Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 5

Sequences. Hand in: February 24, 2009

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1. Let $q \in \mathbb{R}_+$ and $x_n := \sqrt[n]{q}$, $y_n := \sqrt[n]{n}$, $n \in \mathbb{N}$. Do the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge? If so, find the limit.

2. (a) Find a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$\inf\{a_n : n \in \mathbb{N}\} < \liminf\{a_n : n \in \mathbb{N}\} < \limsup\{a_n : n \in \mathbb{N}\} < \sup\{a_n : n \in \mathbb{N}\}.$$

In this case, must the set $\{a_n : n \in \mathbb{N}\}$ have a maximum?

(b) Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and define sequences $(y_k)_{k\in\mathbb{N}}$, $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{R} \cup \{\pm\infty\}$ by

$$y_k := \sup\{x_n : n \geq k\}, \qquad z_k := \inf\{x_n : n \geq k\}, \qquad k \in \mathbb{N}.$$

Show that $(y_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ converge in $\mathbb{R} \cup \{\pm\infty\}$ and that

$$\lim_{k\to\infty} y_k = \limsup x_n, \qquad \lim_{k\to\infty} z_k = \liminf x_n.$$

3. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $a_n \neq 0$ for all $n \in \mathbb{N}$. Show or find a counter-example:

(i) If there exists an $N \in \mathbb{N}$ and $q \in \mathbb{R}$, $q < 1$, such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q, \quad n \in \mathbb{N},\ n \geq N,$$

then $\lim_{n\to\infty} a_n = 0$.

(ii) If there exists an $N \in \mathbb{N}$ and $q \in \mathbb{R}$, $q \leq 1$, such that

$$\left|\frac{a_{n+1}}{a_n}\right| < q, \quad n \in \mathbb{N},\ n \geq N,$$

then $\lim_{n\to\infty} a_n = 0$.

4. The Fibonacci sequence $(a_n)_{n\in\mathbb{N}}$ is defined recursively by

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}, \qquad n \in \mathbb{N}.$$

Moreover, let $\sigma < \tau$ be the solutions of $x^2 - x - 1 = 0$ and

$$x_n = \frac{a_{n+1}}{a_n}, \qquad n \in \mathbb{N}.$$

(a) Show that $(a_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$.

(b) $a_n = \dfrac{1}{\sqrt{5}}(\tau^{n+1} - \sigma^{n+1}), \quad n \in \mathbb{N}.$

(c) $\lim_{n\to\infty} x_n = \tau.$