Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 7

Continuity.

Hand in: September 24, 2009

---

1. For $j = 1, \dots, n$ let $(X_j, \|\cdot\|_j)$ be normed spaces over $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Recall that $(X_1 \times \cdots \times X_n, \|\cdot\|)$ with
$$\|(x_1, \dots, x_n)\| := \|x_1\| + \cdots + \|x_n\|$$
is a normed space over $\mathbb{F}$.

   (a) Show that for all $j = 1, \dots, n$ the projection pr_j is continuous where
$$\mathrm{pr}_j : X_1 \times \cdots \times X_n \to X_j, \quad (x_1, \dots, x_n) \mapsto x_j.$$

   (b) Let $f = (f_1, \dots, f_n) : V \to X_1 \times \cdots X_n$ where V is a normed space (that is $f_j : V \to X_j$ and $f(v) = (f_1(v), \dots, f_n(v))$).
   Show that f is continuous if and only if every f_j is continuous.

   (c) Let X be a normed space, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $f : \mathcal{D}_f \to \mathbb{F}, \quad g : \mathcal{D}_g \to \mathbb{F}$ continuous. Let $\mathcal{D}_{fg} = \mathcal{D}_f \cap \mathcal{D}_g$. Then $fg : \mathcal{D}_{fg} \to \mathbb{F}, \quad (fg)(x) = f(x)g(x)$ is continuous. If $g(x) \neq 0$, $x \in \mathcal{D}_{fg}$, then $\frac{f}{g} : \mathcal{D}_{fg} \to \mathbb{F}$, $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is continuous.

2. Proof the Cauchy criterion (Theorem 5.15):
   Let (X, d_X), (Y, d_Y) be metric spaces, Y complete, $f : X \supseteq \mathcal{D} \to Y$ a function and x_0 a limit point of $\mathcal{D}$. Then f has a limit in x_0 if and only if
$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x, y \in D_f :$$
$$\Big(0 < d_X(x, x_0) < \delta\ \wedge\ 0 < d_X(y, x_0) < \delta \implies d_Y\big(f(x), f(y)\big) < \varepsilon\Big).$$

3. Let (X, d) be a metric space and $f, g : X \to \mathbb{R}$ continuous functions. Show that the following functions are continuous:
$$S : X \to \mathbb{R}, \quad S(x) := \min\{f(x),\ g(x)\},$$
$$T : X \to \mathbb{R}, \quad T(x) := \max\{f(x),\ g(x)\}.$$

4. Where are the following functions are continuous? Proof your answer.

   (a) $f : [0, \infty) \to \mathbb{R}, \quad x \mapsto \sqrt{x}$,

   (b) $g : \mathbb{C} \to \mathbb{R}, \quad z \mapsto |z + \bar{z}^2|$,

   (c) $h : [-1, 1] \cup \{2\} \to \mathbb{R}, \quad x \mapsto \begin{cases} -\sqrt{-x}, & -1 \leq x \leq 0, \\ \sqrt{x}, & 0 < x \leq 1, \\ x, & x = 2. \end{cases}$

   (d) $D : \mathbb{R} \to \mathbb{R}, \quad D(x) := \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$
   Hint: Show that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, that is, for every $x \in \mathbb{R}$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \mathbb{Q}$ such that $\lim\limits_{n \to \infty} x_n = x$.