Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 5

limsup; Euler number; Cauchy's condensation test.

Hand in: September 10, 2009

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1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a normed space and $(b_n)_{n\in\mathbb{N}}$ be defined by

$$b_n := \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Show or find a counterexample:

(a) $(a_n)_{n\in\mathbb{N}}$ converges $\implies$ $(b_n)_{n\in\mathbb{N}}$ converges.
(b) $(b_n)_{n\in\mathbb{N}}$ converges $\implies$ $(a_n)_{n\in\mathbb{N}}$ converges.

2. (a) Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and define sequences $(y_k)_{k\in\mathbb{N}}$, $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{R}\cup\{\pm\infty\}$ by

$$y_k := \sup\{x_n : n \geq k\}, \qquad z_k := \inf\{x_n : n \geq k\}, \qquad k \in \mathbb{N}.$$

Show that $(y_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ converge in $\mathbb{R}\cup\{\pm\infty\}$ and that

$$\lim_{k\to\infty} y_k = \limsup x_n, \qquad \lim_{k\to\infty} z_k = \liminf x_n.$$

(b) Find a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$\inf\{a_n : n\in\mathbb{N}\} < \liminf\{a_n : n\in\mathbb{N}\} < \limsup\{a_n : n\in\mathbb{N}\} < \sup\{a_n : n\in\mathbb{N}\}.$$

In this case, must the set $\{a_n : n\in\mathbb{N}\}$ have a maximum?

3. **Cauchy's condensation test.**

(a) For a monotonically decreasing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^0$ show

$$\sum_{n\in\mathbb{N}} a_n \text{ converges} \iff \sum_{n\in\mathbb{N}} 2^n a_{2^n} \text{ converges.}$$

(b) Do the series $\sum_{n=1}^{\infty}(n\log_2 n)^{-1}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}$ converge? Prove your answer. (Use what you know from the calculus courses about the logarithm.)

4. **The Euler number** e.

For $n\in\mathbb{N}$ let $a_n := \left(1+\frac{1}{n}\right)^n$ and $s_n := \sum_{k=0}^{\infty}\frac{1}{k!}$.

(a) Show that $2^k < k!$ for all $k \geq 4$ and that

$$1 \leq \left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{n}\frac{1}{k!} < 3, \qquad n\in\mathbb{N}.$$

(b) Show that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ converge.
(c) Show that

$$\mathrm{e} := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}, \qquad n\in\mathbb{N}.$$