Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 10

Differentiable functions; l'Hospital's rules; exponential functions. Hand in: April 2, 2009

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1. (a) Find all local and global extrema of

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \frac{2 \sin x}{2 - \cos^2 x}.$$

(b) Let $\mathcal{D} = (a, b)$ a nonempty interval and $f : \mathcal{D} \to \mathbb{R}$ a differentiable function with an isolated global minimum at $x_0 \in \mathcal{D}$. Is the following statement true:
There exist $c, d \in (a, b)$ such that $c < x_0 < d$ and $f'(x) \leq 0, \ x \in (c, x_0)$ and $f'(x) \geq 0, \ x \in (x_0, d)$.

2. **Darboux's Theorem**

Let $\mathcal{D} = (a, b) \subseteq \mathbb{R}$ a nonempty interval, $f : \mathcal{D} \to \mathbb{R}$ a differentiable function and $q \in \mathbb{R}$ such that

$$\inf\{f'(x) : x \in \mathcal{D}\} < q < \sup\{f'(x) : x \in \mathcal{D}\}.$$

Show that there exists a $c \in (a, b)$ such that $f'(c) = q$.

3. Find the following limits, if they exist:

(a) $\lim_{x \to \infty} \left(x - \sqrt[3]{x^3 - x^2 + 1} \right)$, (b) $\lim_{a \to \infty} \left(1 + \frac{x}{a} \right)^a$.

4. **Exponential function.**

For fixed $a \in \mathbb{R}^+ = (0, \infty)$ define the function

$$p_a : \mathbb{C} \to \mathbb{C}, \qquad p_a(z) = \exp(z \log(a)).$$

(a) For $a \in \mathbb{R}^+$ and $q \in \mathbb{Q}$ show

$$p_a(q) = a^q. \tag{$*$}$$

(b) Show that p_a is differentiable and find its derivative.

Recall. For $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$ we have defined

$$a^n := \prod_{n=1}^{n} a, \qquad a^0 := 1, \qquad a^{\frac{1}{n}} := \text{unique positive solution of } x^n = a.$$

Therefore $a^q := \left((a^\sigma)^{\frac{1}{m}} \right)^n$ is defined for all $q = \frac{\sigma n}{m} \in \mathbb{Q}$ with $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\sigma \in \{\pm 1\}$.

Remark. Because of the identity $(*)$ one defines

$$a^z := \exp(z \log(a)), \qquad a \in \mathbb{R}^+, \ z \in \mathbb{C}.$$