Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 7

Continuous functions.

Hand in: March 13, 2009

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1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces over a field $\mathbb{F}$ and let $(Z, \|\cdot\|_Z)$ be a normed space.

   (a) Show that $X \times Y$ together with
   $$\|(x, y)\| := \|x\|_X + \|y\|_Y, \qquad (x, y) \in X \times Y, \tag{$*$}$$
   is a normed space and that the following functions are continuous:
   $$X \to X \times X, \quad x \mapsto (x, 0) \qquad \text{and} \qquad X \to X \times X, \quad x \mapsto (0, x).$$

   (b) Assume that X is a field with a norm. Show that the functions
   $$\begin{aligned} A : X \times X \to X, &\quad (x, y) \to x + y, \\ M : X \times X \to X, &\quad (x, y) \to x \cdot y, \\ I : X \setminus \{0\} \to X, &\quad x \to \frac{1}{x}, \end{aligned}$$
   are continuous when $X \times X$ is equipped with the metric $(*)$. (For the continuity of A it suffices that X is a normed space.)

   (c) Show: $f, g : X \to Z$ continuous, then $f + g$ is continuous. In addition, if Z is a field, then $f \cdot g$ is continuous on X and $\frac{f}{g}$ is continuous on $\{x \in X : g(x) \neq 0\}$.

2. Where are the following functions are continuous? Proof your answer.

   (a) $w : [0, \infty) \to \mathbb{R}, \quad x \mapsto \sqrt{x}$,

   (b) $g : \mathbb{C} \to \mathbb{R}, \quad z \mapsto |z + \bar{z}^2|$,

   (c) $h : [-1, 1] \cup \{2\} \to \mathbb{R}, \quad x \mapsto \begin{cases} -\sqrt{-x}, & -1 \leq x \leq 0, \\ \sqrt{x}, & 0 < x \leq 1, \\ x, & x = 2. \end{cases}$

   (d) $D : \mathbb{R} \to \mathbb{R}, \quad D(x) := \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$

   Hint: Show that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, that is, for every $x \in \mathbb{R}$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \mathbb{Q}$ such that $\lim_{n \to \infty} x_n = x$.

3. Are the following functions uniformly continuous? Prove your answer.

   (a) $f : [0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x}{1+x^2}$,

   (b) $f : [0, \infty) \to \mathbb{R}, \quad f(x) = \frac{1+x^2}{1+x}$.

4. Let $f : [0, \infty) \to [0, \infty)$ be a bounded and continuous function. Show that there exists an p such that $f(p) = p$.