Universidad de los Andes
MATE-2201

Analysis 1

Problem Sheet 3

Supremum; complex numbers. Hand in: February 10, 2009

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1. (a) For every $x \in \mathbb{R}_+$ there exists an $n \in \mathbb{N}_0$ with $n \leq x < n+1$.(Proposition 3.19)
   (b) Every interval in $\mathbb{R}$ contains a rational number. (Proposition 3.20)
   (c) $\mathbb{Q}$ does not have the least upper bound property. (Hint: $\{x \in \mathbb{Q} : x^2 \leq 2\}$)

2. Find the infimum and supremum of the following sets in the ordered field $\mathbb{R}$. Determine if they have a maximum and a minimum.
   (a) $\{x \in \mathbb{R} : \exists\, n \in \mathbb{N} \;\; x = n^2\}$,
   (b) $\big\{x \in \mathbb{R} : \exists\, n \in \mathbb{N} \;\; x = \dfrac{1}{n^3}\big\}$,
   (c) $\big\{x \in \mathbb{R} : \exists\, n \in \mathbb{N} \;\; x = \dfrac{1}{n} + n\big(1 + (-1)^n\big)\big\}$,
   (d) $\big\{x \in \mathbb{R} : x^2 \leq 2\big\} \cap \mathbb{Q}$.

3. (a) Let $X \subset \mathbb{R}$, $X \neq \emptyset$, and $\xi \in \mathbb{R}$ an upper bound of X. Show that
$$\xi = \sup X \qquad \Longleftrightarrow \qquad \forall \varepsilon \in \mathbb{R}_+ \; \exists\, x_\varepsilon \in X \;\; \xi - \varepsilon < x_\varepsilon \leq \xi.$$
   What is the analogous statement for $\inf X$?
   (b) Let $X,\ Y \subset \mathbb{R}$ non empty sets such that
$$\forall\, x \in X \;\; \exists\, y \in Y \;:\; y < x.$$
   Does that imply $\inf Y < \inf X$? Proof your assertion.

4. (a) Show that for every $z \in \mathbb{C} \setminus \{0\}$ there exist exactly two numbers $\zeta_1, \zeta_2 \in \mathbb{C}$ such that $\zeta_1^2 = \zeta_2^2 = z$.
   (b) Let $a, b, c \in \mathbb{C}$, $a \neq 0$. Show that there exists at least one $z \in \mathbb{C}$ such that
$$az^2 + bz + c = 0.$$