Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 9

Power series; exponential and trigonometric functions.
Differentiation.

Hand in: October 16, 2009

1. Find the radius of convergence of

i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (2z)^n}{n}\,,$ ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}\, z^n\,,$

iii) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{n}-1)^n z^n\,,$ iv) $\displaystyle\sum_{n=1}^{\infty} \frac{8^n z^{3n}}{3^n}\,.$

2. Show that the following functions are differentiable and find the derivative. Prove your assertions.

(a) $w : \mathbb{R}_+ \to \mathbb{R},\ x \mapsto \sqrt{x},$

(b) $\cos : \mathbb{R} \to \mathbb{R}, \quad \sin : \mathbb{R} \to \mathbb{R},$

Hint. Prove Euler's formula (Theorem 5.51): $\exp(z) = \cos(z) + \mathrm{i}\sin(z),\ z \in \mathbb{C}.$

3. Show the following properties of the exponential function (Theorem 5.50):

(a) $\exp(\bar{z}) = \overline{\exp(z)}, \quad z \in \mathbb{C},$

(b) $\exp(z+w) = \exp(z)\exp(w), \quad z, w \in \mathbb{C},$

(c) $\exp(n) = \mathrm{e}^n, \quad n \in \mathbb{Z},$

(d) $\exp(z) \neq 0, \quad z \in \mathbb{C},$

(e) $|\exp(\mathrm{i}x)| = 1 \iff x \in \mathbb{R}.$

4. (a) Show the following identities for $x, y \in \mathbb{C}$:

(i) $\sin^2(x) + \cos^2(x) = 1.$

(ii) $\sin(x+y) = \cos(x)\sin(y) + \cos(y)\sin(x),$

(iii) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$

(b) Show that $\{x \in \mathbb{R}_+ \,:\, \cos x = 0\} \neq \emptyset.$

Let $\pi := 2 \cdot \inf\{x \in \mathbb{R}_+ \,:\, \cos x = 0\}.$

(c) For $x \in \mathbb{R}$ show:

(i) $\sin x = 0 \iff \exists\, k \in \mathbb{Z} \quad x = k\pi.$

(ii) $\cos x = 0 \iff \exists\, k \in \mathbb{Z} \quad x = k\pi + \frac{\pi}{2}.$

Hint. Without proof you can use

$$1 - \frac{x^2}{2} \,\le\, \cos x \,\le\, 1 - \frac{x^2}{2} + \frac{x^4}{24}, \qquad x \in (0,3].$$