Universidad de los Andes — Departamento de Matemáticas

MATE-2201: Analysis 1

Final Exam 200920, 7 December 2009

Código: ________________ Nombre: ____________________________

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Sum

Solve the following problems. Give all necessary steps to find the solutions and prove your answers. *Hand in this sheet and all sheets of paper you used. Put your name on everything you hand in.*

Good luck!

8 pts. **Problem 1.** For $n \in \mathbb{N}$ define

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}.$$

(a) Is $(f_n)_{n\in\mathbb{N}}$ pointwise convergent? If so, find the limit function.

(b) Is $(f_n)_{n\in\mathbb{N}}$ uniformly convergent? If so, find the limit function.

8 pts. **Problem 2.** Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Assume that $\lim_{x\to\infty} f(x) =: A \in \mathbb{R}$ exists. Show that f is uniformly continuous.

9 pts. **Problem 3.** Let $x = (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ a bounded sequence with only finitely many cluster values $a_1, \dots, a_m$. Assume that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_k - x_{k+1}| < \varepsilon$ for all $k \geq N$.

(a) Show that for every $\delta > 0$ there exists a $K \in \mathbb{N}$ such that $x_n \in \bigcup_{j=1}^{m} B_\delta(a_j)$ for all $n \geq K$.

(b) Show that the sequence x converges.

9 pts. **Problem 4.** Let X be a topological space.

(a) When is X called compact?

(b) Let X be a compact. Let $f : X \to \mathbb{R}$ a function such that for every $p \in X$ there exists a neighbourhood V_p such that the restriction of f to V_p is bounded. Show that f is bounded.

8 pts. **Problem 5.** Let X be a topological space and $M \subseteq X$ a non-empty subset.

(a) How is the boundary ∂M of M defined?

(b) Show that $\partial M = \partial(X \setminus M)$.

(c) Show that $\partial M = \overline{M} \cap \overline{X \setminus M}$.

(d) Show that ∂M is closed.

8 pts. **Problem 6.** Let X be a metric space.

(a) When is X called totally bounded?

(b) Show: If X is totally bounded, then it is bounded.