Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 8

Continuity and uniform continuity. Hand in: October 9, 2009

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1. Prove Theorem 5.26 and Theorem 5.27:
   Let $I = (a, b)$ a nonempty real interval and $f : I \to \mathbb{R}$ a function.
   (a) Assume that f is continuous. Then f is injective if and only if f is strictly monotonic.
   (b) If f is strictly monotonically increasing or decreasing, then it is invertible and its inverse $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

2. Show that $f : [0, \infty) \to \mathbb{R},\ x \mapsto \sqrt{x}$, is uniformly continuous but not Lipschitz continuous.

3. Do the following sequences of functions converge pointwise? Do they converge uniformely? If they converge, find the limit function.
   (a) $f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n}, \\ 2n - n^2 x, & \frac{1}{n} < x \le \frac{2}{n}, \\ 0, & \text{else.} \end{cases}$
   (b) $f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \dfrac{nx}{1 + nx^2},$
   (c) $f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \dfrac{nx}{1 + n^2x^2},$
   (d) $f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \dfrac{n^2 x}{1 + nx}.$

4. Let $\mathcal{D} \subseteq \mathbb{R}$, $f : \mathcal{D} \to \mathbb{R}$ a function and $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \setminus \{0\}$ a sequence that converges to 0. Define $f_n : \mathcal{D} \to \mathbb{R}$ by $f_n(x) = a_n f(x)$, $x \in \mathcal{D}$.
   (a) $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $g : \mathcal{D} \to \mathbb{R},\ g(x) = 0$.
   (b) $(f_n)_{n\in\mathbb{N}}$ converges uniformely if and only if f is bounded on $\mathcal{D}$.