Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 4

Sequences.

Hand in: September 3, 2009

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1. Let $q \in \mathbb{R}_+$ and $x_n := \sqrt[n]{q}$, $y_n := \sqrt[n]{n}$, $n \in \mathbb{N}$. Do the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge? If so, find the limit.

2. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $a_n \neq 0$ for all $n \in \mathbb{N}$. Show or find a counter-example:

   (i) If there exists an $N \in \mathbb{N}$ and $q \in \mathbb{R}$, $q < 1$, such that
   $$\left|\frac{a_{n+1}}{a_n}\right| \leq q, \quad n \in \mathbb{N},\ n \geq N,$$
   then $\lim_{n\to\infty} a_n = 0$.

   (ii) If there exists an $N \in \mathbb{N}$ and $q \in \mathbb{R}$, $q \leq 1$, such that
   $$\left|\frac{a_{n+1}}{a_n}\right| < q, \quad n \in \mathbb{N},\ n \geq N,$$
   then $\lim_{n\to\infty} a_n = 0$.

3. The Fibonacci sequence $(a_n)_{n\in\mathbb{N}}$ is defined recursively by
   $$a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}, \qquad n \in \mathbb{N}.$$
   Moreover, let $\sigma < \tau$ be the solutions of $x^2 - x - 1 = 0$ and
   $$x_n = \frac{a_{n+1}}{a_n}, \qquad n \in \mathbb{N}.$$
   (a) Show that $(a_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$.
   (b) $a_n = \frac{1}{\sqrt{5}}(\tau^{n+1} - \sigma^{n+1}), \quad n \in \mathbb{N}.$
   (c) $\lim_{n\to\infty} x_n = \sigma.$

4. If it exists, find the value of
   $$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}},$$
   i. e. the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ with
   $$x_1 := 1 \quad \text{and} \quad x_{n+1} := 1 + \frac{1}{x_n},\ n \geq 1.$$