Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 3

Complex numbers; sequences.

Hand in: August 27, 2009

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1. (a) Show that for every $z \in \mathbb{C} \setminus \{0\}$ there exist exactly two numbers $\zeta_1, \zeta_2 \in \mathbb{C}$ such that $\zeta_1^2 = \zeta_2^2 = z$.

   (b) Let $a, b, c \in \mathbb{C}$, $a \neq 0$. Show that there exists at least one $z \in \mathbb{C}$ such that
$$az^2 + bz + c = 0.$$

2. (a) Let (X, d), $X \neq \emptyset$, be a metric space and $M \subseteq X$. Show that the following are equivalent:
   - (i) M is bounded.
   - (ii) $\exists\, x \in X\ \exists\, r > 0 : M \subseteq B_r(x)$.
   - (iii) $\forall\, x \in X\ \exists\, r > 0 : M \subseteq B_r(x)$.

   (b) For $M \subseteq \mathbb{R}$ show that M is bounded as subset of the ordered field $(\mathbb{R}, >)$ if and only if M is bounded as subset of the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$.

3. (a) Let (X, d), $X \neq \emptyset$, be a metric space and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X. Show: If there exists an $a \in X$ such that
$$\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n,$$
then
$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$
Is the converse also true (proof or counter example)?

   (b) Let (X, d) be a metric space and $\varrho : \mathbb{N} \to \mathbb{N}$ a bijection. Show: If $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges, then $(x_{\varrho(n)})_{n\in\mathbb{N}} \subseteq X$ converges and has the same limit.

4. (a) Let $x_n = \sqrt{1 + n^{-1}}$, $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

   (b) Do the following sequences in $\mathbb{R}$ converge? If so, find the limit. Prove your assertions.
   - (i) $(a_n)_{n\in\mathbb{N}}$ with $a_n = \dfrac{2^n}{n!}, \quad n \in \mathbb{N}$,
   - (ii) $(b_n)_{n\in\mathbb{N}}$ with $b_n = \sqrt{1 + n^{-1} + n^{-2}}, \quad n \in \mathbb{N}$,
   - (iii) $(d_n)_{n\in\mathbb{N}}$ with $d_n = \sqrt{n^2 + n + 1} - n, \quad n \in \mathbb{N}$,