Universidad de los Andes
MATE-2201

# Analysis 1

Problem Sheet 6

Sequences and Series; e. Hand in: March 3, 2009

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1. **The Euler number** e**.**

For $n \in \mathbb{N}$ let $a_n := \big(1 + \frac{1}{n}\big)^n$ and $s_n := \sum_{k=0}^{\infty} \frac{1}{k!}$.

(a) Show that $2^k < k!$ for all $k \geq 4$ and that

$$1 \leq \left(1 + \frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!} < 3, \qquad n \in \mathbb{N}.$$

(b) Show that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ converge.

(c) Show that

$$\mathrm{e} := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}, \qquad n \in \mathbb{N}.$$

2. **Cauchy's condensation test.**

(a) For a monotonically decreasing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^0$ show

$$\sum_{n\in\mathbb{N}} a_n \text{ converges} \quad \Longleftrightarrow \quad \sum_{n\in\mathbb{N}} 2^n a_{2^n} \text{ converges.}$$

(b) Does the series $\sum_{n=1}^{\infty} (n \log_2 n)^{-1}$ converge? Prove your answer.
(Use what you know from the calculus courses about the logarithm.)

3. Find the 5-adic and 7-adic fraction of $\frac{1}{5}$. Prove your assertion.

4. Do the following series converge? Prove your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(b) $\displaystyle\sum_{n=2}^{\infty} b_m$, where $b_{2m} := \frac{1}{(2m)^2}$, $b_{2m+1} = -\frac{1}{2m}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \left(a + \frac{1}{n}\right)^n$ where $a \in \mathbb{R}$.