Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 4

Metric spaces and sequences. Hand in: February 17, 2009

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1. (a) Let (X, d), $X \neq \emptyset$, be a metric space and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in X. Show: If there exists an $a \in X$ such that

$$\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n,$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Is the converse also true (proof or counter example)?

(b) Let $n \in \mathbb{N}$ and let

$$\|\cdot\|_{\max} : \mathbb{R}^n \to \mathbb{R}, \quad \|x\|_{\max} = \max\{|x_j| \;:\; j = 1, \ldots, n\} \quad \text{for } x = (x_j)_{j=1}^n \in \mathbb{R}^n.$$

Show that $(\mathbb{R}^n, \|\cdot\|_{\max})$ is a normed space. Sketch the unit ball in the case $n = 2$.

2. (a) Let (X, d), $X \neq \emptyset$, be a metric space and $M \subseteq X$. Show that the following is equivalent:
   (i) M is bounded.
   (ii) $\exists\, x \in X \;\exists\, r > 0 : M \subseteq B_r(x)$.
   (iii) $\forall\, x \in X \;\exists\, r > 0 : M \subseteq B_r(x)$.

(b) For $M \subseteq \mathbb{R}$ show that M is bounded as subset of the ordered field $(\mathbb{R}, >)$ (Definition 1.1) if and only if M is bounded as subset of the metric space $(\mathbb{R}, d)$ (Definition 4.3) where $d(x, y) = |x - y|$.

3. Let (X, d) be a metric space and $\varrho : \mathbb{N} \to \mathbb{N}$ a bijection. Show: If $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges, then also $(x_{\varrho(n)})_{n\in\mathbb{N}} \subseteq X$ converges and has the same limit.

4. (a) Let $x_n = \sqrt{1 + n^{-1}}$, $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

(b) Do the following real sequences converge? If so, find the limit. Prove your assertions.

   (i) $(a_n)_{n\in\mathbb{N}}$ with $a_n = \dfrac{2^n}{n!}, \quad n \in \mathbb{N},$

   (ii) $(b_n)_{n\in\mathbb{N}}$ with $b_n = \sqrt{1 + n^{-1} + n^{-2}}, \quad n \in \mathbb{N},$

   (iii) $(c_n)_{n\in\mathbb{N}}$ with $c_n = \dfrac{1 + 2 + \cdots + n}{n + 2} - \dfrac{n}{n}, \quad n \in \mathbb{N},$

   (iv) $(d_n)_{n\in\mathbb{N}}$ with $d_n = \sqrt{n^2 + n + 1} - n, \quad n \in \mathbb{N},$