HS}}^2 = \sum_{n=M+1}^{N} s_n^2 \to 0$ for $M \to \infty$. □

An important class of examples is given in the following theorem.

Theorem 5.57. *Let $H = L_2(0,1)$ and $T \in L(H)$. Then the following is equivalent:*

(i) *T is a Hilbert-Schmidt operator.*

(ii) *There exists a $k \in L_2(0,1)^2$ such that*

$$(Tx)(t) = \int_0^1 k(s,t)x(s)\ \mathrm{d}s.$$

In this case we write T_k for T.
If one of the equivalent conditions holds, then

$$\|T\| = \Big(\int_0^1\int_0^1 |k(s,t)|^2\ \mathrm{d}s.\ \mathrm{d}t\Big)^2 = \|k\|_{L_2(0,1)^2}.$$

Proof. (ii) $\Longrightarrow$ (i) Let $(\mathrm{e}_n)_n$ be an ONB of $L_2(0,1)$. Then also $(\overline{\mathrm{e}_n})_n$ is an ONB of $L_2(0,1)$ (where $\overline{\mathrm{e}_n}$ denotes the to e_n complex conjugated function) and we find

$$\begin{aligned}\sum_{n=1}^{\infty}\|T\,\mathrm{e}_n\|^2 &= \sum_{n=1}^{\infty}\int_0^1\Big|\int_0^1 k(s,t)\,\mathrm{e}_n(s)\ \mathrm{d}s\Big|^2\ \mathrm{d}t = \sum_{n=1}^{\infty}\int_0^1 |\langle k(\cdot,t)\,,\overline{\mathrm{e}_n}\rangle|^2\ \mathrm{d}t\\ &= \int_0^1\sum_{n=1}^{\infty}|\langle k(\cdot,t)\,,\overline{\mathrm{e}_n}\rangle|^2\ \mathrm{d}t \qquad (5.9)\\ &= \int_0^1 \|k(\cdot,t)\|^2\ \mathrm{d}t \qquad (5.10)\\ &= \int_0^1\int_0^1 |k(s,t)|^2\ \mathrm{d}s\ \mathrm{d}t = \|k\|_{L_2(0,1)^2}.\end{aligned}$$

In (5.9) we have used the monotone convergence theorem to exchange the sum and the integral (Theorem A.18) and in (5.10) we used Parseval's equality (Theorem 4.31). It follows that T is a Hilbert-Schmidt operator and that $\|T\|_{\mathrm{H}S} = \|k\|_{L_2(0,1)^2}$.
(i) $\Longrightarrow$ (ii) By the proof we have an isometry

$$\Psi : L_2(0,1)^2 \to \mathrm{H}S(L_2(0,1)), \qquad \Psi k = T_k.$$

We will show that the range of Ψ is dense in $\mathrm{H}S(H)$. By Lemma 5.56 it suffices to show that $\mathrm{rg}(\Psi)$ contains the finite-rank operators. Let T be of finite rank. Then T is of the form $T = \sum_{n=1}^{n_0}\langle\cdot\,,x_n\rangle y_n$ so that for every $f \in H$

$$Tf(t) = \sum_{n=1}^{n_0}\langle f\,,x_n\rangle y_n(t) = \sum_{n=1}^{n_0}\int_0^1 f(s)x_n(s)y_n(t)\ \mathrm{d}s = \int_0^1\Big(\sum_{n=1}^{n_0}x_n(s)y_n(t)\Big)f(s)\ \mathrm{d}s.$$

This shows that $T \in \operatorname{rg}\Psi$. Fix $S \in \mathrm{H}S(H)$ and choose a sequence $(S_n)_{n\in\mathbb{N}}$ in the range of Ψ. Since Ψ_n is an isometry, it follows that $(\Psi^{-1}S_n)_{n\in\mathbb{N}}$ is Cauchy sequence in H, hence its limit exists. Using the continuity of Ψ we find

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \Psi\Psi^{-1}S_n = \Psi\Big(\lim_{n\to\infty} \Psi^{-1}S_n\Big) \in \operatorname{rg}(\Psi). \qquad \square$$

Theorem 5.58. *Let H_1, H_2 be Hilbert spaces.*

(i) $\Big(\mathrm{H}S(H_1,H_2), \|\cdot\|_{\mathrm{H}S}\Big)$ *is a normed spaces. The norm is induced by the inner product*

$$\langle S\,, T\rangle_{\mathrm{H}S} = \sum_{\alpha} \langle S\,\mathrm{e}_\alpha\,, T\,\mathrm{e}_\alpha\rangle, \quad S, T \in \mathrm{H}S(H_1,H_2),$$

for an arbitrary ONB $(\mathrm{e}_\alpha)_{\alpha\in A}$ of H_1.

(ii) *Let $T \in \mathrm{H}S(H_1,H_2)$ and A a bounded linear operator between appropriate Hilbert spaces. Then AT and TA are Hilbert-Schmidt operators and*

$$\|AT\|_{\mathrm{H}S} \le \|A\|\,\|T\|_{\mathrm{H}S}, \qquad \|TA\|_{\mathrm{H}S} \le \|A\|\,\|T\|_{\mathrm{H}S}.$$

(iii) $\mathrm{H}S(H)$ *is a two-sided ideal in $L(H)$.*

Proof. Note that $(a+b)^2 = a^2+b^2+2ab = a^2+b^2-(a-b)^2+a^2+b^2 \le 2(a^2+b^2)$ for $a, b \in \mathbb{R}$.
(i) Let $S, T \in \mathrm{H}S(H_1,H_2)$ and $\lambda \in \mathbb{C}$. Then obviously $\lambda S \in \mathrm{H}S(H_1,H_2)$. To show that $S+T \in \mathrm{H}S(H_1,H_2)$ fix an ONS $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ of H_1. Using the above remark is follows that

$$\sum_{\lambda\in\Lambda} \|(S+T)\,\mathrm{e}_\lambda\|^2 \le \sum_{\lambda\in\Lambda}(\|S\,\mathrm{e}_\lambda\| + \|T\,\mathrm{e}_\lambda\|)^2 \le 2\sum_{\lambda\in\Lambda} \|S\,\mathrm{e}_\lambda\|^2 + \|T\,\mathrm{e}_\lambda\|^2 < \infty.$$

It follows that $\langle\cdot\,,\cdot\rangle_{\mathrm{H}S}$ is well-defined. The properties of an inner product are clear. In particular, $\|T\|_{\mathrm{H}S} = \langle T\,, T\rangle_{\mathrm{H}S}$ for $T \in \mathrm{H}S(H_1,H_2)$.
(ii) Note that

$$\sum_{\lambda\in\Lambda} \|AT\,\mathrm{e}_\lambda\|^2 \le \|A\|^2 \sum_{\lambda\in\Lambda} \|T\,\mathrm{e}_\lambda\|^2 = \|A\|^2\|T\|^2_{\mathrm{H}S},$$

so AT is a Hilbert-Schmidt operator. It follows that $TA = (A^*T^*)^*$ is also a Hilbert-Schmidt operator with norm $\|TA\|_{\mathrm{H}S} = \|(A^*T^*)^*\|_{\mathrm{H}S} = \|A^*T^*\|_{\mathrm{H}S} \le \|A^*\|\,\|T^*\|_{\mathrm{H}S} \le= \|A\|\,\|T\|_{\mathrm{H}S}$.
(iii) is a consequence of (i) and (ii). $\square$

5.6 Polar decomposition

Theorem 5.59. *Let H be a Hilbert space and $T \in L(H)$ a selfadjoint operator with $T \ge 0$. Then there exists exactly one $R \in L(H)$ such that $R \ge 0$ $R^2 = T$.*
In addition, if $S \in L(H)$ commutes with T, then S commutes with R.

the operator R is called the *root* of T and is denoted by $\sqrt{T}$.

Proof. Without restriction we can assume $\|T\| \leq 1$, hence $0 \leq T \leq \mathrm{id}$. Now assume that a solution $R \in \mathrm{L}(H)$ of $R^2 = T$ exists. Let $A := \mathrm{id} - T$ and $X := \mathrm{id} - R$. Note that

$$\mathrm{id} - A = T = R^2 = (1 - X)^2 = \mathrm{id} - 2X + X^2.$$

Note that $0 \leq R \leq \mathrm{id}$ if and only if $0 \leq X \leq \mathrm{id}$. Hence R is a non-negative solution of $R^2 = T$ if and only if X is a non-negative solution of

$$X = \frac{1}{2}(A + X^2). \tag{5.11}$$

Step 1. Construction of a solution of (5.11).
We define

$$X_0 := \mathrm{id}, \qquad X_n := \frac{1}{2}(A + X_{n-1}^2), \quad n \in \mathbb{N}.$$

Note that every X_n is a polynomial in A with positive coefficients and that $X_n X_m = X_m X_n$ for all $n, m \in \mathbb{N}$. Since A is positive, this implies that all X_n are positive. We will show the following properties of the sequence $(X_n)_{n\in\mathbb{N}}$ by induction.

(i) $X_n - X_{n-1}$ is a polynomial in A with positive coefficients, so that in particular $X_n - X_{n-1} \geq 0$.

(ii) $\|X_n\| \leq 1$.

All assertions are clear in the case $n = 0$ (with $X_{-1} := 0$). Now assume that the assertions are true for some $n \in \mathbb{N}$. Note that

$$\begin{aligned} X_{n+1} - X_n &= \frac{1}{2}(A + X_n^2) - \frac{1}{2}(A + X_{n-1}^2) = \frac{1}{2}(X_n^2 - X_{n-1}^2) \\ &= \frac{1}{2}(X_n - X_{n-1})(X_n + X_{n-1}). \end{aligned}$$

Since by induction hypothesis both terms in the second line are polynomials in A with positive coefficients, (i) is proved for $n + 1$. (ii) follows from $\|X_{n+1}\| \leq \frac{1}{2}(\|A\| + \|X_{n+1}\|) \leq 1$.

Since $(X_n)_{n\in\mathbb{N}}$ is uniformely bounded monotonically increasing sequence in, there exists an $X \in L(H)$ such that $X = \operatorname*{s\text{-}lim}_{n\to\infty} X_n$ and $\|X\| \leq \liminf_{n\to\infty} \|X_n\| \leq 1$ (see Exercise 4.25).

Now let $S \in L(H)$ with $ST = TS$. By definition of A, then also $SA = AS$ and $X_n S = X_n S$ for all X_n since the X_n are polynomials in A. For every $x \in H$ we therefore obtain

$$0 \leq \|SXx - XSx\| = \lim_{n\to\infty} \|SX_n x - X_n Sx\| = \lim_{n\to\infty} \|SX_n x - SX_n x\| = 0.$$

Since all X_n commute with T, it follows that $X_n X = X_n X$ for all $n \in \mathbb{N}$, so that for all $x \in X$

$$\|(X_n^2 - X^2)x\| = \|(X_n - X)(X_n + X)x\| \leq 2\|(X_n - X)^2 x\| \longrightarrow 0, \quad n \to \infty,$$

which shows that $X^2 = \operatorname*{s\text{-}lim}_{n\to\infty} X_n^2$. Therefore X solves (5.11) because

$$X = \operatorname*{s\text{-}lim}_{n\to\infty} X_n = \operatorname*{s\text{-}lim}_{n\to\infty} \frac{1}{2}(A + X_n^2) = \frac{1}{2}(A + \operatorname*{s\text{-}lim}_{n\to\infty} X_n^2) = \frac{1}{2}(A + X^2).$$

Setting $R = \mathrm{id} - X$ we obtain a bounded selfadjoint solution of $R^2 = T$ with $0 \le R \le \mathrm{id}$.

Step 2. Uniqueness of the solution.
Let $R' \in L(H)$ be solution of $R^2 = T$ with $R' \ge 0$. Then R and R' commute because

$$R'A = R'(R')^2 = (R')^2R' = AR'.$$

It follows that

$$(R - R')R(R - R') + (R - R')R'(R - R') = (R^2 - R'^2)(R - R') = 0.$$

Since both operators on the left hand side are non-negative, it follows that both of them are 0 and therefore

$$(R - R')^4 = (R - R')R(R - R') - (R - R')R'(R - R') = 0.$$

Since $R - R'$ is normal, it follows that $\|(R - R')\|^4 = \|(R - R')\|^4$. □

Corollary 5.60. *If $S, T \in L(H)$ are positive and $ST = TS$, then also ST is positive.*

Proof. By Theorem 5.59 the root of T exists, is selfadjoiunt and commutes with S. Hence for all $x \in H$

$$\langle STx\,, x\rangle = \langle S\sqrt{T}\sqrt{T}x\,, x\rangle = \langle \sqrt{T}S\sqrt{T}x\,, x\rangle = \langle S\sqrt{T}x\,, \sqrt{T}x\rangle \ge 0.$$ □

Definition 5.61. For $T \in L(H)$ we define $|T| := (T^*T)^{\frac{1}{2}}$.

Definition 5.62. Let H_1, H_2 be Hilbert spaces and $U \in L(H_1, H_2)$. U is called a *partial isometry* if $U|_{(\ker U)^\perp}$ is an isometry. $\ker U \perp$ is called its *initial space*.

Note that U is an partial isometry if and only if

$$U|_{(\ker U)^\perp} : (\ker U)^\perp \to \mathrm{rg}(U)$$

is unitary.

Theorem 5.63 (Polar decomposition). *Let H_1, H_2 be Hilbert spaces and $T \in L(H_1, H_2)$. Then there exists a partial isometry $U \in L(H_1, H_2)$ such that $T = U|T|$. If in addition the initial space of U is $(\ker T)^\perp$, then U is unique.*

Proof. Note that $\|\,|T|x\,\|^2 = \|Tx\|^2$ for all $x \in H_1$ because

$$\|\,|T|x\,\|^2 = \langle |T|x\,, |T|x\rangle = \langle (T^*T)^{\frac{1}{2}}x\,, (T^*T)^{\frac{1}{2}}x\rangle = \langle T^*Tx\,, x\rangle = \langle Tx\,, Tx\rangle = \|Tx\|^2.$$

We define

$$U : \mathrm{rg}(|T|) \to \mathrm{rg}(T), \qquad U(|T|x) = Tx.$$

U is well-defined because for $x, y \in H_1$ with $|T|x = |T|y$ it follows that $\|Tx - Ty\| = \||T|x - |T|y\| = 0$ hence $Tx = Ty$. U is and isometry because $\|Tx\| = \|\,|T|x\,\|$ for all $x \in H$ as shown above. In particular, $\|U\| = 1$ and has a unique continuous extension to $\overline{\mathrm{rg}(|T|)} \to \overline{\mathrm{rg}(T)}$. Now we extend U to H_1 by setting $Ux = 0$ for all $x \in \overline{\mathrm{rg}(|T|)}^\perp = \ker(|T|) = \ker T$. □

Appendix A

$\mathscr{L}_p$ spaces

Spaces of integrable functions play an important role in applications. As the norm of a function $f : [a.b] \to \mathbb{K}$ one could consider

$$\|f\| := \int_a^b |f(t)| \, \mathrm{d}t,$$

or more generally, for some $1 \le p < \infty$

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

It can be shown that $\|\cdot\|_p$ is a norm on $C([a,])$. However the space of continuous functions $C([a,b])$ is not complete for the norm $\|\cdot\|_1$. For example, let

$$f_n : [0,2] \to \mathbb{R}, \qquad f_n(t) = \begin{cases} t^n, & 0 \le t \le 1, \\ 1, & 1 < t \le 2. \end{cases}$$

All f_n are continuous and it is easy to check that $\|f_n - f_m\|_1 \to 0$ for $n, m \to \infty$. So the f_n form a Cauchy sequence, but it is not convergent. (If it were, then there must exist a continuous function g such that

$$\int_0^2 |f_n(t) - g(t)| \, \mathrm{d}t = \int_0^1 |f_n(t) - g(t)| \, \mathrm{d}t + \int_1^2 |f_n(t) - g(t)| \, \mathrm{d}t \longrightarrow 0$$

for $n \to \infty$. Hence $g(t) = 0$ for $t \in (0,1)$ and $g(t) = 0$ for $t \in (1,2)$ which is impossible for a continuous function by the intermediate value theorem.

If we extend the space of functions to the Riemann integrable functions $\mathcal{R}([a,b])$, then the sequence above does converge to $\chi_{[1,2]}$. But there are several other problems with the space of Riemann integrabel functions.

For example, let $\mathbb{Q} \cap [0,1] = \{q_n : n \in \mathbb{N}\}$. Then all characteristic functions $\chi_n := \chi_{\{q_1,\dots,q_n\}}$ are Riemann integrable, $\|\chi_n\|_1 = 0$, the χ_n form a Cauchy sequence, the pointwise limit exists and is

$\chi_{\mathbb{Q}\cap[0,1]}$ which is *not* Riemann integrabel. This example shows that $\|\cdot\|_1$ is only a seminorm on $\mathcal{R}([a,b])$, that Cauchy sequences do not need to converge and that in general pointwise limit and integral cannot be exchanged. The pointwise limit of a sequence of Riemann integrable functions does not need to be Riemann integrable.

Recall that the Riemann integral of a function $f:[a,b]\to\mathbb{R}$ is obtained as the limit of Riemann sums when the interval $[a,b]$ is divided in small pieces. Lebesgue's approach is to divide the range of the function in small pieces and then measure the "size" of the pre-image. Hence admissible are functions whose pre-images of intervals can be measured in some sense.

A.1 A reminder on measure theory

Definition A.1. Let T be a set and $\Sigma \subset \mathbb{P}T$ a family of subsets of T.

(i) $\Sigma \subset \mathbb{P}T$ is called a *ring* if for all $A, B \in \Sigma$ also $A \cup B$ and $A \setminus B$ belong to Σ.
(ii) $\Sigma \subset \mathbb{P}T$ is called a *σ-ring* if it is a ring and $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$ for all $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$.
(iii) $\Sigma \subset \mathbb{P}T$ is called an *algebra* if Σ is a ring and $T \in \Sigma$, that is

  (a) $\emptyset \in \Sigma$,
  (b) $A \in \Sigma \implies T \setminus A \in \Sigma$,
  (c) $A, B \in \Sigma \implies A \cup B \in \Sigma$.

(iv) $\Sigma \subset \mathbb{P}T$ is called a *σ-algebra* if it is a algebra and $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$ for all $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$.

Note that for $A, B \in \Sigma$ also $A \cap B = A \setminus (T \setminus B) \in \Sigma$ if Σ is an algebra.

Remark. Σ is indeed a *ring* in the algebraic sense if one sets $A + B := (A \cup B) \setminus (A \cap B)$ and $A \cdot B := A \cap B$.

Definition A.2. Let T be a set with a σ-algebra Σ. A *measure* on Σ is a function $\mu : \Sigma \to [0,\infty]$ such that

(i) $\mu(\emptyset) = 0$,
(ii) $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with pairwise disjoint $A_n \implies \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Obviously, the intersection of rings is again a ring and $\mathbb{P}T$ is a ring. Hence, given a family $\mathcal{U}$ of subsets of T, there exists a smallest ring containing $\mathcal{U}$, namely the intersection of all rings that contain $\mathcal{U}$. This ring is called the *ring generated by* $\mathcal{U}$. Analogously the σ-ring, the algebra and the σ-algebra generated by $\mathcal{U}$ are obtained.

Example A.3. The smallest σ-algebra containing all intervals of $\mathbb{R}$ is called the *Borel sets*.

More generally, let $(T, \mathcal{O})$ be a topological space. Then the Borel sets is the σ-algebra generated by $\mathcal{O}$.

The aim is to assign a measure $\mu(U)$ to every Borel set $U \subseteq \mathbb{R}$ such that the measure of intervals is its length.

Definition A.4. Let T be a set with a ring Σ of sets. A pre-measure μ on (T, Σ) is a function

$$\mu : \Sigma \to [0, \infty]$$

such that $\mu(\emptyset) = 0$ and

$$(A_n)_{n\in\mathbb{N}} \subseteq \Sigma, \ \text{ pairwise disjoint and } \bigcup_{n\in\mathbb{N}} A_n \in \Sigma \quad \Longrightarrow \quad \mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

Note that a pre-measure is monotonic: if $A, B \in \Sigma$ and $A \subseteq B$, then $\mu(A) \le \mu(B)$.

Example A.5. Let $\mathcal{A}$ be the set of all finite unions of finite intervals and define $\mu(A) := \int_{\mathbb{R}} \chi_A \,\mathrm{d}x$ where χ_A is the characteristic function of A. Then μ is a pre-measure on $\mathcal{A}$.

Proof. Obviously $\mathcal{A}$ is a ring, for every $A \in \mathcal{A}$ the characteristic function χ_A is Riemann integrable and $\mu(\emptyset) = 0$. Now let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint with $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. Obviously, $\mu(\bigcup_{n=1}^n A_n) = \sum_{n=1}^n \mu(A_n)$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$ define $B_n := A \setminus (A_1 \cup \dots \cup A_n)$. Obviously, $B_n \in \mathcal{A}$ and

$$\mu(A) = \mu(\bigcup_{k=1}^n A_k) + \mu(B_n).$$

To prove that $\mu(A) = \mu(\bigcup_{k=1}^\infty A_k)$ it suffices to show that $\lim\limits_{n\to\infty} \mu(B_n) = 0$. Fix $\varepsilon > 0$. Since $B \in \mathcal{A}$ there exists an compact set $C_n \subseteq B_n$ with $\mu(B_n \setminus C_n) < 2^{-n}\varepsilon$. Let $D_n := C_1 \cap \dots \cap C_n$. Then all D_n are compact and $D_n \subseteq C_n \subseteq B_n$. By construction, $B_1 \supseteq B_2 \supseteq B_3 \dots$, hence

$$\begin{aligned}\mu(B_n \setminus D_n) &= \mu(B_n \setminus (C_1 \cap \dots \cap C_n) = \mu(\bigcup_{k=1}^n (B_n \setminus C_k)) \le \mu(\bigcup_{k=1}^n (B_k \setminus C_k)) \\ &\le \sum_{k=1}^n \mu(B_k \setminus C_k) < \sum_{k=1}^n 2^{-k}\varepsilon = \varepsilon.\end{aligned}$$

On the other hand, $\bigcap_{n=1}^\infty D_n \subseteq \bigcap_{n=1}^\infty B_n = \emptyset$. Since all D_n are compact and $D_1 \supseteq D_2 \supseteq \dots$, there exists an $K \in \mathbb{N}$ such that $D_n = \emptyset$, $n \ge K$. Hence $\mu(B_n) = \mu(B_n \setminus D_n) < \varepsilon$ for all $n \ge k$. □

In order to measure all Borel sets, we have to show that the pre-measure of Example A.5 can be extended to the Borel sets.

Theorem A.6 (Hahn). *Let T be a set, $\mathcal{A}$ a ring on T and $\tilde{\mu}$ a pre-measure on $\mathcal{A}$. Let $\Sigma(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$. Then*

$$\mu(A) := \inf \Big\{ \sum_{n=1}^\infty \tilde{\mu}(A_n) \ : \ (A_n)_{n\in\mathbb{N}} \subseteq \Sigma, \ A \subseteq \bigcup_{n=1}^\infty A_n \Big\}.$$

If $\tilde{\mu}$ is σ-finite, i. e., if there exist $A_n \in \Sigma$ with $\mu(A_n) < \infty$ and $T = \bigcup_{n=1}^\infty A_n$, then the extension μ is unique.

For the proof, we first show that $\tilde{\mu}$ can be extended to an *outer measure* μ^* on $\mathbb{P}T$. Then, by the lemma of Carathéodory, the restriction of the outer measure to the set of the μ^*-measurable sets is a measure.

Definition A.7. Let $\mathcal{A}$ be a σ-algebra on T. An *outer measure* on $\mathcal{A}$ is a function $\mu^* : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu^*(\emptyset) = 0$,

(ii) $A, B \in \mathcal{A}$ with $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

(iii) $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A} \implies \mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

A set $A \in \mathcal{A}$ is called μ^*-measurable if

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \setminus A), \qquad Z \in \mathcal{A}.$$

Lemma A.8. *With the assumptions of Hahn's theorem (Theorem A.6)*

$$\mu^*(A) := \inf \Big\{ \sum_{n=1}^{\infty} \tilde{\mu}(A_n) \ : \ (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A},\ A \subseteq \bigcup_{n=1}^{\infty} A_n \Big\}$$

defines an outer measure on $\mathbb{P}T$. In addition, $\mu(A) = \mu^*(A)$ for $A \in \mathcal{A}$.

Proof. Properties (i) and (ii) of an outer measure are clear. Now let $(A_n)_{n\in\mathbb{N}} \subseteq \mathbb{P}T$ and $\varepsilon > 0$. Then there exists a family $(B_n^j)_{n,j\in\mathbb{N}} \subseteq \mathcal{A}$ such that $A_n \subseteq \bigcup_{j=1}^{\infty} B_n^j$ and

$$\sum_{j=1}^{\infty} \mu(B_n^j) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}, \qquad n \in \mathbb{N}.$$

By construction $A := \bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n,j\in\mathbb{N}} B_n^j$ and

$$\mu^*(A) \leq \sum_{n,j=1}^{\infty} \mu(B_n^j) \leq \sum_{n=1} \mu^*(A_n) + \varepsilon.$$

Note that $(B_n^j)_{n,j\in\mathbb{N}}$ is countable, hence we have proved $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.
Now let $A \in \mathcal{A}$. Clearly, $\mu(A) \leq \mu^*(A)$ holds. Now fix $\varepsilon > 0$ and choose $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \varepsilon. \qquad n \in \mathbb{N}.$$

Since $A = \bigcup_{n\in\mathbb{N}} (A_n \cap A)$ and μ is a pre-measure on $\mathcal{A}$, it follows that

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n \cap A) \leq \sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \varepsilon. \qquad n \in \mathbb{N}.$$

Since this is true for all $\varepsilon > 0$, it follows that $\mu(A) \leq \mu^*(A)$. □

Lemma A.9 (Carathéodory). *Let μ^* be an outer measure on $\mathbb{P}T$. Then the set $\mathcal{M}$ of all μ^*-measurable sets is a σ-algebra and μ^* is a measure on $\mathcal{M}$.*

Proof. ...... □

Proof of Theorem A.6. It suffices to show that the set of the μ^*-measurable sets contains $\mathcal{A}$ ... □

Hahn's theorem gives the desired measure on the Borel sets.

Definition A.10 (Lebesgue completion). Let (T, Σ, μ) be a measure space. $A \subseteq T$ is a *zero set* if there exists a $B \subseteq T$ with $\mu(B) = 0$ and $A \subseteq B$ (note that A does not necessarily belong to Σ). The σ-algebra generated by Σ and the zero sets is called the *Lebesgue completion*.

The measure on the completion of the Borel sets in $\mathbb{R}$ is the *Lebesgue measure*, usually denoted by λ. .

A.2 Integration

In the following, I is always an interval in $\mathbb{R}$.

Definition A.11. A function $f : I \to \mathbb{R}$ is called *measurable* if for every $(a, b) \subseteq \mathbb{R}$ its preimage $f^{-1}((a, b))$ is a Borel set.

More generally, let (T, Σ_T, μ_T) and (S, Σ_S, μ_S) be measure spaces. A function $f : T \to S$ is called *measurable* if for every $U \in \Sigma_S$ also $f^{-1}(U) \in \Sigma_T$.

Example A.12. Let E be a Borel set. Then the characteristic function χ_E is measurable.

Definition A.13. Let (T, Σ, μ) be a measure space. A function $f : T \to \mathbb{C}$ is called a *simple function* if there are $E_k \in \Sigma$ and $\alpha_k \in \mathbb{C}$ such that

$$f = \sum_{k=1}^{n} \alpha_k \chi_{E_k}.$$

It is easy to see that simple functions are measurable. Note, however, that the sum representation of a simple function is not unique.

The next theorem lists important properties of measurable functions.

Theorem A.14. *Let I be an interval in $\mathbb{R}$ and f_n, f, $g : I \to \mathbb{R}$ be functions.*

(i) *If f and g are measurable, then so are $f + g$, fg, $\frac{f}{g}$ (if it exists), $\max\{f, g\}$ and $\min\{f, g\}$.*

(ii) *Every continuous function is measurable.*

(iii) *If all f_n are measurable and f is their pointwise limit (i. e. $f(t) = \lim_{n\to\infty} f_n(t) = f(t)$, $t \in I$), then f is measurable.*

(iv) *If f is measurable, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of simple functions that converges pointwise to f. If in addition $f \geq 0$, then the sequence can be chosen such that $\varphi_n(t) \nearrow f(t)$, $t \in I$.*

(v) *If f is measurable and bounded, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of simple functions that converges uniformly to f.*

The theorem says that the set of the measurable functions are a vector space and that it is stable under taking pointwise limits.

Next we introduce the integral for positive functions.

Definition A.15. Let (T, Σ, μ) be a measure space.

(i) Let $f = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ with $E_k \in \Sigma$ and $\alpha_k \in [0, \infty]$ a simple function. We define its *integral* as

$$\int_T f \,\mathrm{d}\mu = \sum_{k=1}^{n} \alpha_k \mu(E_k).$$

(ii) Let $f : T \to [0, \infty]$ be a measurable function. Choose a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of simple functions with $\varphi_1 \leq \varphi_2 \leq \dots$ that converges pointwise to f. We define the *integral* of f by

$$\int_T f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_T \varphi_n \,\mathrm{d}\mu.$$

Of course, it must be proved that the integral in (i) does not depend on the sum representation of the simple function, and that the limit in (ii) does not depend on the chosen sequence of simple functions.

Definition A.16. Let I be an interval in $\mathbb{R}$.

(i) A function $f : I \to [0, \infty]$ is called (*Lebesgue*) *integrable* if $\int_I f \,\mathrm{d}\lambda < \infty$.

(ii) A function $f : I \to \mathbb{R}$ is called (*Lebesgue*) *integrable* if $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$ are integrable. In this case

$$\int_I f \,\mathrm{d}\lambda := \int_I f^+ \,\mathrm{d}\lambda - \int_I f^- \,\mathrm{d}\lambda.$$

(iii) A function $f : I \to \mathbb{C}$ is called (*Lebesgue*) *integrable* if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are integrable. In this case

$$\int_I f \,\mathrm{d}\lambda := \int_I \mathrm{Re}(f) \,\mathrm{d}\lambda + \mathrm{i} \int_I \mathrm{Im}(f) \,\mathrm{d}\lambda.$$

The Lebesgue integral has the following properties.

Lemma A.17. (i) *If f, g are Lebesgue integrable and $\alpha \in \mathbb{K}$, then*

$$\int_I (\alpha f + g) \,\mathrm{d}\lambda = \alpha \int_I f \,\mathrm{d}\lambda + \int_I g \,\mathrm{d}\lambda.$$

(ii) *If f is Lebesgue integrable then*

$$\Big|\int_I f \,\mathrm{d}\lambda\Big| \le \int_I |f| \,\mathrm{d}\lambda.$$

For Lebesgue integrals much stronger convergence theorems hold than for the Riemann integral. The most important convergence theorems are the following.

Theorem A.18 (Monotone convergence theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : I \to [0,\infty]$ with $0 \le f_1 \le f_2 \le \cdots$. Then*

$$f : I \to [0,\infty], \qquad f(t) := \lim_{n\to\infty} f_n(t)$$

is measurable and

$$\int_I f \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_I f_n \,\mathrm{d}\lambda.$$

The monotone convergence theorem is also called Beppo Levi theorem.

A sequence $(f_n)_{n\in\mathbb{N}}$ converges to f λ-a.e. if the set, where the sequence does not converge to f, has measure zero.

Theorem A.19 (Dominated convergence theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and assume that there exists a measurable function f such that $f(t) = \lim\limits_{n\to\infty} f_n(t)$ λ-a.e. If there exists an integrable function g with $|f_n| \le g$ λ-a.e., then f is integrable and*

$$\int_I f \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_I f_n \,\mathrm{d}\lambda.$$

The dominated convergence theorem is also called Lebesgue's convergence theorem.

Theorem A.20 (Fatou's lemma). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions and assume that there exists a measurable function f such that $f(t) = \lim\limits_{n\to\infty} f_n(t)$ λ-a.e. If there exists a C such that $\int_I f_n \,\mathrm{d}\lambda \le C$ for all $n \in \mathbb{N}$, then f is integrable and*

$$\int_I f \,\mathrm{d}\lambda \le \liminf_{n\to\infty} \int_I f_n \,\mathrm{d}\lambda.$$

A.3 $\mathscr{L}_p$ spaces

In the following, Ω is always an open subset of $\mathbb{R}^n$.

Definition A.21. For $1 \le p < \infty$ we define

$$\mathscr{L}_p(\Omega) := \Big\{ f : \Omega \to \mathbb{K} : f \text{ measurable}, \int_\Omega |f|^p \,\mathrm{d}\lambda < \infty \Big\},$$

$$\|f\|_p := \Big(\int_\Omega |f|^p \Big)^{\frac{1}{p}}, \qquad f \in \mathscr{L}_p(\Omega).$$

Definition A.22. For a measurable function $f : \Omega \to \mathbb{K}$ we define the essential supremum

$$\begin{aligned} \operatorname{ess\,sup} f &:= \inf\{C \in \mathbb{R} : f(t) \le C \text{ for } \lambda\text{-almost all } t\}, \\ \operatorname{ess\,inf} f &:= -\operatorname{ess\,sup}(-f) \end{aligned}$$

and

$$\begin{aligned} \mathscr{L}_\infty(\Omega) &:= \{f : \Omega \to \mathbb{K} : f \text{ measurable, } \operatorname{ess\,sup}|f| < \infty\}, \\ \|f\|_\infty &:= \operatorname{ess\,sup}|f|, \qquad f \in \mathscr{L}_\infty(\Omega). \end{aligned}$$

It is easy to see that $\mathscr{L}_\infty(\Omega)$ is a vector space. For $1 \le p < \infty$ this follows from

$$\begin{aligned} \int_\Omega |f+g|^p \,\mathrm{d}\lambda &\le \int_\Omega (|f| + |g|)^p \,\mathrm{d}\lambda \le \int_\Omega (2\max\{|f|,\, |g|\})^p \,\mathrm{d}\lambda \\ &\le 2^p \int_\Omega \max\{|f|^p,\, |g|^p\} \,\mathrm{d}\lambda \le 2^p \int_\Omega |f|^p + |g|^p \,\mathrm{d}\lambda \\ &= 2^p(\|f\|_p^p + \|g\|_p^p) < \infty. \end{aligned}$$

That $\lambda f \in \mathscr{L}_p$ for $\lambda \in \mathbb{K}$ and $f \in \mathscr{L}_p$ is clear.

That $\|\cdot\|_p$ is a seminorm on $\mathscr{L}_p$ follows from the Minkowski inequality:

Theorem A.23 (Minkowski inequality). *Let $1 \le p \le \infty$ and $f, g \in \mathscr{L}_p(\Omega)$. Then*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

For the proof of the Minkowski inequality, Hölder's inequality is used.

Theorem A.24. *Let $1 \le p \le \infty$ and q the conjugated exponent, i. e., $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathscr{L}_p(\Omega)$ and $g \in \mathscr{L}_q(\Omega)$, then $fg \in \mathscr{L}_1(\Omega)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Note that $\mathscr{L}_p(\Omega)$ is only a seminormed space, because there are non-zero functions f with $\|f\|_p = 0$.

Theorem A.25. *$(\mathscr{L}_p(\Omega),\ \|\cdot\|_p)$ is complete.*

Definition A.26. Let $\mathcal{N}_p(\Omega) := \{f \in \mathscr{L}_p : \|f\|_p = 0\}$. Then $L_p := \mathscr{L}_p(\Omega)/\mathcal{N}(\Omega)$ is a complete normed space.

Usually an equivalence class $[f] \in L_p(\Omega)$ is simply denoted by f.

Often one is interested in dense subspaces of $L_p(\Omega)$.

Theorem A.27. *Let $1 \le p < \infty$ and $\Omega \in \mathbb{R}^n$ open. Then the* test functions

$$C_0^\infty(\Omega) := \mathscr{D}(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) \subset \Omega \ \textit{ is compact}\}.$$

form a dense subset of $L_p(\Omega)$.

Appendix B

Exercises

Exercises for Chapter 1

1. **Banach's fixed point theorem.** Let M be a metric space. A map $f : M \to M$ is called a *contraction* if there exists a $\gamma < 1$ such that

$$d(f(x),\, f(y)) \le \gamma\, d(x,\, y), \qquad x, y \in M.$$

Show that every contraction f on a complete normed space M has exactly one fixed point, that is, there exists exactly one $x_0 \in M$ such that $f(x_0) = x_0$.

2. Let X be a normed space. Then the following is equivalent:

   (i) X is complete.

   (ii) Every absolutely convergent series in X converges in X.

3. Let X be a normed space. Show:

   (a) Every finite-dimensional subspace of X is closed.

   (b) If V is a finite-dimensional subspace of X and W is a closed subspace of X, then

$$V + W := \{v + w : v \in V,\ w \in W\}$$

   is a closed subspace of X.

4. Let T be a set and $\ell_\infty(T)$ be the space of all functions $x : T \to \mathbb{K}$ with

$$\|x\|_\infty := \sup\{|x(t)| : t \in T\} < \infty.$$

Show that $(\ell_\infty(T), \|\cdot\|_\infty)$ is a Banach space.

5. Let the sequence spaces d, c_0, c be defined as in Example 1.15.

   (a) Show that $(c_0, \|\cdot\|_\infty)$ and $(c, \|\cdot\|_\infty)$ are Banach spaces.

(b) Show that $(d, \|\cdot\|_\infty)$ is a normed space, but that it is not complete.

6. Sea X un espacio normado con $\dim X \geq 1$ y S, T operadores lineales en X tales que $ST - TS = id$. Muestre que al menos uno de estos operadores no es acotado. Ayuda: Muestre que $ST^{n+1} - T^{n+1}S = (n+1)T^n$.

7. Sean X y Y espacios normados con X de dimensión finita. Muestre que toda función lineal $T : X \to Y$ es acotada.

8. (a) Sea $X = C([a, b])$ con la norma $\|\cdot\|_\infty$. Muestre que

$$T : X \to \mathbb{C}, \qquad Tx = \int_a^b x(t)\ \mathrm{d}t$$

es un operador lineal y acotado. ¿Cuál es su norma?

(b) Ahora considere X con la norma

$$\|x\|_p := \Big(\int_a^b |x(t)|^p\ \mathrm{d}t\Big)^{1/p}, \qquad x \in X,$$

para $1 \leq p < \infty$. ¿Sigue siendo T acotado? Si es así, calcule su norma.

9. Sea $1 \leq p < \infty$. Para $z = (z_n)_{n\in\mathbb{N}} \in \ell_\infty$ sea $T : \ell_p \to \ell_p$ definido por $(Tx)_n = x_n z_n$ para $x = (x_n)_{n\in\mathbb{N}} \in \ell_p$. Muestre que $T \in L(\ell_p)$ y calcule $\|T\|$.

Exercises for Chapter 2

1. Demuestre el teorema de Hahn-Banach para espacios vectoriales complejos.

   Sugerencia: Para un espacio vectorial sobre los complejos X muestre que:

   (a) Sea $\varphi : X \to \mathbb{R}$ un funcional $\mathbb{R}$-lineal, entonces

   $$V_\varphi : X \to \mathbb{C}, \quad V_\varphi(x) := \varphi(x) - \mathrm{i}\varphi(\mathrm{i}x),$$

   es un funcional $\mathbb{C}$-lineal sobre X con $\operatorname{Re} V_\varphi = \varphi$.

   (b) Sea $\lambda : X \to \mathbb{C}$ un funcional $\mathbb{C}$-lineal con $\operatorname{Re}\lambda = \varphi$, entonces $V_\varphi = \lambda$.

   (c) Sea p un funcional sublineal sobre X y φ, V_φ definido como en el punto anterior, entonces

   $$|\varphi(x)| \leq p(x) \iff |V_\varphi(x)| \leq p(x), \qquad x \in X.$$

   (d) $\|\varphi\| = \|V_\varphi\|$.

2. En $X = \ell_2(\mathbb{N})$ considere el subespacio

$$U = \{(x_n)_{n\in\mathbb{N}} : x_n = 0 \text{ excepto para un número finito de índices } n\}.$$

Sea V el complemento algebraico de U en X, i. e., U es un subespacio tal que $U + V = X$ y $U \cap V = \{0\}$. Muestre que

$$\varphi : X \to \mathbb{K}, \quad \varphi(x) = \sum_{n=0}^{\infty} u_n \qquad \text{para } x = u + v \text{ con } u \in U,\ v \in V.$$

es un funcional lineal bien definido y no acotado.

3. (a) Sea $c \subseteq \ell_\infty$ el conjunto de las sucesiones convergentes. Muestre que el funcional

$$\varphi_0 : c \to \mathbb{K}, \quad x = (x_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} x_n$$

es continuo y calcule su norma.

(b) Sea $\ell_\infty(\mathbb{N}, \mathbb{R})$ el conjunto de todas las sucesiones acotadas en $\mathbb{R}$ con la norma del supremo. Muestre que existe $\varphi \in (\ell_\infty(\mathbb{N}, \mathbb{R}))'$ tal que

$$\liminf_{n\to\infty} x_n \le \varphi(x) \le \limsup_{n\to\infty} x_n, \qquad x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty.$$

4. Sea X un espacio normado, $f : X \to \mathbb{K}$ un funcional lineal no nulo y $K = \ker f$

(a) Muestre que $\dim(X/K) = 1$.

(b) Muestre que f es continuo si y solo si $\ker f$ es cerrado.

5. Un *isomorfismo entre espacios normados* X y Y es un homeomorfismo lineal. Pruebe las siguientes afirmaciones.

(a) Si $T : X \to Y$ es un isomorfismo [isométrico] entre los espacios normados X y Y, entonces $T' : Y' \to X'$ es un isomorfismo [isométrico]. Si X y Y son espacios de Banach, el converso también vale.

(b) Si un espacio normado Y es isomorfo a un espacio de Banach reflexivo X, entonces Y es un espacio de Banach reflexivo.

6. Sea X un espacio normado separable y $(x'_n)_{n\in\mathbb{N}}$ una sucesión acotada en X'. Entonces existe una subsucesión $(x'_{n_k})_{k\in\mathbb{N}}$ y $x'_0 \in X'$ tal que

$$\lim_{k\to\infty} x'_{n_k}(x) = x'_0(x), \qquad x \in X.$$

Es cierto esto sin la hipótesis de que X sea separable?

7. Sea X un espacio normado y M un subespacio de X. Sea

$$L = \{f \in X' \mid f(x) = 0 \ \text{ para todo } x \in M\}.$$

Muestre que L es un subespacio cerrado de X' y que M' es isométricamente isomorfo a X'/L.

8. Sea X un espacio compacto, $C_{\mathbb{R}}(X)$ el conjunto de funciones continuas real-evaluadas sobre X y $Y \subset X$ un subconjunto cerrado.

(a) Considere el mapa $\rho : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ definido por $\rho(f) = f|_Y$. Muestre que $I := \ker(\rho)$ es un subespacio cerrado de $C_{\mathbb{R}}(X)$.

(b) Sea $\widetilde{\rho} : C_{\mathbb{R}}(X)/I \to C_{\mathbb{R}}(Y)$ el mapa inducido en el espacio cociente. Pruebe que $\widetilde{\rho}$ es una isometría.

(c) Demuestre que $\operatorname{rg}(\rho)$ es completo.

(d) Use el teorma de Stone-Weierstraß para concluir el teorema de Tietze: Sea X un espacio compacto de Hausdorff y $Y \subseteq X$ un subconjunto cerrado. Entonces cada función continua $f : Y \to \mathbb{R}$ tiene una extensión continua $\widetilde{f} : X \to \mathbb{R}$ con $\|\widetilde{f}\|_{C(X)} = \|f\|_{C(Y)}$.

9. Muestre que en l_1 la convergencia débil y la convergencia en norma coinciden.

Exercises for Chapter 3

1. (a) Todo espacio métrico completo con infinitos elementos y ningún punto aislado es no enumerable.

   (b) Toda base algebraica de un espacio de Banach infinito dimensional es no enumerable.

2. (a) Sea X un espacio de Banach, Y un espacio normado y $(T_n)_{n\in\mathbb{N}} \subseteq L(X,Y)$. Suponga que para todo $x \in X$ el límite $Tx := \lim_{n\in\mathbb{N}} T_n x$ existe. Entonces $T \in L(X,Y)$.

   (b) Sean X, Y espacios de Banach, Y reflexivo, y $(T_n)_{n\in\mathbb{N}} \subseteq L(X,Y)$ tal que $(\varphi(T_n x))_{n\in\mathbb{N}}$ converge para todo $x \in X$ y $\varphi \in Y'$. Entonces existe un $T \in L(X,Y)$ tal que $T_n \xrightarrow{w} T$.

3. Muestre que la hipótesis de completitud en el principio de acotación uniforme es necesaria.

4. Sea $[a,b] \subseteq \mathbb{R}$, $n \in \mathbb{N}$ y tome $a \le t_1^{(n)} < \cdots < t_n^{(n)} \le b$ y $\alpha_k^{(n)} \in \mathbb{K}$, $k = 1, \ldots, n$. Para $f \in C([a,b])$ se define

$$Q_n(f) := \sum_{k=1}^{n} \alpha_k^{(n)}\ f(t_k^{(n)}).$$

   Muestre que los siguientes enunciados son equivalentes:

   (a) $Q_n(f) \to \displaystyle\int_a^b f(t)\ \mathrm{d}t,\ n \to \infty$, para todo $f \in C[a,b]$.

   (b) $Q_n(p) \to \displaystyle\int_a^b p(t)\ \mathrm{d}t,\ n \to \infty$, para todo polinomio $p : [a,b] \to \mathbb{K}$ y $\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |\alpha_k^{(n)}| < \infty$.

Sean X, Y, Z espacios de Banach y $T : X \supseteq \mathcal{D}(T) \to Y$ un operador lineal.

(a) Sea $S : X \supseteq \mathcal{D}(S) \to Y$ un operador lineal. Entonces la *suma de operadores* $S + T$ se define como

$$\mathcal{D}(S+T) := \mathcal{D}(S) \cap \mathcal{D}(T), \qquad (S+T)x := Sx + Tx.$$

(b) Sea $R : Y \supseteq \mathcal{D}(R) \to Z$ un operador lineal. Entonces el *producto de operadores* o *composición* RT se define como

$$\mathcal{D}(RT) := \{x \in \mathcal{D}(T) : Tx \in \mathcal{D}(R)\}, \qquad (RT)x := R(Tx).$$

5. Sean X, Y, Z espacios de Banach, $R \in L(X, Y)$, $T : X \supseteq \mathcal{D}(T) \to Y$, $S : Y \supseteq \mathcal{D}(S) \to Z$ operadores lineales cerrados. Muestre que:

(a) $R + T$ es un operador lineal cerrado.

(b) SR es cerrado.

(c) Si S es contínuamente invertible (i. e., $S^{-1} : \operatorname{rg}(S) \to Y$ existe y es contínuo), entonces ST es cerrado.

Muestre además que estas afirmaciones siguen siendo válidas cambiando “cerrado” por “clausurable”

6. Sea $X = \ell_2(\mathbb{N})$ y

$$T : X \supseteq \mathcal{D}(T) \to X, \qquad Tx = (nx_n)_{n\in\mathbb{N}} \quad \text{para} \quad x = (x_n)_{n\in\mathbb{N}}.$$

Diga si T es cerrado con:

(a) $\mathcal{D}(T) = \{x = (x_n)_{n\in\mathbb{N}} \in \ell_2(\mathbb{N}) : (nx_n)_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})\}$,

(b) $\mathcal{D}(T) = d = \{x = (x_n)_{n\in\mathbb{N}} \in \ell_2(\mathbb{N}) : x_n \neq 0 \text{ para solo finitos } n\}$.

7. Sea X un espacio de Banach, $n \in \mathbb{N}$ y T un operador lineal densamente definido de X en K^n. Muestre que T es cerrado si y solo si $T \in L(X, \mathbb{K}^n)$.

8. Sean X y Y espacios normados y $T : X \supseteq \mathcal{D}(T) \to Y$ un operador lineal cerrado.

(a) Sea $K \subset X$ compacto. Muestre que $T(K)$ es cerrado en Y.

(b) Muestre que si F es un compacto en Y entonces $T^{-1}(F)$ es cerrado en X.

(c) ¿Si A es cerrado en X, es cierto que $T(A)$ es cerrado?

9. Sea X un espacio normado. Una sucesión $(x_n)_{n\in\mathbb{N}} \subseteq X$ es una *sucesión débil de Cauchy* si para todo $\varphi \in X'$ la sucesión $(\varphi(x_n))_{n\in\mathbb{N}}$ es una sucesión de Cauchy en $\mathbb{K}$.

(a) Sea $x = (x_n)_{n\in\mathbb{N}}$ una sucesión acotada en X. Muestre que x es una sucesión débil de Cauchy si y solo si existe un subconjunto denso U' de X' tal que $(\varphi(x_n))_{n\in\mathbb{N}}$ es una sucesión de Cauchy para todo $\varphi \in U'$.

(b) Toda sucesión débil de Cauchy en X es acotada.

10. Sea X un espacio de Banach, $(x_n)_{n\in\mathbb{N}} \subseteq X$, $(\varphi_n)_{n\in\mathbb{N}} \subseteq X'$, y $x_0 \in X$, $\varphi_0 \in X'$ tal que $x_n \xrightarrow{\|\cdot\|} x_0$ y $\varphi_n \xrightarrow{w*} \varphi_0$. Muestre que $\lim\limits_{n\to\infty} \varphi_n(x_n) = \varphi_0(x_0)$.

11. Sea X un espacio normado.

    (a) Muestre que $(X, \|\cdot\|)' = (X, \sigma(X, X'))'$. Es decir: un funcional lineal $\varphi : X \to \mathbb{K}$ es continua con respecto a la topología inducida por $\|\cdot\|$ si y sólo si es continua con respecto a la topología débil.

    (b) Sean $(x_n)_{n\in\mathbb{N}} \subseteq X$, $x_0 \in X$ y $(\varphi_n)_{n\in\mathbb{N}} \subseteq X'$, $\varphi_0 \in X'$ tal que $x_n \xrightarrow{w} x_0$ y $\varphi_n \xrightarrow{w*} \varphi_0$. Muestre

    $$\|x_0\| \leq \liminf_{n\to\infty} \|x_n\|, \qquad \|\varphi_0\| \leq \liminf_{n\to\infty} \|\varphi_n\|.$$

    (c) Sean $S = \{x \in X : \|x\| = 1\}$ la esfera unitaria y $K = \{x \in X : \|x\| = 1\}$ la bola unitaria cerrada en X. ¿Siempre son débilmente cerradas (prueba o contraejemplo)?

12. Para $n \in \mathbb{N}$ sea $e_n = (0, \dots, 1, 0, \dots)$ la sucesión que tiene 1 en la posición n y 0 en el resto.

    (a) Muestre que $(e_n)_{n\in\mathbb{N}}$ no es convergente débilmente en ℓ_1.

    (b) Muestre que $(e_n)_{n\in\mathbb{N}}$ es $w*$ convergente en ℓ_1.

13. Sea X un espacio vectorial y $M \subseteq X$ un subconjunto convexo, balanceado y absorbente. Muestre que el funcional de Minkowski p_M es una seminorma en X.

Exercises for Chapter 4

1. Sea X un espacio pre-Hilbert, $U \subseteq H$ un subespacio denso y $x_0 \in X$ tal que $\langle x_0\,, u\rangle = 0$ para todo $u \in U$. Muestre que $x_0 = 0$.

2. Sea $w \in C([0,1], \mathbb{R})$. Para $x, y \in C([0,1])$ se define

   $$\langle x\,, y\rangle_w := \int_0^1 x(t)\overline{y(t)}w(t)\ \mathrm{d}t.$$

   Halle una condición necesaria y suficiente spobre w para que $\langle\cdot\,,\cdot\rangle_w$ sea un producto interno. Bajo qué condición la norma inducida por $\langle\cdot\,,\cdot\rangle_w$ es equivalente a la norma usual de L_2?

3. Let H be Hilbert space, $(x_n)_{n\in\mathbb{N}} \subseteq H$ and $x_0 \in H$. Then the following is equivalent:

   (a) $x_n \to x_0$.

   (b) $\|x_n\| \to x_0$ and $x_n \xrightarrow{w} x_0$.

4. Ejemplo de una proyección no acotada. Sea $\mathcal{H} = l_2$ y e_i el vector usual $e_i^j = \delta_i^j$. Defina

$$L_1 := \overline{span\{e_{2n+1} : n \in \mathbb{N}_0\}}$$

y

$$L_2 := \overline{span\left\{e_1 + \frac{1}{2}e_2, e_3 + \frac{1}{2^2}e_4, e_5 + \frac{1}{2^3}e_6, \dots\right\}}.$$

(a) Muestre que $L_1 \cap L_2 = \{0\}$.

(b) Muestre que $\overline{L_1 \oplus L_2} = \mathcal{H}$.

(c) Muestre que $L_1 \oplus L_2 \neq \mathcal{H}$.

(d) Defina el operador $P_0 : L_1 \oplus L_2 \to L_1 \oplus L_2$, $P_0(x + y) = x$. Muestre que P_0 es una proyección no acotada.

5. Para $\lambda \in \mathbb{R}$ defina $f_\lambda : \mathbb{R} \to \mathbb{C}$, $f_\lambda(s) = \mathrm{e}^{\mathrm{i}\lambda s}$ y sea $X = \mathrm{span}\{f_\lambda : \lambda \in \mathbb{R}\}$. Muestre que

$$\langle f\,, g\rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s)\overline{g(s)}\;\mathrm{d}s$$

define un producto interior en X. Muestre que la completación de X no es separable. ($\|f_\lambda - f_{\lambda'}\| =?$)

Los elementos en la completación de X se llaman *funciones casi periódicas*.

6. ¿Existe algún producto interno $\langle\cdot\,,\,\cdot\rangle$ en $C[0,1]$ tal que $\langle x, x\rangle = \|x\|_\infty^2$ para todo $x \in C[0,1]$?

7. Sea X un espacio pre-Hilbert. Muestre los siguientes resultados

(a) Sean $x, y \in X$ con $x \perp y$, entonces

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

¿El converso es cierto en general? ¿Hay algún caso para el que se tenga?

(b) Si $x \neq 0$, $y \neq 0$ y $x \perp y$ muestre que el conjunto $\{x, y\}$ es linealmente independiente. ¿Como se puede generalizar este resultado?

(c) $x \perp y$, si y solo si $\|x + \alpha y\| \geq \|x\|$ para todo escalar α.

8. Let H be a Hilbert space, $Y \subseteq H$ a subspace and $\varphi_0 \in Y'$. Show that there exists exactly one extension $\varphi \in H'$ of φ_0 with $\|\varphi_0\| = \|\varphi\|$.

9. Sea X un espacio pre-Hilbert y $U \subseteq X$ un subespacio. Muestre que

(a) $\overline{U} \neq U^{\perp\perp}$. ¿Se tiene alguna contenencia?

(b) $\overline{U} \oplus U^\perp \neq X$

10. Sea $1 \leq p \leq \infty$. Para $f \in L_p(\mathbb{R})$ y $s \in \mathbb{R}$ defina $T_s : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ como $(T_s f)(t) := f(t - s)$. Claramente los T_s son isometrias lineales.

(a) Sea $1 \le p < \infty$. Muestre que $T_s \xrightarrow{s} \mathrm{id}$ para $s \to 0$. Los T_s convergen en norma?

(b) Los T_s convergen en norma o convergen fuertemente en el caso $p = \infty$?

11. Muestre que $W^m(\Omega)$, $H^m(\Omega)$ y $H_0^m(\Omega)$ son espacios de Hilbert.

Para el problema 4.10: Para $\Omega \subseteq \mathbb{R}$ definimos el conjunto de *funciones de prueba*

$$\mathscr{D}(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp}(\varphi) \subseteq \Omega \ \text{ es compacto}\}.$$

Para un multi-índice $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ se define $|\alpha| = \alpha_1 + \cdots + \alpha_n$ y $D^\alpha \varphi = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \varphi$ si la derivada existe.

Sea $f \in L_2(\Omega)$. Una función $g \in L_2(\Omega)$ se llama la *derivada débil* α-ésima de f si

$$\langle g, \varphi \rangle = (-)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \qquad \varphi \in \mathscr{D}(\Omega).$$

Note que la derivada débil es única si existe; se denota por $D^{(\alpha)} f$.

Para $m \in \mathbb{N}$ definimos el *espacio de Sobolev*

$$W^m(\Omega) := \{f \in L_2(\Omega) : D^{(\alpha)} f \in L_2(\Omega), |\alpha| \le m\}.$$

$W^m(\Omega)$ es un producto interior con

$$\langle f, g \rangle_{W^m} := \sum_{|\alpha| \le m} \langle D^{(\alpha)} f, D^{(\alpha)} g \rangle_2.$$

Además, definimos los espacios

$$H^m(\Omega) := \overline{C^m(\Omega) \cap W^m(\Omega)} \qquad \text{and} \qquad H_0^m(\Omega) := \overline{\mathscr{D}(\Omega)}$$

donde la clausura es tomada con respecto a la norma inducida por $\langle \cdot, \cdot \rangle_{W^m}$.

12. Sea H un espacio de Hilbert and $B : H \times H \to \mathbb{K}$ sesquilineal. En $H \times H$ considere la norma $\|(x, y)\| := \sqrt{\|x\|^2 + \|y\|^2}$.

(a) Muestre que las siguientes son equivalentes:

(i) B es contínua.

(ii) B is parcialmente contínua, es decir, para cada x_0 fijo, $y \mapsto B(x_0, y)$ es contínua para cada y_0 fijo, $x \mapsto B(x, y_0)$ es contínua.

(iii) B es acotado, es decir, existe $M \in \mathbb{R}$ tal que $\|B(x, y)\| \le M\|x\|\|y\|$ para todo $x, y \in H$.

(b) Si B es contínuo, entonces existe $T \in L(H)$ tal que

$$B(x, y) = \langle Tx, y \rangle, \qquad x, y \in H.$$

(c) Si además existe $m > 0$ tal que $B(x, x) \ge m\|x\|^2$, $x \in H$, entonces T es invertible y $\|T^{-1}\| \le m^{-1}$.

13. Sea H un espacio de Hilbert. Muestre que para toda sucesión $(x_n)_n \subseteq H$ acotada, existe una subsucesión $(x_{n_k})_k$ tal que la sucesión $(y_m)_m$ donde,

$$y_m = \frac{1}{m}\sum_{k=1}^{m} x_{n_k},$$

converge.

14. Sea X un espacio normado, $(x_n)_{n\in\mathbb{N}} \subseteq X$ y $x \in X$. Las siguientes son equivalentes:

   (a) $\sum_{n\in\mathbb{N}} x_n$ converge incondicionalmente a x.

   (b) Para todo $\varepsilon > 0$ existe un conjunto finito $A \subseteq \mathbb{N}$ tal que para todo conjunto finito B con $A \subseteq B \subseteq \mathbb{N}$

$$\Big\| \sum_{b\in B} x_b - x \Big\| < \varepsilon.$$

15. Sea H un espacio de Hilbert. Si $P : H \to H$ es un operador lineal, las siguientes son equivalentes:

   (a) P es una proyección ortogonal.

   (b) $P^2 = P$ y $\langle Px \,, y\rangle = \langle x \,, Py\rangle$.

16. Sea H un espacio de Hilbert, $V, W \subseteq H$ subespacios cerrados y P_V, P_W sus correspondientes proyecciones ortogonales.

   (a) Muestre que

$$V \subseteq W \qquad \Longleftrightarrow \qquad P_V = P_V P_W = P_W P_V.$$

   (b) Muestre que las siguientes afirmaciones son equivalentes:

      (i) $P_V P_W = 0$.

      (ii) $V \perp W$.

      (iii) $P_V + P_W$ es una proyección ortogonal.

   Muestre que $\mathrm{rg}(P_V + P_W) = V \oplus W$ si alguna de las condiciones anteriores se tiene.

17. Sea H un espacio de Hilbert y P_0, P_1 las proyecciones ortogonales sobre H_0, $H_1 \subseteq H$. Entonces las siguientes afirmaciones son equivalenteas:

   (a) $H_0 \subseteq H_1$,

   (b) $\|P_0 x\| \leq \|P_1 x\|, \quad x \in H$.

   (c) $\langle P_0 x \,, x\rangle \leq \langle P_1 x \,, x\rangle, \quad x \in H$.

   (d) $P_0 P_1 = P_0$.

18. Sea H un espacio de Hilbert separable, $(x_n)_{n\in\mathbb{N}}$ una base ortonormal de H, y, $(y_n)_{n\in\mathbb{N}}$ una sucesión tal que:

$$\sum_{n=1}^{\infty} ||x_n - y_n|| < 1$$

y $z \perp y_n$, para todo $n \in \mathbb{N}$, entonces $z = 0$.

19. Sea H un espacio de Hilbert complejo y $T : H \to H$ un operador lineal acotado. Muestre que T es normal si y solo si $||T^*x|| = ||Tx||$ para todo $x \in H$. En este caso, muestre que $||T^2|| = ||T||^2$.

20. **Haar functions.** Let $\psi = \chi_{[0,\,1/2)} - \chi_{[1/2,\,1)}$. For $n, k \in \mathbb{Z}$ define

$$\psi_{n,k} : \mathbb{R} \to \mathbb{R}, \qquad \psi_{n,k}(t) = 2^{k/2}\,\psi(2^k t - n).$$

For $k \in \mathbb{N}_0$ and $n \in \{0,\, 1,\, 2,\, \dots,\, 2^k - 1\}$ let

$$h_{2^k+n} : [0,1] \to \mathbb{R}, \qquad \begin{cases} h_{2^k+n}(t) = \psi_{k,n}(t), & \text{for } \ t \in [0,1), \\ h_{2^k+n}(1) = \lim\limits_{t\to 1^-} \psi_{k,n}(t). \end{cases}$$

and $h_0(t) = 1,\ t \in [0,1]$.

(a) $(h_j)_{j\in\mathbb{N}_0}$ is a orthonormal system in $L_2[0,1]$ and $(\psi_{n,k})_{n,k\in\mathbb{Z}}$ is a orthonormal system in $L_2(\mathbb{R})$.

(b) $T : L_2[0,1] \to L_2[0,1],\ Tf = \sum_{j=0}^{2^k-1} \langle f\,, h_j\rangle h_j$ is a orthonormal projection on the subspace

$$U = \{f \in L_2[0,1] : f \text{ const. in intervals } [r2^{-k}, (r+1)2^{-k}) \text{ with } r \in \mathbb{N}_0\}.$$

(c) For $f \in C[0,1]$, the series $\sum_{j=0}^{\infty} \langle f\,, h_j\rangle h_j$ converges uniformely to f.

(d) $(h_j)_{j\in\mathbb{N}_0}$ is an orthonormal basis of $L_2[0,1]$.

(e) $(\psi_{k,n})_{k,n\in\mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{R})$.

21. Sea H un espacio de Hilbert, $V, W \subseteq H$ subespacios cerrados y P_V, P_W sus correspondientes proyecciones ortogonales.

(a) Muestre que

$$V \subseteq W \quad \iff \quad P_V = P_V P_W = P_W P_V.$$

(b) Muestre que las siguientes afirmaciones son equivalentes:

(i) $P_V P_W = 0$.

(ii) $V \perp W$.

(iii) $P_V + P_W$ es una proyección ortogonal.

Muestre que $\mathrm{rg}(P_V + P_W) = V \oplus W$ si alguna de las condiciones anteriores se tiene.

22. Sea H un espacio de Hilbert y P_0, P_1 las proyecciones ortogonales sobre H_0, $H_1 \subseteq H$. Entonces las siguientes afirmaciones son equivalenteas:

    (i) $H_0 \subseteq H_1$,

    (ii) $\|P_0 x\| \leq \|P_1 x\|, \quad x \in H$.

    (iii) $\langle P_0 x\,, x\rangle \leq \langle P_1 x\,, x\rangle, \quad x \in H$.

    (iv) $P_0 P_1 = P_0$.

23. Sea H un espacio de Hilbert separable, $(x_n)_{n\in\mathbb{N}}$ una base ortonormal de H, y, $(y_n)_{n\in\mathbb{N}}$ una sucesión tal que:

$$\sum_{n=1}^{\infty} ||x_n - y_n|| < 1$$

    y $z \perp y_n$, para todo $n \in \mathbb{N}$, entonces $z = 0$.

24. Sea H un espacio de Hilbert complejo y $T : H \to H$ un operador lineal acotado. Muestre que T es normal si y solo si $||T^* x|| = ||Tx||$ para todo $x \in H$. En este caso, muestre que $||T^2|| = ||T||^2$.

25. Sea H un espacio de Hilbert y $(T_n)_{n\in\mathbb{N}}$ una sucesión acotada y monótonamente creciente de operadores autoadjuntos. Muestre que la sucesión converge en el sentido fuerte a un operador autoadjunto.

26. Sea $(P_n)_{n\in\mathbb{N}}$ una sucesión monótona de proyecciones ortogonales en un espacio de Hilbert H. Muestre que $(P_n)_{n\in\mathbb{N}}$ converge en el sentido fuerte a una proyección ortogonal P y además

    (a) rg $P = \overline{\bigcup_{n\in\mathbb{N}} rgP_n}$ si P_n es creciente.

    (b) rg $P = \overline{\bigcap_{n\in\mathbb{N}} rgP_n}$ si P_n es decreciente.

27. Sean H_1, H_2 y H_3 espacios de Hilbert y $S(H_1 \to H_2)$ y $T(H_2 \to H_3)$ operadores lineales densamente definidos.

    (a) Si $T \in L(H_2, H_3)$ entonces TS es densamente definido y $(TS)^* = S^* T^*$.

    (b) Si S es inyectivo y $S^{-1} \in L(H_2, H_1)$ entonces TS es densamente definido y $(TS)^* = S^* T^*$.

    (c) Si S es inyectivo y $S^{-1} \in L(H_2, H_1)$ entonces S^* es inyectivo y $(R^*)^{-1} = (R^{-1})^*$

28. Sean H_1, H_2 espacios de Hilbert y $U : H_1 \times H_2 \to H_2 \times H_1$, $U(x, y) = (-y, x)$. Entonces

    (a) U es unitario.

    (b) Si $T(H_1 \to H_2)$ es densamente definido,

$$G(T^*) = [U(G(T))]^{\perp} = U(G(T)^{\perp}).$$

    (c) T^* es cerrado.

    (d) Si T es clausurable, T^* es densamente definido y $T^{**} = \overline{T}$.

Exercises for Chapter 5

1. (a) Sea $X = C([0,1])$ y $a \in C([0,1])$. Muestre que

$$A : X \to X, \qquad (Ax)(t) = a(t)x(t)$$

es un operador lineal acotado. Encuentre $\|A\|$, $\sigma(A)$, $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)$ y $\sigma_{\mathrm{r}}(A)$.

(b) Sea $H = \{f \in C([0,1]) : x(0) = 0\}$ y

$$S : H \to H, \qquad (Sx)(t) = \int_0^t x(s)\,\mathrm{d}s.$$

Encuentre $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$, $\sigma_{\mathrm{c}}(S)$ y $\sigma_{\mathrm{r}}(S)$.

2. Sea $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ una sucesión acotada, y,

$$T : \ell^1 \to \ell^1, \qquad T\left((x_n)_{n\in\mathbb{N}}\right) = (\lambda_n x_n)_{n\in\mathbb{N}}.$$

Encuentre $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ y $\sigma_{\mathrm{r}}(T)$. Muestre además que, para todo $K \subseteq \mathbb{C}$ compacto no vacío, existe un operador $T \in L(\ell^1)$ cuyo espectro es K.

3. Sea X un espacio de Banach $S, T \in L(X)$. Muestre que $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$.

Hint. Muestre que $\mathrm{id} - ST$ es invertible si y solo si $\mathrm{id} - TS$ es invertible, encontrando una relación entre $(\mathrm{id} - TS)^{-1}$ y $(\mathrm{id} - ST)^{-1}$. Suponga $||T||\,||S|| < 1$ y mire si la relación en este caso es válida en general.

4. Encuentre el espectro puntual, el espectro continuo y el espectro residual de los operadores:

$$\begin{aligned}
&R : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}), \quad R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots),\\
&L : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}), \quad L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots),\\
&T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N}), \quad T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).
\end{aligned}$$

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