Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 9

Uniform continuity; power series; $\sin$ and $\cos$; π. Hand in: March 26, 2009

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1. Find the radius of convergence of

$$\text{i)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n (2z)^n}{n}, \qquad \text{ii)} \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n,$$

$$\text{iii)} \quad \sum_{n=1}^{\infty} (\sqrt{n}-1)^n z^n, \qquad \text{iv)} \quad \sum_{n=1}^{\infty} 8^n z^{3n}.$$

2. Show that the following functions are differentiable and find the derivative.

(a) $w : \mathbb{R}_+ \to \mathbb{R},\ x \mapsto \sqrt{x}$,

(b) $\cos : \mathbb{R} \to \mathbb{R}, \quad \sin : \mathbb{R} \to \mathbb{R}$.

Hint. Prove Euler's formula:

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z), \qquad z \in \mathbb{C}.$$

3. For $k \in \mathbb{N}$ let $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(x) := \begin{cases} x^k \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

For which k is f_k differentiable? If f_k is differentiable, is then its derivative $f_k' : \mathbb{R} \to \mathbb{R}$, $x \mapsto f_k'(x)$, continuous?

4. (a) Show that for all $x, y \in \mathbb{C}$ the following identities are true:

(i) $\sin^2(x) + \cos^2(x) = 1$.

(ii) $\sin(x+y) = \cos(x)\sin(y) + \cos(y)\sin(x)$,

(iii) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$,

(b) Show that $\{x \in \mathbb{R}_+ \,:\, \cos x = 0\} \neq \emptyset$.

Let $\pi := 2 \cdot \inf\{x \in \mathbb{R}_+ \,:\, \cos x = 0\}$.

(c) For $x \in \mathbb{R}$ show:

(i) $\sin x = 0 \iff \exists\, k \in \mathbb{Z} \quad x = k\pi$.

(ii) $\cos x = 0 \iff \exists\, k \in \mathbb{Z} \quad x = k\pi + \frac{\pi}{2}$.

Hint. Without proof you can use

$$1 - \frac{x^2}{2} \le \cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24}, \qquad x \in (0,3].$$