THE PHRAGMÉN LINDELÖF THEOREM AND INTERPOLATION THEORY

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Abstract. In this note we present the classic theorem of Phragmén Lindelöf now known as the three lines theorem. We show as this result can be used in the proof of the Riesz-Thorin interpolation Theorem, a classical result in interpolation of operators on Banach spaces.
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1. The Riesz-Thorin interpolation Theorem and some of its consequences

Let $(M,\mathcal{M},\mu)$ and $(N,\mathcal{N},\eta)$ be a measure spaces and T be an operator mapping a linear space of μ-measurable functions D on $(M,\mathcal{M},\mu)$ into η-measurable functions on $(N,\mathcal{N},\eta)$. We assume that D contains all characteristic functions of sets of finite measure. Let us assume that T extends to a bounded operator (this extension will be denoted by T) from $L^{p_i}(\mu)$ to $L^{q_i}(\eta)$, $i=0,1$ and $1\leq p_i,q_i\leq\infty$. With notation above, the Riesz-Thorin interpolation Theorem can be formulated of the following way:

Theorem (Riesz-Thorin Interpolation Theorem). Suppose that a linear operator T on D extends to a bounded from $L^{p_i}(\mu)$ into $L^{q_i}(\eta)$, $i=0,1$ with norm k_i. Then T extends to a bounded operator from $L^{p_t}(\mu)$ into $L^{q_t}(\eta)$ where

$$\frac{1}{p_t}=\frac{1-t}{p_0}+\frac{t}{p_1} \tag{1.1}$$

and

$$\frac{1}{q_t}=\frac{1-t}{q_0}+\frac{t}{q_1}, \tag{1.2}$$

for all $0\leq t\leq 1$, with norm $k_t\leq k_0^{1-t}k_1^t$.

In order to illustrate this theorem we consider two consequences. First, let us consider the operator Fourier transform on $\mathbb{R}^n$, $\mathscr{F}:\mathscr{S}\to\mathscr{S}$ which is continuous bijection on the Schwartz space $\mathscr{S}$ defined by those functions f satisfying

$$\sup_{x\in\mathbb{R}^n}(1+|x|^2)^{\frac{n}{2}}|\partial_x^\alpha f(x)|<\infty. \tag{1.3}$$

The Fourier transform is defined by the formula

$$\mathscr{F}(f)(\xi)=\int_{\mathbb{R}^n}e^{-i2\pi\langle x,\xi\rangle}f(x)dx, \tag{1.4}$$

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where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$, and $f\in\mathscr{S}$. It is easy to see that $\|\mathscr{F}f\|_{L^\infty}\leq\|f\|_{L^1}$. Hence, $\mathscr{F}$ extends to a bounded operator from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. On the other hand, the well known Plancherel Theorem states that the Fourier transform extends to a bounded operator on $L^2(\mathbb{R}^n)$ and $\|\mathscr{F}f\|_{L^2}=\|f\|_{L^2}$. With this in mind the Riesz-Thorin interpolation theorem with $p_0=1$, $p_1=2$, $q_0=\infty$ and $q_1=2$ implies that the Fourier transform $\mathscr{F}$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ into $L^{p'}(\mathbb{R}^n)$ for all $1<p\leq 2$, with norm $k_p\leq 1$. Here $1/p+1/p'=1$. In this case

$$\|\mathscr{F}(f)\|_{L^{p'}}\leq\|f\|_{L^p} \tag{1.5}$$

which is known as the Hausdorff-Young inequality. Now, we consider the case of the convolution operator $T_f:\mathscr{S}\to\mathscr{S}$ given by

$$T_f(g)(x)=f*g(x):=\int_{\mathbb{R}}f(y)g(x-y)dy. \tag{1.6}$$

Here the function f associated to the operator T_f is a fixed function on L^p, $1<p<\infty$. An immediate application of the Hölder inequality gives that T extends to a bounded operator from $L^{p'}$ into L^∞ with norm $\leq\|f|_{L^p}$. T_f is also a bounded operator from L^1 into L^p with operator norm bounded by $\|f\|_{L^p}$. In fact, by using the Minkowski integral inequality we obtain

$$\begin{aligned}(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}f(x-y)g(y)dy\right|^{\frac{1}{p}}dx)^{\frac{1}{p}} &\leq\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x-y)g(y)|^p dx\right)dy\\ &=\|f\|_{L^p}\|g\|_{L^1}.\end{aligned}$$

Finally, by using the Riesz-Thorin interpolation theorem we deduce that $\mathscr{F}$ extends to a bounded operator from L^{p_t} into L^{q_t} where $1/p_t=1-t+t/p'$ and $1/q_t=(1-t)/p$, $0\leq t\leq 1$. As a consequence of this fact $\mathscr{F}$ is a bounded operator from L^r into L^q for all $1/q=1/r+1/p-1$. This is know as the Young inequality. With this result we end this section.

2. The Phragmén-Lindelöf Theorem and the Proof of the Riesz-Thorin interpolation Theorem

As an application of complex analysis to interpolation theory we present a well known proof of the Riesz-Thorin interpolation theorem by using the three lines theorem. Now, we present this theorem with its proof and later we prove the Riesz-Thorin Theorem.

Theorem (Phragmén-Lindelöf Theorem). Suppose F is a bounded and continuous function on the strip $S=\{z=x+iy:-\infty<y<\infty \text{ and } 0\leq x\leq 1\}$. If F is holomorphic in the interior of S and $|F(iy)|\leq k_0$, $|F(1+iy)|\leq k_1$, then for all $0\leq x\leq 1$, $|F(x+iy)|\leq k_0^{1-x}k_1^x$.

Proof. We assume that $k_0,k_1>0$. If we put $G(z)=\frac{F(z)}{k_0^{1-z}k_1^z}$ we reduce the problem to the case $k_i=1$ and we only need to prove that $|G(z)|\leq 1$. However we will working with F instead of G in our arguments. If we assume that

$$\lim_{|y|\to\infty} |F(x+iy)| = 0$$

then we can find $y_0 > 0$ such that, if $|y| \geq y_0$ then $|F(x+iy)| \leq 1$. Now, if we apply the maximum principle to the region $A = \{z = x+iy : 0 \leq x \leq 1 \text{ and } |y| \leq y_0\}$, we obtain that $|f(z)| \leq 1$ for every $z \in A$ and therefore $|F(z)| \leq 1$ for every $z \in S$. In the general case where F does not satisfy the precedent condition, we define for every $n \in \mathbb{N}$ the function $F_n(z) = F(z)e^{(z^2-1)/n}$, $z \in S$. If $z = x+iy$ then $|F_n(z)| \leq |F(x+iy)|e^{(x^2-1)/n}e^{-y^2/n}$. Clearly $|F_n(x+iy)| \to 0$ as $|y| \to \infty$, uniformly on x. So, by the precedent argument we have that $|F_n(z)| \leq 1$ for every $z \in S$. Now, if we take limit in both sides when $n \to \infty$ we obtain the desired inequality. □

Now, we prove the Riesz-Thorin interpolation theorem. In order to do this we define $\alpha_j = 1/p_j$, $\beta_j = 1/q_j$, $j = 0, 1$. $\alpha = 1/p_t$ and $\beta = 1/q_t$, $p = p_t$ and $q = q_t$. With this in mind, if $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$ and $\beta(z) = (1-z)\beta_0 + z\beta_1$, $z \in \mathbb{C}$, $\alpha(j) = \alpha_j$, $\beta(j) = \beta_j$, $j = 0, 1$ and $\alpha(t) = \alpha$, $\beta(t) = \beta$. Since

$$\|h\|_{L^q} = \sup\{|\int_N hv\, d\eta|\},$$

Where the sup is taken on all simple functions v satisfying $\|g\|_{L^{q'}} = 1$. So, we have that, if f is a simple function on M, then

$$\|Tf\|_{L^q} = \sup\{|\int_N (Tf)g\, d\eta|\},$$

Where the sup is taken on all simple functions g satisfying $\|g\|_{L^{q'}} = 1$. If we assume that $\|f\|_{L^p} = 1$ where f is a simple function, then we only need to prove that the integral

$$I = \int_N (Tf)g$$

has norm less than or equal to $k_0^{1-t}k_1^t$. Let us write

$$f = \sum_{j=1}^{m} a_j 1_{E_j}, \quad g = \sum_{k=1}^{m} a_k 1_{F_k}. \tag{2.1}$$

If $\theta_j = \arg(a_j)$ and $\phi_k = \arg(b_k)$, for every $z \in \mathbb{C}$, let us define

$$f_z = \sum_{j=1}^{m} |a_j|^{\alpha(z)/\alpha} e^{\theta_j} 1_{E_j}, \quad g = \sum_{k=1}^{m} |b_k|^{(1-\beta(z))/(1-\beta)} e^{i\phi_k} 1_{F_k}. \tag{2.2}$$

So, we obtain an entire function if we write

$$F(z) = \int_N (Tf_z)g_z d\eta = \sum_{j,k=1}^{n} |a_j|^{\alpha(z)/\alpha} |b_k|^{(1-\beta(z))/(1-\beta)} \rho_{jk}, \tag{2.3}$$

with $\rho_{jk} = e^{i(\theta_j+\phi_k)} \int_N (T1_{E_j})1_{E_k} d\eta$. Every term $|a_j|^{\alpha(z)/\alpha}|b_k|^{(1-\beta(z))/(1-\beta)}\rho_{jk}$ in the sum is a bounded function on S, therefore the function F is bounded on S.

F is continuous function on S and holomorphic on the interior of S. Moreover $|f_{x_i+iy}|^{p_i} = |f|^p$, $|g_{x_i+iy}|^{q_i'} = |g|^{q'}$ if $x_0 = 0$, $x_1 = 1$. Hence, if $x = 0, 1$ we have

$$\begin{aligned} |F_{x_i+iy}| &\leq \|Tf_{x_i+iy}\|_{L_i^q} \|g_{x_i+iy}\|_{L^{q_i'}} \\ &\leq k_i \|f_{x_i+iy}\|_{L^{p_i}} \|g_{x_i+iy}\|_{L^{q_i'}} = k_i \|f\|_{L^p} \|g_x\|_{L^{q'}} = k_i. \end{aligned}$$

Thus, by the Phragmén-Lindelöf theorem we obtain that $|F(x+iy)| \leq k_0^{1-x} k_1^{1-x}$ for every $z = x+iy \in S$. In particular $|F(t)| = |\int_N (Tf) g \, d\mu| \leq k_0^{1-t} k_1^t$. Hence the operator norm of the extension of T to a bounded operator from L^p into L^q is less than or equal to $k_0^{1-t} k_1^t$. So, we end the proof.

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References

1. Stein, E. Weiss, G. Introduction to Fourier analysis on euclidean spaces. Princeton University Press. (1971)

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