Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 10

Differentiation; exponential functions. Hand in: October 23, 2009

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1. For $k \in \mathbb{N}$ let $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(x) := \begin{cases} x^k \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

For which k is f_k differentiable? For which k is f_k continuously differentiable?

2. Let $f : [a, b] \to [a, b]$ be continuous and differentiable in (a, b) with $f'(x) \neq 1$, $x \in (a, b)$. Show that there exists exactly one $p \in [a, b]$ such that $f(p) = p$.

3. (a) *Darboux's Theorem.* Let $f : \mathcal{D} \to \mathbb{R}$ be a differentiable function of the nonempty interval $\mathcal{D} = (a, b) \subseteq \mathbb{R}$. Show that for every $q \in \mathbb{R}$ with

$$\inf\{f'(x) : x \in \mathcal{D}\} < q < \sup\{f'(x) : x \in \mathcal{D}\}.$$

there exists a $c \in (a, b)$ such that $f'(c) = q$.

(b) Let $\mathcal{D} = (a, b)$ a nonempty interval and $f : \mathcal{D} \to \mathbb{R}$ a differentiable function with an isolated global minimum at $x_0 \in \mathcal{D}$. Is the following statement true:
There exist $c, d \in (a, b)$ such that $c < x_0 < d$ and $f'(x) \leq 0,\ x \in (c, x_0)$ and $f'(x) \geq 0,\ x \in (x_0, d)$.

4. **Exponential functions.** For fixed $a \in \mathbb{R}^+ = (0, \infty)$ define the function

$$p_a : \mathbb{C} \to \mathbb{C}, \qquad p_a(z) = \exp(z \ln(a)).$$

(a) For $a \in \mathbb{R}^+$ and $q \in \mathbb{Q}$ show

$$p_a(q) = a^q. \tag{$*$}$$

(b) Show that p_a is differentiable and find its derivative.

Recall. For $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$ we have defined

$$a^n := \prod_{n=1}^{n} a, \qquad a^0 := 1, \qquad a^{\frac{1}{n}} := \text{unique positive solution of } x^n = a.$$

Therefore $a^q := \big((a^\sigma)^{\frac{1}{m}}\big)^n$ is defined for all $q = \frac{\sigma n}{m} \in \mathbb{Q}$ with $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\sigma \in \{\pm 1\}$.

Remark. Because of the identity $(*)$ one defines

$$a^z := \exp(z \ln(a)), \qquad a \in \mathbb{R}^+,\ z \in \mathbb{C}.$$