

Analysis 1

Problem Sheet 14

Topology.

Hand in: May 7, 2009

1. If $f : [-1, 1] \rightarrow \mathbb{R}$ is continuous, then

$$\lim_{t \rightarrow 0} \int_{-1}^1 \frac{t}{t^2 + x^2} f(x) dx = \pi f(0).$$

2. Let (X, d) a metric space and define $\mathcal{O} \subseteq \mathbb{P}X$ by

$$U \in \mathcal{O} \iff \forall p \in U \exists \varepsilon \geq 0 \ B_\varepsilon(p) \subseteq U.$$

Show that (X, \mathcal{O}) is a topological space with the Hausdorff property.

Show that for $r > 0$ and $a \in X$ the open ball $B_r(a)$ is open and the closed ball $K_r(a)$ is closed. Let $S_r(a) := \{x \in X : d(x, a) = r\}$. Show that

$$\partial B_r(a) \subseteq S_r(a) \quad \text{and} \quad \overline{B_r(a)} \subseteq K_r(a). \quad (*)$$

Is equality in (*) true?

3. (a) Find the interior and the closure of

$$M := \{(x, \sin x^{-1}) : x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathbb{R}^2.$$

(b) Let (X, \mathcal{O}) be topological space and $M \subseteq X$. Can $(\partial M)^\circ = \emptyset$ be concluded?

4. Show that every open subset of \mathbb{R} is the disjoint union of at most countably many open intervals.
5. Let $K, A \subseteq \mathbb{R}^n$ such that K is compact and A is closed. Then there are $p \in K$ and $a \in A$ such that

$$|p - a| \leq |q - x|, \quad q \in K, \ x \in A.$$