Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 11

Inequalities. Riemann-Stieltjes integral; Hand in: April 16, 2009

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1. (a) Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$. Show:
$$\sum_{k=1}^{\infty} a_n \text{ converges} \quad \Longrightarrow \quad \sum_{k=1}^{\infty} \frac{\sqrt{a_n}}{n} \text{ converges.}$$
   (b) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$. Show that $(\mathbb{F}^n, \|\cdot\|_2)$ is a normed space[1] where
$$\|x\|_2 := \sqrt{|x_1|^2 + \cdots + |x_n|^2}, \qquad x = (x_j)_{j=1}^n \in \mathbb{F}^n.$$

2. Let $a < c < b \in \mathbb{R}$, $\alpha : [a,b] \to \mathbb{R}$ a monotonic functions and $f, g : [a,b] \to \mathbb{R}$ bounded functions.
   (a) Show that f is Riemann-Stieltjes integrable with respect to α if and only if the restrictions $f_1 := f|_{[a,c]}$ and $f_2 := f|_{[c,b]}$ are so and that in this case:
$$\int_a^b f(x)\,\mathrm{d}\alpha = \int_a^c f_1(x)\,\mathrm{d}\alpha + \int_c^b f_2(x)\,\mathrm{d}\alpha\,.$$
   (b) Suppose there exists a set $M = \{a_1, \ldots, a_n\} \subset [a,b]$ such that α is continuous in M and $f(x) = g(x)$ for all $x \in [a,b] \setminus M$. Then $f \in \mathcal{R}(\alpha)$ if and only if $g \in \mathcal{R}(\alpha)$; in this case
$$\int_a^b f(x)\,\mathrm{d}\alpha = \int_a^b g(x)\,\mathrm{d}\alpha.$$

3. (a) Let $a \in \mathbb{R}_+$ and let $f : \mathbb{R} \to \mathbb{R}$, $f = \exp$. Use Riemann sums $s(f,P)$ and $S(f,P)$ to find $\int_0^a \exp(x)\,\mathrm{d}x$.
   (b) For $k, m \in \mathbb{N}$ find the integrals
$$\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,\mathrm{d}x, \qquad \int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,\mathrm{d}x.$$

4. (a) Does $\lim_{x\to\infty} \int_0^x \frac{\sin t}{t}\mathrm{d}t$ exist?
   (b) Does $\int_0^1 D(t)\mathrm{d}t$ exist, where D is the Dirichlet function
$$D : [0,1] \to \mathbb{R}, \quad D(t) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

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[1] Generally: Let $p \in (1,\infty)$. Show that $(\mathbb{F}^n, \|\cdot\|_p)$ is a normed space where $\|x\|_p := \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$.