Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 2

Ordered fields; supremum. Hand in: August 20, 2009

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1. Let $(K,+,\cdot,>)$ be an ordered field and a, x, x', y, $y' \in K$. Show the following statements from Corollary 3.9. Justify every step.

   (iii) $x < y \implies x + a < y + a$,
   (iv) $x < y \;\wedge\; x' < y' \implies x + x' < y + y'$,
   (v) $x < y \;\wedge\; a > 0 \implies a \cdot x < a \cdot y$,
   $x < y \;\wedge\; a < 0 \implies a \cdot x > a \cdot y$,
   (vi) $0 \le x < y \;\wedge\; 0 \le x' < y' \implies 0 \le x' \cdot x < y' \cdot y$,
   (ix) $x > 0 \implies x^{-1} > 0$,
   (x) $0 < x < y \implies 0 < y^{-1} < x^{-1}$,
   (xi) $x > 0 \wedge y < 0 \implies xy < 0$.

2. Find the infimum and supremum of the following sets in the ordered field $\mathbb{R}$. Determine if they have a maximum and a minimum.

   (a) $\{x \in \mathbb{R} : \exists n \in \mathbb{N} \;\; x = n^2\}$,
   (b) $\left\{\dfrac{|x|}{1+|x|} : x \in \mathbb{R}\right\}$,
   (c) $\left\{x \in \mathbb{R} : \exists n \in \mathbb{N} \;\; x = \dfrac{1}{n} + n\big(1 + (-1)^n\big)\right\}$,
   (d) $\big\{x \in \mathbb{R} : x^2 \le 2\big\} \cap \mathbb{Q}$.

3. (a) For every $x \in \mathbb{R}_+$ there exists an $n \in \mathbb{N}_0$ with $n \le x < n+1$. (Proposition 3.19).
   (b) Every interval in $\mathbb{R}$ contains a rational number. (Proposition 3.20).
   (c) $\mathbb{Q}$ does not have the least upper bound property.

4. (a) Let $X \subset \mathbb{R}$, $X \neq \emptyset$, and $\xi \in \mathbb{R}$ an upper bound of X. Show that

   $$\xi = \sup X \quad \iff \quad \forall \varepsilon \in \mathbb{R}_+ \; \exists x_\varepsilon \in X \;\; \xi - \varepsilon < x_\varepsilon \le \xi.$$

   What is the analogous statement for $\inf X$?
   (b) Let X, $Y \subset \mathbb{R}$ non empty sets such that

   $$\forall x \in X \;\; \exists y \in Y \; : \; y < x.$$

   Does that imply $\inf Y < \inf X$? Proof your assertion.