Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 14

Basic Topology.

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1. If $f : [-1,1] \to \mathbb{R}$ is continuous, then

$$\lim_{t\to 0} \int_{-1}^{1} \frac{t}{t^2+x^2}\, f(x)\,\mathrm{d}x = \pi f(0).$$

2. Let (X,d) a metric space and define $\mathcal{O} \subseteq \mathbb{P}X$ by

$$U \in \mathcal{O} \quad :\Longleftrightarrow \quad \forall p \in U \quad \exists \varepsilon > 0 \quad B_\varepsilon(p) \subseteq U.$$

   Show that $(X,\mathcal{O})$ is a topological space with the Hausdorff property.

   Show that for $r > 0$ and $a \in X$ the open ball $B_r(a)$ is open and the closed ball $K_r(a)$ is closed. Let $S_r(a) := \{x \in X : d(x,a) = r\}$. Show that

$$\partial B_r(a) \subseteq S_r(a) \quad \text{and} \quad \overline{B_r(a)} \subseteq K_r(a). \tag{$*$}$$

   Is equality in $(*)$ true?

3. (a) Find the interior and the closure of

$$M := \{(x, \sin x^{-1}) : x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathbb{R}^2.$$

   (b) Let $(X,\mathcal{O})$ be topological space and $M \subseteq X$. Can $(\partial M)^\circ = \emptyset$ be concluded?

4. Show that every open subset of $\mathbb{R}$ is the disjoint union of at most countably many open intervals.

5. Let K, $A \subseteq \mathbb{R}^n$ such that K is compact and A is closed. Then there are $p \in K$ and $a \in A$ such that

$$|p-a| \le |q-x|, \qquad q \in K,\ x \in A.$$