Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 13

Taylor polynomials and series. Hand in: November 12, 2009

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1. Let $D \subset \mathbb{R}$ be an interval, $p \in D$, $n \in \mathbb{N}_0$ and $f \in C^n(D, \mathbb{C})$. Let P be a polynomial of degree $\leq n$ such that

$$P^{[k]}(p) = f^{[k]}(p), \quad k = 0, 1, \ldots, n.$$

Show that $P = j_p^n f$ where $j_p^n f$ is the nth Taylor polynomial of f in p.

2. (a) Find the Taylor series at $p = 2$ and determine its radius of convergence of

$$f(x) = \frac{1}{(x-3)(x-5)}$$

(b) Find the limit (without using l'Hospital's rule)

$$\lim_{x \to 0} \frac{x - \sin x}{\mathrm{e}^x - 1 - x - x^2/2}.$$

3. (a) Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$, $f(x) = -\log(\cos x)$. Show that

$$\left| f(x) - \frac{x^2}{2} \right| \leq \frac{2}{3}\,|x|^3, \quad x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

4. (a) Show that the following function is arbitrarily often differentiable and find its Taylor series at 0. What is its radius of convergence? Where is the Taylor series equal to φ?

$$\varphi : \mathbb{R} \to \mathbb{R}, \qquad \varphi(x) = \begin{cases} \exp(-x^{-2}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(b) Show that the following function is arbitrarily often differentiable and find its Taylor series at 0. What is its radius of convergence?

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = \sum_{n=0}^{\infty} \frac{\cos(n^2 x)}{2^n}.$$