Contents

Chapter 1

Introduction

In general relativity, the stage of physics is a four-dimensional differentiable manifold with a Lorentzian metric (M, g), the spacetime. Events in spacetime are points in M. Spacetime, however, is not an immutable stage for physical action to take place, but is itself part of physics. The relation between the geometry of spacetime and the energy contained in it is given by Einstein's equation

$$R_{ab} - \tfrac{1}{2}\, R\, G_{ab} \;=\; 8\pi\, T_{ab}, \tag{1.1}$$

where the left hand side is the so-called Einstein tensor, which involves the metric g and derivatives thereof, and hence describes the geometry of spacetime. The right hand side is the stress-energy tensor arising from the energy distribution in spacetime. System (1.1) is a nonlinear partial differential equation for the components $g_{\mu\nu}$ of the metric g. A differentiable curve γ in spacetime is called timelike if $g(\xi, \xi) < 0$ for all tangent vectors ξ on the curve (if the metric g has signature $(-{+}{+}{+})$). A spacetime is called stationary if it admits an isometry Φ_t whose orbits are timelike curves. It has been shown that all stationary electrovac solutions of Einstein's equation are given by a three parameter family, the so-called Kerr-Newman metric. If the Kerr-Newman metric describes the spacetime outside a black hole, then the three parameters M, Q and a in the Kerr-Newman metric have the physical interpretation as the mass M, the electric charge Q and the angular moment per mass $a = J/M$ of the black hole. That the field outside a stationary black hole is determined by three parameters only has been summarised by J. A. Wheeler in the statement that "a black hole has no hair". For this and other results on black holes and general relativity we refer primarily to [Wal84], [FN98] and the references therein.

In this work we consider a spin-$\frac{1}{2}$ particle in the Kerr-Newman background metric. Such particles are described by a four component spinor Ψ subject to the Dirac equation. In Kerr-Newman spacetime, the Dirac equation is a coupled system of partial differential equations which can be written in the form

$$(\widehat{\mathfrak{R}} + \widehat{\mathfrak{A}})\widehat{\Psi} \;=\; 0, \qquad (t,\, r,\, \vartheta,\, \varphi) \in (-\infty,\, \infty) \times (r_+,\, \infty) \times (0,\, \pi) \times (-\pi,\, \pi), \tag{1.2}$$

see [Pag76] and [Cha98]; the explicit form of the differential expressions $\widehat{\mathfrak{R}}$ and $\widehat{\mathfrak{A}}$ is given in (2.5). A priori it is not clear how this formal differential expression can be implemented in an operator theoretical context. Physical considerations imply that this operator should act on an $\mathscr{L}^2$-space since the solutions $\widehat{\Psi}$ are to be interpreted as the possible wave functions of a fermion. Taking into account the functional determinant arising from the Kerr-Newman metric, the integration weight in the $\mathscr{L}^2$-space should be $\sin\vartheta\, \Sigma(r, \vartheta) = \sin\vartheta(r^2 + a^2 \cos^2\vartheta)$, see (2.2).

The left hand side of the equation in (1.2) is well defined on the space of all smooth functions with compact support. However, it is not clear if the operator defined in this way is essentially selfadjoint in the $\mathscr{L}^2$-space described above (or any other suitable Hilbert space), thus providing a unique canonical description of the physical situation.

In this work we follow the approach of Chandrasekhar by applying a suitable ansatz for $\widehat{\Psi}$ such that the Dirac equation is separated into the following coupled system of differential equations (see (2.8))

$$(\mathfrak{R}^{(d)} - \lambda)\begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0, \qquad (\mathfrak{A}^{(d)} - \lambda)\begin{pmatrix} S_- \\ S_+ \end{pmatrix} = 0; \tag{1.3}$$

the first one, the so-called radial equation, is an ordinary differential equation with respect to the radial coordinate r on (r_+, ∞), while the second one, the so-called angular equation, is an ordinary differential equation with respect to the angular coordinate ϑ in the interval $(0, \pi)$. The full solution of the Dirac equation is then given by

$$\Psi(t, r, \vartheta, \varphi) \;=\; \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{e}^{-\mathrm{i}(k+\frac{1}{2})\varphi} \begin{pmatrix} X_-(r)\, S_+(\vartheta) \\ X_+(r)\, S_-(\vartheta) \\ X_+(r)\, S_+(\vartheta) \\ X_-(r)\, S_-(\vartheta) \end{pmatrix}$$

for $(t, r, \vartheta, \varphi) \in (-\infty, \infty) \times (r_+, \infty) \times (0, \pi) \times (-\pi, \pi)$.
The quantity ω is interpreted as the energy of the fermion in the Kerr-Newman metric as measured by a distant observer. In the special case $a = 0$, i.e., if the spacetime is spherically symmetric, the number $k \in \mathbb{Z}$ is the z-component of the total angular momentum $\vec{\mathfrak{J}}$ of the fermion. In this case, the operator associated with $\mathfrak{A}^{(d)}$ can be identified with the spin-orbit operator $\mathfrak{K}$ in usual relativistic quantum mechanics, and the coupling parameter λ is an eigenvalue of $\mathfrak{K}$, see section 3.1.

Both the angular equation and the radial equation in (1.3) admit an operator theoretical realisation in a Hilbert space. It has been shown in [BM99] that the formal differential operator representing the radial equation gives rise to an essentially selfadjoint operator in a weighted $\mathscr{L}^2$-space whose essential spectrum comprises the whole real axis. We show that the spectrum of the angular operator consists only of eigenvalues; so far, only numerical approximations are known in the literature [SFC83], [Cha84].

The aim of this work is to establish analytical bounds for the eigenvalues of the angular operator $\mathcal{A}$ in terms of the physical parameters a, m and ω. To this end, we first realise the formal differential expression $\mathfrak{A}^{(d)}$ in the case $k \in \mathbb{R} \setminus (-1, 0)$ as a selfadjoint operator $\mathcal{A}$ in a suitable $\mathscr{L}^2$-space and show that the spectrum consists of isolated eigenvalues only. Then we apply various techniques that give rise to different kinds of bounds: First, we derive a lower bound for the modulus of the eigenvalues of $\mathcal{A}$ by means of an off-diagonalisation of the angular operator. Then we apply a variational principle for operator valued functions to obtain a formula for the eigenvalues of the angular operator $\mathcal{A}$ in a certain right half plane which yields upper and lower bounds for these eigenvalues of $\mathcal{A}$. Finally, for certain values of $a\omega$ and k, we establish another lower bound for the modulus of the eigenvalues of $\mathcal{A}$ that differs substantially from the bounds derived by the methods above. The proof relies on the fact that $\mathcal{A}$ is unitarily equivalent to a block operator matrix $\mathcal{A}_U$ such that the spectra of the diagonal entries of $\mathcal{A}_U$ do not overlap. Observe that the first two techniques apply not only to the angular operator, but to a wider class of block operator matrices. The results of these methods are compared with bounds for the eigenvalues obtained from standard perturbation theory and with numerical results in the above mentioned papers.

The thesis is organised as follows. In chapter 2 we investigate the angular equation and its operator theoretical implementation in a Hilbert space. After a certain transformation of $\mathfrak{A}^{(d)}$ we obtain the formal differential expression $\mathfrak{A}$ on the interval $(0,\pi)$ which is formally symmetric in the Hilbert space $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2$. With $\mathfrak{A}$ we associate the minimal angular operator

$$\mathcal{D}(\mathcal{A}^{\min}) \;:=\; \mathcal{C}_0^\infty(0,\pi), \qquad \mathcal{A}^{\min}\Psi \;:=\; \mathfrak{A}\Psi. \tag{1.4}$$

In section 2.1 we show that $\mathcal{A}^{\min}$ is essentially selfadjoint for all wave numbers $k \in \mathbb{Z}$ (we obtain this result even for all $k \in \mathbb{R}\setminus(-1,\,0)$). The unique selfadjoint extension of $\mathcal{A}^{\min}$ is denoted by $\mathcal{A}$; it has the block operator matrix representation

$$\mathcal{D}(\mathcal{A}) \;=\; \mathcal{D}(B^*)\oplus\mathcal{D}(B), \qquad \mathcal{A} \;=\; \begin{pmatrix} A & B \\ B^* & D\end{pmatrix} \tag{1.5}$$

with the bounded multiplication operators

$$\mathcal{D}(A) \;=\; \mathcal{D}(D) \;=\; \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta), \qquad A \;=\; -D \;=\; am\cos\vartheta \tag{1.6}$$

and the closed first order differential operator

$$\begin{aligned} \mathcal{D}(B) \;&=\; \{f \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) \,:\, f \text{ is absolutely continuous, } \mathfrak{B}_+ f \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)\}, \\ Bf \;=\; \mathfrak{B}_+ f \;&:=\; \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big) f. \end{aligned} \tag{1.7}$$

In section 2.2 we show that the spectrum of $\mathcal{A}$ consists only of simple eigenvalues without accumulation points in $(-\infty,\,\infty)$ and that it is neither bounded from below nor from above.

In chapter 2.3 we establish several symmetry properties of the angular operator with respect to the physical parameters a, m, ω and k that will prove useful in the subsequent chapters.

In chapter 3 we apply an abstract off-diagonalisation method for block operator matrices to obtain lower bounds for the modulus of the eigenvalues of $\mathcal{A}$. In section 3.1 we consider the case $a = 0$ where the eigenvalues of $\mathcal{A}$ are known explicitly (see lemma 3.3):

$$\sigma_p(\mathcal{A}) \;=\; \{\lambda_n = \operatorname{sign}(n)(|k+\tfrac12| - \tfrac12 + n) \,:\, n \in \mathbb{Z}\setminus\{0\}\} \qquad \text{if} \;\; a = 0.$$

In the rest of this chapter we establish lower bounds for the eigenvalues λ of $\mathcal{A}$ in the case $a \neq 0$, first by using standard perturbation theory in section 3.2, then with the help of an off-diagonalisation method in section 3.3. Based on the off-diagonalisation method, corollary 3.18 in section 3.3.1 yields a lower bound for the modulus of the eigenvalues of block operator matrices of type (1.5) under the assumption that A and D are bounded and that B and B^* are boundedly invertible. Remark 3.32 shows that in the special case of the angular operator the same lower bound can be obtained from standard perturbation theory; however, the off-diagonalisation also yields a lower bound for the modulus of the eigenvalues if one of the operators A or D is only relatively bounded with respect to B^* or B. In section 3.3.2 we apply the off-diagonalisation method to the angular operator. To this end, we first show that the off-diagonal entries B and B^* of the angular operator are indeed boundedly invertible. Using the explicit form of their inverses, we derive a lower bound for $\|B^{-1}\|^{-1}$, see lemma 3.30 for a rather rough estimate, and lemma 3.34 for a refined estimate which is obtained by an iteration process. Another lower bound for $\|B^{-1}\|^{-1}$ is provided in section 4.2.2 where estimates for the eigenvalues of BB^* are obtained by Sturm's comparison theorem. For most values of the parameters a, k, m and ω the latter estimate gives sharper lower bound for the modulus of the eigenvalues λ than the bounds obtained by the iteration method; nevertheless, there are situations where the bounds obtained by the iteration method are tighter, see, e.g., figure 6.1.

Finally, we show with the help of the off-diagonalisation method in combination with the special form of the entries of the angular operator that under certain conditions on am and k there is an interval that contains no eigenvalues of $\mathcal{A}$.

In chapter 4 we obtain a variational characterisation of the eigenvalues of $\mathcal{A}$ to the right of the spectrum of D by applying the variational principle from [EL04]. From this formula, upper and lower bounds in terms of the eigenvalues of BB^* are deduced.
Note that the classical variational principle based on the Rayleigh functional (see, e.g., [RS78]) does not apply here since the operator $\mathcal{A}$ is not semibounded. In [GS99] a variational principle for eigenvalues of operator matrices in a gap of the essential spectrum was proved where the authors did not assume that the operator was semibounded. However, they assumed that the spectra of the operators on the diagonal do not overlap, so that, roughly speaking, the given decomposition of the Hilbert space is close to the decomposition of the Hilbert space into spectral subspaces of the operator matrix under consideration. In the case of the angular operator, however, the spectra of the diagonal entries coincide so that the result of [GS99] does not apply either. In recent works, various types of block operator matrices and their spectral properties have been considered, for a survey we refer the reader to [Tre00]. In section 4.1 we consider so-called off-diagonal dominant selfadjoint block operator matrices

$$\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}, \qquad \mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}^*) \oplus \mathcal{D}(T_{12}),$$

where all T_{ij}, $i,j \in \{1,2\}$ are closed and T_{11} and T_{22} are relatively bounded with respect to T_{12}^* and T_{12}, respectively. The term "off-diagonal dominant" refers to the fact that the diagonal entries are dominated by the off-diagonal entries. Note that the selfadjointness of $\mathcal{T}$ implies that the restrictions of T_{11} to $\mathcal{D}(T_{12}^*)$ and of T_{22} to $\mathcal{D}(T_{12})$ are symmetric. Further we assume T_{11} to be semibounded from below, and we suppose that there exists a $c_2 \in \mathbb{R}$ such that $(c_2, \infty) \subseteq \rho(T_{22})$. Observe that we do not require that the spectra of these operators are separated.
For λ to the right of c_2 we associate the Schur complement

$$S_1(\lambda) = T_{11} - \lambda - T_{12}(T_{22} - \lambda)^{-1} T_{12}^*, \qquad \lambda \in (c_2, \infty),$$

with the block operator matrix $\mathcal{T}$. Since the operators T_{11} and T_{12} may be unbounded, the domain of $S_1(\lambda)$ has to be chosen carefully. In corollary 4.9 we show that the spectrum of the Schur complement and the spectrum of $\mathcal{T}$ to the right of c_2 coincide if the Schur complement with an appropriate domain is selfadjoint and if T_{12} is surjective. A sufficient condition for the existence of a selfadjoint Schur complement $S_1(\lambda)$ is that T_{22} is bounded and that T_{11} is relatively bounded with respect to $T_{12}(T_{22} - \lambda)^{-1} T_{12}^*$ with relative bound less than 1. Under these assumptions we prove a variational principle that gives rise to upper and lower bounds for the eigenvalues of $\mathcal{T}$ in an interval (c_2, λ_e) having empty intersection with the essential spectrum of $\mathcal{T}$.
In the special case where in addition to the above mentioned assumptions we also suppose that the spectrum of the operator $\mathcal{T}_0 = \left(\begin{smallmatrix} 0 & T_{12} \\ T_{12}^* & 0 \end{smallmatrix}\right)$ consists of isolated eigenvalues only, that there is a bound $b > 0$ such that $\|T_{12}^* x\| \geq b\|x\|$ for all $x \in \mathcal{D}(T_{12}^*)$, and that the Schur complement $S_1(\lambda)$ is selfadjoint with domain $\mathcal{D}(S_1(\lambda)) =: \mathcal{D}(S_1)$ independent of λ, the variational principle gives rise to the following estimate of the eigenvalues of $\mathcal{T}$ in (c_2, λ_e), see theorem 4.25:

$$\begin{aligned} \lambda_n &\leq \frac{\alpha_{21}}{2}\sqrt{\nu_{n+n_0}} + \sqrt{\nu_{n+n_0} + \tfrac{1}{4}(\alpha_{21}\sqrt{\nu_{n+n_0}} + \|T_{22}\| + \alpha)^2} + \tfrac{1}{2}(\alpha + c_2), \quad 1 \leq n \leq N, \\ \lambda_n &\geq \sqrt{\nu_{n+n_0}} + \tfrac{1}{2}(c_1 - \|T_{22}\|), \qquad 1 \leq n \leq N. \end{aligned} \tag{1.8}$$

where λ_n is the nth eigenvalue of the operator $\mathcal{T}$ greater than c_2. The numbers $\sqrt{\nu_n}$ are the eigenvalues of $\mathcal{T}_0$ greater than 0, n_0 is an index shift due to the variational principle, the numbers α

and α_{21} arise from the relative boundedness of T_{11} with respect to T_{12}^*, and c_1 is a lower bound for T_{11}. In the case $T_{11} = 0$, $T_{22} = 0$ we obtain $\lambda_n = \sqrt{\nu_n}$; therefore inequalities (1.8) can be regarded as a perturbation result for a certain class of off-diagonal dominant block operator matrices with an unbounded perturbation of one diagonal entry.
The upper bound in (1.8) can be further improved if we assume that both T_{11} and T_{22} are bounded. Under the further condition that $T_{11} = -T_{22}$, satisfied by the angular operator, we obtain the following two-sided estimate, see theorem 4.28,

$$\sqrt{\nu_{n_0+n}} - \|T_{22}\| \;\leq\; \lambda_n \;\leq\; \sqrt{\nu_{n_0+n}} + \|T_{22}\|, \qquad 1 \leq n \leq N. \tag{1.9}$$

In the case of bounded T_{11} and T_{22}, also standard perturbation theory is applicable; we compare the bounds (1.9) with the corresponding results of standard perturbation theory.
Finally, in section 4.2, we apply the theorems of section 4.1 to the angular operator $\mathcal{A}$. All above mentioned assumptions on $\mathcal{T}$ and its entries are satisfied by $\mathcal{A}$. In section 4.2.2 we obtain estimates for the eigenvalues of $\mathcal{B} := \mathcal{A} - \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$ with the help of Sturm's comparison theorem applied to the second order differential expression associated with the operator BB^*. Inserting these bounds into (1.9) we get explicit bounds for the eigenvalues of $\mathcal{A}$ to the right of $\|D\| = |am|$ in terms of the physical parameters a, m, ω and k.

A completely different approach to obtain a lower bound for the modulus of the eigenvalues of $\mathcal{A}$ is used in chapter 5. There we apply a unitary transformation U to $\mathcal{A}$ to obtain the unitarily equivalent operator

$$U\mathcal{A}U^{-1} =: \mathcal{A}_U = \begin{pmatrix} -D_U & B_U \\ B_U^* & D_U \end{pmatrix} = \begin{pmatrix} \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + am\cos\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + am\cos\vartheta & -(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta) \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_U) := U\mathcal{D}(\mathcal{A})$$

on the Hilbert space $\mathcal{H}_U := \mathcal{H}_{U,1} \oplus \mathcal{H}_{U,2} := U(\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2)$. Under certain assumptions on k and $a\omega$ the entries $\pm D_U$ on the diagonal of the transformed operator have separated spectra. Operator matrices of this type have been investigated in [LT98] and [LT01]. However, all the entries in $\mathcal{A}_U$ are unbounded, and it is not at all clear that $\mathcal{A}_U$ is still a block operator matrix, that is, that its domain can be written as a direct sum $\mathcal{D}(\mathcal{A}_U) = \mathcal{D}_{U,1} \oplus \mathcal{D}_{U,2}$ for suitable linear manifolds $\mathcal{D}_{U,1} \subseteq \mathcal{H}_{U,1}$ and $\mathcal{D}_{U,2} \subseteq \mathcal{H}_{U,2}$. In fact, remark 5.4 shows that $\mathcal{A}_U$ is not a block operator matrix if $k \in \{-1, 0\}$. Since all the entries of $\mathcal{A}_U$ are unbounded, we introduce sesquilinear forms associated with the operators constituting $\mathcal{A}_U$. The eigenvalue equation $(\mathcal{A}_U - \lambda)\Psi_U = 0$ gives rise to a linear system of equations in $\mathbb{R}^2$, cf. the proof of theorem 5.9. The essential assumption for the proof of theorem 5.10, which is met by the transformed angular operator $\mathcal{A}_U$ in the case $a\omega\,\mathrm{sign}(k+\frac{1}{2}) \geq |k+\frac{1}{2}|$, is that the spectra of the diagonal entries of $\mathcal{A}_U$ do not intersect. The bound obtained by this method is proportional to $\sqrt{a\omega}$ for $a\omega\,\mathrm{sign}(k+\frac{1}{2})$ sufficiently large and it is independent of am, whereas all other estimates obtained for the eigenvalues of $\mathcal{A}$ in this work involve a term $\pm|am|$ since am is always treated as a perturbation parameter. The drawback of this estimate is that it holds in the case $a\omega\,\mathrm{sign}(k+\frac{1}{2}) \geq -|k+\frac{1}{2}|$ only; otherwise the spectra of the diagonal entries in $\mathcal{A}_U$ are not separated.

Finally, in chapter 6, the analytical bounds proved in this work are compared with numerical values for the eigenvalues of $\mathcal{A}$ provided in the literature. Furthermore, we use the continued fraction equation for the eigenvalues given in [SFC83] to produce numerical values with the help of a short Maple programme. All numerical values lie within the analytical bounds. A priori, it is not easy to decide which of the various analytic lower bounds for the modulus of the eigenvalues is the sharpest; for fixed m, ω and k, figure 6.1 shows that for each of the four different lower bounds shown in the plot there exists an interval for the Kerr parameter a where it gives a larger lower bound than the other three.

Notation

In this work, $\mathcal{H}$ always denotes a Hilbert space; for scalar products $(\cdot\,,\cdot)$ on Hilbert spaces $\mathcal{H}$ we use the physical convention $(\alpha u,\, \beta v) = \overline{\alpha}\,\beta\,(u,\, v)$. If not stated otherwise, we always assume that the Hilbert spaces are infinite dimensional. The space of all linear operators in $\mathcal{H}$ is denoted by $\mathscr{L}(\mathcal{H})$ and the space of all closed linear operators in $\mathcal{H}$ is denoted by $\mathscr{C}(\mathcal{H})$. If $\mathcal{H}$ is an $\mathscr{L}^2$-space, we denote the norm on $\mathcal{H}$ by $\|\cdot\|_2$. For formal 2×2 differential expressions we use capital Gothic types $\mathfrak{A}$, $\mathfrak{B}$, $\dots$; block operator matrices are denoted by calligraphic types $\mathcal{A}$, $\mathcal{B}$, $\dots$ and their entries by Roman types A, $B\dots$. The domain of a linear operator A is usually denoted by $\mathcal{D}(A)$. Sometimes we use the notation $A(\mathcal{H}_1 \to \mathcal{H}_2)$ for a linear operator A with domain in the Hilbert space $\mathcal{H}_1$ and values in the Hilbert space $\mathcal{H}_2$.
Sesquilinear forms are denoted by small Gothic types $\mathfrak{b}$, $\mathfrak{d}$, $\dots$. Small Greek letters and small Gothic types are used for one-dimensional formal differential expressions.
Throughout the text, the letters $\mathfrak{A}$ and $\mathcal{A}$ with various super- and subscripts are reserved for the formal differential expression and operator theoretical realisations of the angular part of the Dirac equation. To the off-diagonal entries of the angular operator, the letter B is assigned. Sometimes it is convenient to express the dependence of B on the wave number k explicitly by writing B_k.
For a list of symbols we refer to the appendix, pp. 149.

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Chapter 2

The angular equation

The aim of this chapter is to implement the formal differential expression representing the angular part of the Dirac equation in the Kerr-Newman background as a selfadjoint block operator matrix $\mathcal{A}$ acting on a suitable Hilbert space. Furthermore, a qualitative description of the spectrum of $\mathcal{A}$ is given, and various symmetries with respect to the physical parameters are investigated.

The so-called *Kerr-Newman metric* is the most general stationary electrovac solution of the Einstein equation (1.1); in Boyer-Lindquist coordinates it is given by

$$\begin{aligned} \mathrm{d}s^2 = &-\left(\frac{\Delta - a^2\sin^2\vartheta}{\Sigma}\right)\mathrm{d}t^2 - \frac{2a\sin^2\vartheta(r^2+a^2-\Delta)}{\Sigma}\,\mathrm{d}t\mathrm{d}\varphi \\ &+ \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\vartheta}{\Sigma}\right]\sin^2\vartheta\,\mathrm{d}\varphi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\vartheta^2. \end{aligned} \tag{2.1}$$

Sometimes the metric is also denoted by $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, where x^μ and x^ν run through the spacetime coordinates t, r, ϑ, φ; the coefficients $g_{\mu\nu}$ of the metric can be read off from (2.1).
The functions Δ and Σ are defined by

$$\begin{aligned} \Delta(r) &:= r^2 - 2Mr + a^2 + Q^2 = (r-M)^2 + a^2 + Q^2 - M^2, \\ \Sigma(r,\vartheta) &:= r^2 + a^2\cos^2\vartheta. \end{aligned}$$

The functional determinant of the metric g is given by

$$g(r,\vartheta) := \det((g_{\mu\nu}(r,\vartheta))_{\mu,\nu}) = -\sin^2\vartheta\,\Sigma(r,\vartheta)^2 = -\sin^2\vartheta\,(r^2+a^2\cos^2\vartheta)^2. \tag{2.2}$$

Note that in the case $a = 0$ this expression is the negative functional determinant of the usual polar coordinates in $\mathbb{R}^3$.
The family (2.1) of spacetime metrics depends on the three real parameters M, Q and a. If the metric describes the spacetime in the exterior of a black hole, then these parameters have the interpretation as the mass, electric charge and angular momentum per unit mass of the black hole. We define

$$r_\pm := \begin{cases} M \pm \sqrt{M^2 - a^2 - Q^2} & \text{if } M^2 - a^2 - Q^2 \geq 0, \\ 0 & \text{if } M^2 - a^2 - Q^2 < 0, \end{cases}$$

so that $\Delta > 0$ on (r_+, ∞) and $\Delta(r) = 0$ if and only if $r \in \{r_-, r_+\}$, provided that Δ has a zero. It can be shown that in the case $r_+ > 0$ the singularity in the metric at $r_\pm$ is a coordinate singularity which can be removed by using a different coordinate system. However, the points of spacetime with $r = r_+$ form a so-called *event horizon*, i.e., particles can cross the event horizon from the outside into the region with $r < r_+$, but nothing can cross the horizon from the inside to the outer region. Therefore, if the *black hole condition*

$$M^2 - a^2 - Q^2 \;\geq\; 0 \tag{2.3}$$

holds, the true singularity at $\Sigma = 0$ is hidden behind the event horizon. In the case $M^2-a^2-Q^2 = 0$ the metric (2.1) is called the *extreme Kerr-Newman metric*. If $M^2 - a^2 - Q^2 \geq 0$, then the Kerr-Newman metric for $(r, t, \vartheta, \varphi) \in (r_+, \infty) \times (-\infty, \infty) \times (0, \pi) \times (-\pi, \pi)$ is interpreted as the spacetime outside of a massive, charged, rotating black hole with mass M, electric charge Q and angular momentum aM. The parameter a is also referred to as the *Kerr-Newman parameter*. If $M^2-a^2-Q^2 < 0$, then the function Δ has no zero. Therefore, a spacetime described by (2.1) would contain a so-called naked singularity which is supposed to be forbidden by the cosmic censorship conjecture. For more details on the Kerr-Newman black holes and general relativity we refer above all to the textbook [Wal84] and the monograph [FN98].

In the following we consider a spin-$\frac{1}{2}$ particle with mass m and charge e in the Kerr-Newman background. In general, the behaviour of fermions is governed by the *Dirac equation*, a linear system of four differential equations. In the Kerr-Newman metric, the Dirac equation is formally given by the coupled system of partial differential equations (see, e.g., [Pag76], [Cha98])

$$(\widehat{\mathfrak{R}} + \widehat{\mathfrak{A}})\widehat{\Psi} = 0 \tag{2.4}$$

where

$$\widehat{\mathfrak{R}} \;:=\; \begin{pmatrix} \mathrm{i}mr & 0 & \sqrt{\Delta}\mathfrak{R}_+^{t,\varphi} & 0 \\ 0 & -\mathrm{i}mr & 0 & \sqrt{\Delta}\mathfrak{R}_-^{t,\varphi} \\ \sqrt{\Delta}\mathfrak{R}_-^{t,\varphi} & 0 & -\mathrm{i}mr & 0 \\ 0 & \sqrt{\Delta}\mathfrak{R}_+^{t,\varphi} & 0 & \mathrm{i}mr \end{pmatrix}, \quad \widehat{\mathfrak{A}} \;:=\; \begin{pmatrix} -\mathfrak{D} & 0 & 0 & \mathfrak{L}_+^{t,\varphi} \\ 0 & \mathfrak{D} & -\mathfrak{L}_-^{t,\varphi} & 0 \\ 0 & \mathfrak{L}_+^{t,\varphi} & -\mathfrak{D} & 0 \\ -\mathfrak{L}_-^{t,\varphi} & 0 & 0 & \mathfrak{D} \end{pmatrix} \tag{2.5}$$

and

$$\begin{aligned} \mathfrak{D} &:= am\cos\vartheta, \\ \mathfrak{R}_\pm^{t,\varphi} &:= \frac{\partial}{\partial r} \pm \frac{\mathrm{i}}{\Delta}\left[(r^2+a^2)\,\mathrm{i}\frac{\partial}{\partial t} + a\,\mathrm{i}\frac{\partial}{\partial\varphi} + eQr\right] \;=:\; \frac{\partial}{\partial r} \pm \mathrm{i}\,\Omega^{t,\varphi}(r) \qquad &&\text{on } (r_+, \infty), \\ \mathfrak{L}_\pm^{t,\varphi} &:= \frac{\partial}{\partial\vartheta} + \frac{\cot\vartheta}{2} \mp \mathrm{i}\left[a\sin\vartheta\frac{\partial}{\partial t} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right] &&\text{on } (0,\pi). \end{aligned}$$

We would like to emphasise that at this stage the Dirac equation is a formal equation only. The choice of its realisation in an operator theoretical context has yet to be made, see the discussion at the end of this section and also in section 2.1 where an operator associated to the angular part of the Dirac equation is established.
It is clear, however, that for massive fermions, that is, for $m \neq 0$, the formal operator on the left hand side of (2.4) cannot be formally selfadjoint in any space of square integrable functions because of the nonvanishing complex multiplication operators on the diagonal of the matrix $\widehat{\mathfrak{R}}$. To overcome that obstacle to (formal) symmetry, we multiply equation (2.4) from the left by the invertible matrix

$$V_0 \;:=\; \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}$$

and we obtain that equation (2.4) is equivalent to

$$(\widehat{\mathfrak{R}}_s + \widehat{\mathfrak{A}}_s)\widehat{\Psi} \;=\; 0, \tag{2.6}$$

with

$$\widehat{\mathfrak{R}}_s \;:=\; V_0\,\widehat{\mathfrak{R}} \;=\; \begin{pmatrix} -\mathrm{i}\,\mathfrak{R}_-^{t,\varphi} & 0 & -\frac{mr}{\sqrt{\Delta}} & 0 \\ 0 & \mathrm{i}\,\mathfrak{R}_+^{t,\varphi} & 0 & -\frac{mr}{\sqrt{\Delta}} \\ -\frac{mr}{\sqrt{\Delta}} & 0 & \mathrm{i}\,\mathfrak{R}_+^{t,\varphi} & 0 \\ 0 & -\frac{mr}{\sqrt{\Delta}} & 0 & -\mathrm{i}\,\mathfrak{R}_-^{t,\varphi} \end{pmatrix},$$

$$\widehat{\mathfrak{A}}_s \;:=\; V_0\,\widehat{\mathfrak{A}} \;=\; \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 0 & -\mathrm{i}\,\mathfrak{L}_+^{t,\varphi} & \mathrm{i}\,\mathfrak{D} & 0 \\ -\mathrm{i}\mathfrak{L}_-^{t,\varphi} & 0 & 0 & \mathrm{i}\mathfrak{D} \\ -\mathrm{i}\,\mathfrak{D} & 0 & 0 & \mathrm{i}\,\mathfrak{L}_+^{t,\varphi} \\ 0 & -\mathrm{i}\,\mathfrak{D} & \mathrm{i}\,\mathfrak{L}_-^{t,\varphi} & 0 \end{pmatrix}.$$

A straightforward computation shows that $\widehat{\mathfrak{R}}_s$ and $\widehat{\mathfrak{A}}_s$, and consequently $\widehat{\mathfrak{H}}_s := \widehat{\mathfrak{R}}_s + \widehat{\mathfrak{A}}_s$, are formally symmetric on the weighted $\mathscr{L}^2$-space $\mathscr{L}^2((r_+,\,\infty)\times(0,\,\pi)\times(-\pi,\,\pi),\ \sin\vartheta\,\mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi)^4$.

Chandrasekhar showed that this system of partial differential equations can be separated into a system of ordinary differential equations, see [Cha98]. To this end, we employ the ansatz

$$\begin{aligned} \widehat{\Psi}(t,r,\vartheta,\varphi) \;&=:\; \mathrm{e}^{-\mathrm{i}\,\omega t}\mathrm{e}^{-\mathrm{i}(k+\frac{1}{2})\varphi}\Psi(r,\vartheta) \\ &=:\; \mathrm{e}^{-\mathrm{i}\,\omega t}\mathrm{e}^{-\mathrm{i}(k+\frac{1}{2})\varphi} \begin{pmatrix} \Psi_1(r,\vartheta) \\ \Psi_2(r,\vartheta) \\ \Psi_3(r,\vartheta) \\ \Psi_4(r,\vartheta) \end{pmatrix} \;=:\; \mathrm{e}^{-\mathrm{i}\,\omega t}\mathrm{e}^{-\mathrm{i}(k+\frac{1}{2})\varphi} \begin{pmatrix} X_-(r)S_+(\vartheta) \\ X_+(r)S_-(\vartheta) \\ X_+(r)S_+(\vartheta) \\ X_-(r)S_-(\vartheta) \end{pmatrix}, \end{aligned} \tag{2.7}$$

so that the system (2.4) of partial differential equations decouples into the following system of ordinary differential equations with coupling parameter λ (the superscript (d) labels the operators "decoupled"):

$$(\mathfrak{R}^{(d)} - \lambda)\begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0, \qquad (\mathfrak{A}^{(d)} - \lambda)\begin{pmatrix} S_- \\ S_+ \end{pmatrix} = 0, \tag{2.8}$$

where

$$\mathfrak{R}^{(d)} = \begin{pmatrix} -\mathrm{i}mr & \sqrt{\Delta}\,\mathfrak{R}_- \\ \sqrt{\Delta}\,\mathfrak{R}_+ & \mathrm{i}mr \end{pmatrix} \qquad \text{and} \qquad \mathfrak{A}^{(d)} = \begin{pmatrix} -\mathfrak{D} & \mathfrak{L}_- \\ -\mathfrak{L}_+ & \mathfrak{D} \end{pmatrix}$$

with

$$\mathfrak{R}_\pm = \frac{\mathrm{d}}{\mathrm{d}r} \pm \frac{\mathrm{i}}{\Delta}\left[\omega(r^2+a^2) + a\,(k+\tfrac{1}{2}) + eQr\right] \;=:\; \frac{\mathrm{d}}{\mathrm{d}r} \pm \mathrm{i}\,\Omega(r), \tag{2.9}$$

$$\mathfrak{L}_\pm = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{\cot\vartheta}{2} \mp \left[a\omega\sin\vartheta + \frac{k+\frac{1}{2}}{\sin\vartheta}\right] \tag{2.10}$$

are obtained from $\mathfrak{R}^{t,\varphi}$ and $\mathfrak{L}^{t,\varphi}$ by replacing the differential operators $\mathrm{i}\frac{\partial}{\partial t}$ and $\mathrm{i}\frac{\partial}{\partial\varphi}$ by the multiplication operators ω and $(k+\frac{1}{2})$, respectively.

To clarify the structure of $\widehat{\mathfrak{H}}_s = \widehat{\mathfrak{R}}_s + \widehat{\mathfrak{A}}_s$ that accounts for the fact that the Dirac equation can be separated into a radial and a angular equation, we transform $\widehat{\mathfrak{H}}_s$ with the unitary matrix

$$V := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 & \mathrm{i} \\ \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

and obtain that the Dirac equation (2.6) is equivalent to

$$\big(V\widehat{\mathfrak{R}}_s V^{-1} + V\widehat{\mathfrak{A}}_s V^{-1}\big) V\Psi = 0, \tag{2.11}$$

with the 4×4-matrices

$$V\widehat{\mathfrak{R}}_s V^{-1} = \begin{pmatrix} \frac{rm}{\sqrt{\Delta}} - \Omega(r) & 0 & -\frac{\mathrm{d}}{\mathrm{d}r} & 0 \\ 0 & \frac{rm}{\sqrt{\Delta}} - \Omega(r) & 0 & -\frac{\mathrm{d}}{\mathrm{d}r} \\ \frac{\mathrm{d}}{\mathrm{d}r} & 0 & -\big(\frac{rm}{\sqrt{\Delta}} + \Omega(r)\big) & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}r} & 0 & -\big(\frac{rm}{\sqrt{\Delta}} + \Omega(r)\big) \end{pmatrix}$$

$$= \begin{pmatrix} \Big(\frac{rm}{\sqrt{\Delta}} - \Omega(r)\Big)\, I_2 & -\frac{\mathrm{d}}{\mathrm{d}r}\, I_2 \\ \frac{\mathrm{d}}{\mathrm{d}r}\, I_2 & -\Big(\frac{rm}{\sqrt{\Delta}} - \Omega(r)\Big)\, I_2 \end{pmatrix},$$

$$V\widehat{\mathfrak{A}}_s V^{-1} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 0 & 0 & -\mathfrak{D} & \mathfrak{L}_- \\ 0 & 0 & -\mathfrak{L}_+ & \mathfrak{D} \\ -\mathfrak{D} & \mathfrak{L}_- & 0 & 0 \\ -\mathfrak{L}_+ & \mathfrak{D} & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 0 & \mathfrak{A}^{(d)} \\ \mathfrak{A}^{(d)} & 0 \end{pmatrix},$$

$$V\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}\,(\Psi_2 - \Psi_4) \\ \mathrm{i}\,(\Psi_1 - \Psi_3) \\ -(\Psi_2 + \Psi_4) \\ -(\Psi_1 + \Psi_3) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}\,(X_+ - X_-)S_- \\ -\mathrm{i}\,(X_+ - X_-)S_+ \\ -(X_+ + X_-)S_- \\ -(X_+ + X_-)S_+ \end{pmatrix}.$$

Let $\binom{S_-}{S_+}$ be a nontrivial solution of the equation for $\binom{S_-}{S_+}$ in (2.8). Since by assumption the functions $X_\pm$ do not depend on ϑ, it follows from the Dirac equation (2.11) that

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} & -\big(\frac{rm}{\sqrt{\Delta}} + \Omega(r)\big) \\ \frac{rm}{\sqrt{\Delta}} - \Omega(r) & -\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{2}}\,(X_+ - X_-) \\ -\frac{1}{\sqrt{2}}\,(X_+ + X_-) \end{pmatrix} = 0. \tag{2.12}$$

Application of the unitary transformation $W := \frac{1}{\sqrt{2}}\big(\begin{smallmatrix} \mathrm{i} & -1 \\ -\mathrm{i} & -1 \end{smallmatrix}\big)$ shows that the above equation is equivalent to

$$\begin{aligned}
0 &= W \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} & -\left(\frac{rm}{\sqrt{\Delta}} + \Omega(r)\right) \\ \frac{rm}{\sqrt{\Delta}} - \Omega(r) & -\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} \end{pmatrix} W^{-1}\, W \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{2}}\,(X_+ - X_-) \\ -\frac{1}{\sqrt{2}}\,(X_+ + X_-) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\lambda}{\sqrt{\Delta}} + \frac{\mathrm{i}rm}{\sqrt{\Delta}} & -\frac{\mathrm{d}}{\mathrm{d}r} + \mathrm{i}\Omega(r) \\ -\frac{\mathrm{d}}{\mathrm{d}r} - \mathrm{i}\,\Omega(r) & \frac{\lambda}{\sqrt{\Delta}} - \frac{\mathrm{i}rm}{\sqrt{\Delta}} \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \\
&= -\frac{1}{\sqrt{\Delta}} \left(\begin{pmatrix} -\mathrm{i}\,rm & \sqrt{\Delta}\,\mathfrak{R}_- \\ \sqrt{\Delta}\,\mathfrak{R}_+ & \mathrm{i}\,rm \end{pmatrix} - \lambda \right) \begin{pmatrix} X_+ \\ X_- \end{pmatrix},
\end{aligned}$$

which is equivalent to the first equation in (2.8). Remember that the second equation in (2.8) is satisfied by our assumption on $\binom{S_-}{S_+}$.
Conversely, if we have solutions of (2.8), then we obtain a solution $\widehat{\Psi}$ of the Dirac equation by (2.7).

Remark 2.1. Note that for $m \neq 0$ the differential expression $\mathfrak{R}^{(d)}$ is not formally symmetric in $\mathscr{L}^2$-spaces due to the multiplication operator $-im$ on its diagonal. However, it is possible to write the radial equation $(\mathfrak{R}^{(d)} - \lambda)\binom{X_+}{X_-} = 0$ as an eigenvalue equation with ω as the eigenvalue parameter such that the corresponding radial operator becomes formally symmetric, see [BM99]. To this end we extract from formula (2.11) the equation

$$\begin{aligned}
0 &= \begin{pmatrix} \frac{rm}{\sqrt{\Delta}} - \Omega(r) & -\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} \\ \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} & -\left(\frac{rm}{\sqrt{\Delta}} + \Omega(r)\right) \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{2}}\,(X_+ - X_-) \\ -\frac{1}{\sqrt{2}}(X_+ + X_-) \end{pmatrix} \\
&= \begin{pmatrix} \frac{rm}{\sqrt{\Delta}} - \frac{1}{\Delta}(a(k+\frac{1}{2}) + eQr) - \frac{\omega(r^2+a^2)}{\Delta} & -\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} \\ \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\lambda}{\sqrt{\Delta}} & -\frac{rm}{\sqrt{\Delta}} - \frac{1}{\Delta}(a(k+\frac{1}{2}) + eQr) - \frac{\omega(r^2+a^2)}{\Delta} \end{pmatrix} W^{-1} \begin{pmatrix} X_+ \\ X_- \end{pmatrix}
\end{aligned}$$

which, considered as a mere equation, coincides with (2.12) since only the rows of the matrices are interchanged. The operator realisations, however, of the left hand sides of both equations differ substantially because exchanging of the rows of matrix can in general not be achieved by a unitary transformation.
In order to get rid of the factor in front of the eigenvalue parameter ω, we introduce the new radial coordinate x defined by $\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{r^2+a^2}{\Delta(r)}$ for $r \in (r_+, \infty)$. The new coordinate x is uniquely defined up to an additive constant x_0; it is given explicitly by

$$x(r) = \begin{cases} r + 2r_+ \ln(r - r_+) - \frac{r_+^2 + a^2}{r - r_+} + x_0 & \text{if } r_+ = r_-, \\ r + \frac{a^2 + r_+^2}{r_+ - r_-} \ln(r - r_+) - \frac{a^2 + r_-^2}{r_+ - r_-} \ln(r - r_-) + x_0 & \text{if } r_+ \neq r_-. \end{cases}$$

After multiplication of the radial equation above from the left by $\frac{\Delta(r)}{r^2+a^2}$ we obtain the equation

$$\begin{pmatrix} \frac{rm\sqrt{\Delta}}{r^2+a^2} - \frac{1}{r^2+a^2}(a(k+\frac{1}{2}) + eQr) - \omega & -\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\lambda\sqrt{\Delta}}{r^2+a^2} \\ \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\lambda\sqrt{\Delta}}{r^2+a^2} & -\frac{rm\sqrt{\Delta}}{r^2+a^2} - \frac{1}{r^2+a^2}(a(k+\frac{1}{2}) + eQr) - \omega \end{pmatrix} W^{-1} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0$$

which is an eigenvalue equation with eigenvalue parameter ω for a formally symmetric operator on the Hilbert space $\mathscr{L}^2((-\infty, \infty), \mathrm{d}x)$. The above system is a Dirac system, thus providing a convenient starting point for the investigation of the spectrum of the radial part of the Dirac equation as carried out by Belgiorno and Martellini in [BM99]. In this paper, the authors have proved that there is a unique selfadjoint operator representing the radial equation and that its essential spectrum covers the whole real line. $\Diamond$

In the next section we show that the angular equation has a representation as an eigenvalue equation for a selfadjoint operator on Hilbert space with eigenvalue λ.

If there is a number λ such that the system (2.8) can be satisfied by functions $X_\pm$ and $S_\pm$ enjoying certain integrability properties, then the system consisting of the Kerr-Newman black hole and the fermionic particle is stable. In this work we establish analytic bounds for the eigenvalues of the radial operator that might prove useful in the investigation of the full coupled problem (2.8).

The properties that have to be postulated for the solutions $S_\pm$ and $X_\pm$ can be deduced either from physical or from mathematical considerations. In the first case, one argues that $\widehat{\Psi}$ describes the state of one particle, hence for every given t it must be square integrable on $(r_+, \infty) \times (0, \pi) \times (-\pi, \pi)$ with respect to the integration weight induced by the metric $\mathrm{d}s^2$, see also section 2.1.1. Hence the angular and radial operators under consideration are supposed to be acting on complex $\mathscr{L}^2$-spaces with the integration weight induced by the metric. It turns out that with these weights the operators are formally symmetric.
From a mathematical point of view, given the formal radial and angular equation only, one would consider the formal differential expressions $\mathfrak{A}$ and $\mathfrak{R}$ or transforms thereof on some $\mathscr{L}^2$-spaces where the integration weights have to be chosen such that they are formally symmetric and admit selfadjoint realisations. Of course, the integration weights obtained by this procedure coincide with those obtained by a physical reasoning.

For results on the radial part of the Dirac equation we refer to [BM99] and the recent paper [Sch04] and references therein. Here, we deal with the angular operator only. The first aim is to implement the formal differential expression $\mathfrak{A}^{(d)}$ as a selfadjoint operator $\mathcal{A}$. Then we give a qualitative description of the spectrum of $\mathcal{A}$ in section 2.2.

2.1 Operatortheoretical realisation of $\mathfrak{A}$

In this subsection we collect some basic definitions and facts concerning differential operators, see, e.g., [Wei87]. We consider formal differential expressions of the form

$$\begin{aligned} \tau f(x) \;=\; r^{-1}(x)\Bigg\{ & \sum_{j=0}^{[\frac{n}{2}]} (-1)^j \left(p_j(x) f^{(j)}(x)\right)^{(j)} \\ & + \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \left[\left(q_j(x) f^{(j)}(x)\right)^{(j+1)} - \left(q_j^*(x) f^{(j+1)}(x)\right)^{(j)} \right] \Bigg\} \end{aligned} \tag{2.13}$$

on an interval $(a, b) \subseteq \mathbb{R}$, where r, p_j and q_j are $m \times m$-matrix valued functions on (a, b) such that $r(x)$ is positive definite and $p_j(x)$, $j = 0, \ldots, [\frac{n}{2}]$, are Hermitian for almost all $x \in (a, b)$. Further we assume that p_j, q_j, $j = 0, \ldots, [\frac{n}{2}]$, and r are measurable on (a, b). In addition we require for odd $n =: 2k + 1$ that

(i) q_k is absolutely continuous and $\widehat{q}_k(x) := (q_k - q_k^*)(x)$ is regular for every $x \in (a, b)$,

(ii) $|\widehat{q}_k^{\,-1}|$, $|\widehat{q}_k^{\,-1}(p_k + q_k')|$, $|\widehat{q}_k^{\,-1} q_{k-1}|$, $|p_j|$, $|q_j|$ for $j = 0, \ldots, k-1$ and $|r|$ are locally integrable on (a, b).

For even $n =: 2k$ we suppose that the following conditions hold:

(i) $p_k(x)$ is regular for almost all $x \in (a, b)$;

(ii) $|p_k^{-1}|$, $|p_k^{-1} q_{k-1}|$, $|p_{k-1} - q_{k-1}^* p_k^{-1} q_{k-1}|$, $|p_j|$, $|q_j|$, $j = 0, 1, \ldots, k-2$, and $|r|$ are locally integrable on (a, b).

If r is continuous and the coefficients q_j, p_j, $j = 0, \ldots, n$ are sufficiently often continuously differentiable, then the following *minimal operator associated with* τ is well defined in the Hilbert space $\mathscr{L}^2((a, b), \mathrm{d}x)$:

$$\mathcal{D}(T^{\min}) \;:=\; \mathcal{C}_0^\infty(0, \pi)^m, \qquad T^{\min} f \;:=\; \tau f. \tag{2.14}$$

Remark 2.2. In the general case, the minimal operator associated with τ is defined as

$$\begin{aligned} \mathcal{D}(T^{\{\min\}}) \;:=\; \{ & f \in \mathcal{H} : f \text{ has compact support}, f^{\{0\}}, \ldots, f^{\{n-1\}} \text{ are absolutely continuous} \\ & \text{and } \tau f \in \mathcal{H} \}. \end{aligned}$$

For the general definition of the *quasi-derivatives* $f^{\{j\}}$ we refer the reader to [Wei87, chap. 2]. In this work, we are only interested in the case $nm = 2$: in the next section we show that the formal angular operator $\mathfrak{A}^{(d)}$ of (2.8) is a differential expression of type (2.13) satisfying the above mentioned conditions for $n = 1$ and $m = 2$. The case $n = 2$, $m = 1$ arises in section 4.2.2. For $nm = 2$, the quasi-derivatives are given by

$$f^{\{0\}} \;:=\; f, \qquad f^{\{1\}} \;:=\; \left(\widehat{q}\,\tfrac{\mathrm{d}}{\mathrm{d}x} + (q_0' + p_0)\right) f \;=\; r\tau f \qquad \text{if } n = 1, m = 2,$$

$$\begin{aligned} & f^{\{0\}} \;:=\; f, \qquad f^{\{1\}} \;:=\; \left(p_1 \tfrac{\mathrm{d}}{\mathrm{d}x} - q_0\right) f, \\ & f^{\{2\}} := \left(-\tfrac{\mathrm{d}}{\mathrm{d}x} - q_0^* p_1^{-1}\right) f^{\{1\}} + \left(p_0 - q_0^* p_1^{-1} q_0\right) f^{\{0\}} \;=\; r\tau f \qquad \text{if } n = 2, m = 1. \end{aligned}$$

If the coefficients of τ are such that $T^{\min}$ is well defined, then we have $\overline{T^{\{min\}}} = \overline{T^{\min}}$. $\Diamond$

In the following we recall some basic definitions.

Definition 2.3. For an interval $(a,b) \subseteq \mathbb{R}$ consider a vector valued $f : (a,b) \longrightarrow \mathbb{C}^m$. Then we say that f *lies right in* $\mathscr{L}^2((a,b),\mathrm{d}x)^m$ if for every $c \in (a,b)$ we have $f|_{(c,b)} \in \mathscr{L}^2((c,b),\mathrm{d}x)^m$. Analogously, f is said to *lie left in* $\mathscr{L}^2((a,b),\mathrm{d}x)^m$ if for every $c \in (a,b)$ we have $f|_{(a,c)} \in \mathscr{L}^2((a,c),\mathrm{d}x)^m$.

Definition 2.4. If for some $\lambda \in \mathbb{C}$ every solution of $(\tau - \lambda)f = 0$ lies right in $\mathscr{L}^2((a,b),\mathrm{d}x)^m$, then τ is called *quasi-regular at* b. Quasi-regularity at a is defined analogously. The differential expression τ is called *quasi-regular*, if it is quasi-regular at a and b.

Definition 2.5. Let (a,b) be an interval in $\mathbb{R}$ and τ be a formal differential expression on (a,b) as in (2.13). We say that τ is in the *limit circle case at* b, if for every $\lambda \in \mathbb{C}$ all solutions of $(\tau-\lambda)f = 0$ lie right in $\mathscr{L}^2((a,b),\mathrm{d}x)$. τ is in the *limit point case at* b if for every $\lambda \in \mathbb{C}$ there is at least one solution of $(\tau - \lambda)f = 0$ that does not lie right in $\mathscr{L}^2((a,b),\mathrm{d}x)$.
The notions *limit point case at* a and *limit circle case at* a are defined analogously.

Weyl's alternative states that in the case $nm = 2$ these are the only cases that can occur for real differential expressions τ. We cite the theorem in the version of [Wei87, theorem 5.6].

Theorem 2.6. *Let τ be a differential expression as in* (2.13) *with real coefficients and $p := nm = 2$. Then exactly one of the following two cases holds.*

(i) *For every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda)f = 0$ lie right in $\mathscr{L}^2((a,b), r(x)\,\mathrm{d}x)^m$.*

(ii) *For every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a (up to a multiplicative constant) unique solution f of $(\tau - \lambda)f = 0$ which lies right in $\mathscr{L}^2((a,b), r(x)\,\mathrm{d}x)^m$.*

The same result holds with "left" replaced by "right".

Theorem 2.7. *Let τ be a differential expression as in* (2.13) *with real coefficients such that the minimal operator $T^{\min}$ is well defined and let $p := nm = 2$. If τ is in the limit point case both at a and b, then the closure of the minimal operator associated to τ is selfadjoint.*

The proofs of theorems 2.6 and 2.7 may be found in [Wei87].

2.1.1 Transformation of spacetime coordinates

From the metric (2.1) we obtain the functional determinant $g(r,\vartheta) = -\sin^2\vartheta\,\Sigma^2(r,\vartheta)$. The factor $\Sigma(r,\vartheta) = r^2 + a^2\cos^2\vartheta$ is strictly positive and bounded; here strictly positive means that there exists a constant $c > 0$ such that $\Sigma(r,\vartheta) \geq c$ for all $\vartheta \in [0,\pi]$ and all $r \in [r_+,\infty)$. Hence for fixed $r \in (r_+,\infty)$, the functional determinant g can be estimated from above and from below by some positive constant multiple of $\sin^2\vartheta$, which suggests that the Hilbert space for $\mathfrak{A}^{(d)}$ to operate on is the weighted space $\mathscr{L}^2((0,\pi),\mathrm{d}\cos\vartheta)^2 = \mathscr{L}^2((0,\pi),\sin\vartheta\,\mathrm{d}\vartheta)^2$. Indeed, one can show that in this space $\mathfrak{A}^{(d)}$ is formally symmetric.
This symmetry becomes more apparent if we transform the given spectral problem into a problem in the Hilbert space $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2$. To this end, consider the isometry

$$j : \mathscr{L}^2((0,\pi),\sin\vartheta\,\mathrm{d}\vartheta) \longrightarrow \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta), \qquad f \mapsto \sqrt{\sin\vartheta}\,f.$$

We have $j^* = j^{-1}$ and linear operators T on $\mathscr{L}^2((0,\pi),\sin\vartheta\,\mathrm{d}\vartheta)$ transform according to the following commutative diagramme:

$$\begin{array}{ccc} \mathscr{L}^2((0,\pi),\sin\vartheta\,\mathrm{d}\vartheta) & \xrightarrow{T} & \mathscr{L}^2((0,\pi),\sin\vartheta\,\mathrm{d}\vartheta) \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) & \xrightarrow{jTj^{-1}} & \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) \end{array}$$

This shows that multiplication operators remain unchanged under this transformation, but differential operators change. Consider the linear operator $T = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{\cot\vartheta}{2}$ as an operator acting on $\mathscr{L}^2((0,\pi), \sin\vartheta\,\mathrm{d}\vartheta)$. Then, for arbitrary $f \in \mathcal{D}(T)$, we have

$$\begin{aligned}
(jTj^{-1})(jf)(\vartheta) &= \sqrt{\sin\vartheta}\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{\cot\vartheta}{2}\right)\left(\frac{1}{\sqrt{\sin\vartheta}}\cdot (jf)(\vartheta)\right) \\
&= \sqrt{\sin\vartheta}\left(\frac{1}{\sqrt{\sin\vartheta}}\frac{\mathrm{d}}{\mathrm{d}\vartheta} - \frac{\cos\vartheta}{2(\sin\vartheta)^{\frac{3}{2}}} + \frac{\cot\vartheta}{2\sqrt{\sin\vartheta}}\right)(jf)(\vartheta) \\
&= \frac{\mathrm{d}}{\mathrm{d}\vartheta}(jf)(\vartheta).
\end{aligned}$$

Thus we have shown

$$jTj^{-1} = \frac{\mathrm{d}}{\mathrm{d}\vartheta} \qquad \text{on} \quad j(\mathcal{D}(T)) \subseteq \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta).$$

If we transform the formal matrix differential operator $\mathfrak{A}^{(d)}$ with $J := j \oplus j = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$, we obtain

$$\mathfrak{A} = J\mathfrak{A}^{(d)}J^{-1} = \begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix}$$

which acts on the Hilbert space $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2$. To simplify the following calculations we write $\mathfrak{A}$ as the sum of the unbounded operator $\mathfrak{A}_u$ and the bounded operator $\mathfrak{A}_b$ given by

$$\mathfrak{A}_u := \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} & 0 \end{pmatrix} \qquad \text{and} \qquad \mathfrak{A}_b := \begin{pmatrix} -am\cos\vartheta & a\omega\sin\vartheta \\ a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix}.$$

Now, if in (2.13) we set $n = 1,\ m = 2,\ a = 0,\ b = \pi,\ r(x) = 1$ and

$$q_0(\vartheta) = \frac{1}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad p_{u,0}(\vartheta) = \frac{k+\frac{1}{2}}{\sin\vartheta}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$p_0(\vartheta) = \frac{k+\frac{1}{2}}{\sin\vartheta}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} -am\cos\vartheta & a\omega\sin\vartheta \\ a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix},$$

we find that the formal expressions $\mathfrak{A}_u$ and $\mathfrak{A}$ are of the form (2.13), so they fit into the general framework discussed at the beginning of section 2.1.

2.1.2 Realisation of $\mathfrak{A}$ as a selfadjoint operator on $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2$

In the following we always assume $a, m, \omega \in \mathbb{R}$ and $k \in \mathbb{Z}$ if not stated explicitly otherwise. Then it is easy to see that both $\mathfrak{A}$ and $\mathfrak{A}_u$ are formally symmetric on the space $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. First we work only with $\mathfrak{A}_u$ because the calculations are simpler. The minimal operator associated with $\mathfrak{A}_u$ is

$$\mathcal{D}(\mathcal{A}_u^{\min}) := C_0^\infty(0,\pi)^2, \qquad \mathcal{A}_u^{\min}\Psi := \mathfrak{A}_u\Psi = \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} & 0 \end{pmatrix}\Psi. \tag{2.15}$$

According to [Wei87, theorems 3.7 and 3.9], the operator $\mathcal{A}_u^{\min}$ is symmetric, hence it is closable. Let $\mathcal{A}_u$ be the closure of $\mathcal{A}_u^{\min}$. Then the following holds:

Lemma 2.8. *The operator $\mathcal{A}_u$ is selfadjoint if and only if $k \in \mathbb{R} \setminus (-1, 0)$.*

Proof. We show that $\mathfrak{A}_u$ is in the limit point case at 0 and at π. A fundamental system of the differential equation $\mathfrak{A}_u\Psi = 0$ is

$$\Psi_1(\vartheta) = \left(\tan\tfrac{\vartheta}{2}\right)^{k+\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \Psi_2(\vartheta) = \left(\tan\tfrac{\vartheta}{2}\right)^{-(k+\frac{1}{2})}\begin{pmatrix}0\\1\end{pmatrix}. \tag{2.16}$$

Now we have to examine the square integrability of these solutions. Let $k \geq 0$, $c \in (\frac{\pi}{2}, \pi)$ and $d \in (0, \frac{\pi}{2})$ arbitrary. Then it follows that

$$\int_c^\pi |\Psi_1(\vartheta)|^2\,\mathrm{d}\vartheta \;=\; \int_c^\pi \left(\tan\tfrac{\vartheta}{2}\right)^{2k+1}\mathrm{d}\vartheta \;\geq\; \int_c^\pi \tan\tfrac{\vartheta}{2}\,\mathrm{d}\vartheta \;=\; -2\ln\cos\tfrac{\vartheta}{2}\Big|_c^\pi \;=\; \infty,$$

and

$$\int_0^d |\Psi_2(\vartheta)|^2\,\mathrm{d}\vartheta \;=\; \int_0^d \left(\cot\tfrac{\vartheta}{2}\right)^{2k+1}\mathrm{d}\vartheta \;\geq\; \int_0^d \cot\tfrac{\vartheta}{2}\,\mathrm{d}\vartheta \;=\; 2\ln\sin\tfrac{\vartheta}{2}\Big|_0^d \;=\; \infty.$$

On the other hand, we have the estimates

$$\int_0^c |\Psi_1(\vartheta)|^2\;\mathrm{d}\vartheta \;=\; \int_0^c (\tan\tfrac{\vartheta}{2})^{2k+1}\;\mathrm{d}\vartheta \;\leq\; (\tan\tfrac{c}{2})^{2k+1}\int_0^c \mathrm{d}\vartheta < \infty,$$
$$\int_d^\pi |\Psi_2(\vartheta)|^2\;\mathrm{d}\vartheta \;=\; \int_d^\pi (\cot\tfrac{\vartheta}{2})^{2k+1}\;\mathrm{d}\vartheta \;\leq\; (\cot\tfrac{d}{2})^{2k+1}\int_d^\pi \mathrm{d}\vartheta < \infty.$$

Hence in the case $k \geq 0$ the solution Ψ_1 lies left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ but it does not lie right in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$, whereas the solution Ψ_2 lies right, but not left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2$. For $k \leq -1$ the same holds true for Ψ_1 and Ψ_2 exchanged.
Using Weyl's alternative we conclude that for $k \in \mathbb{R}\setminus(-1,0)$ the formal expression $\mathfrak{A}_u$ is in the limit point case both at 0 and at π, hence it follows from theorem 2.7 that the closure of $\mathcal{A}_u^{\min}$ is selfadjoint.
To prove that $\mathcal{A}_u$ is not selfadjoint for $k \in (-1,0)$, we show that in this case the solutions Ψ_1 and Ψ_2 lie in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2$, thus $\mathfrak{A}_u$ is in the limit circle case both at 0 and π. Then, by theorem 2.7, the assertion is proved. We give a proof only for $\Psi_1 \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ in the case $k \in (-1, -\frac{1}{2}]$; the remaining cases can be treated analogously. By assumption, we have $2k+1 \in (-1, 0]$. Hence it follows from $\sin\frac{\vartheta}{2} \geq \frac{\vartheta}{\pi} > 0$, $\vartheta \in (0,\pi)$, and the monotonicity of the cosine and tangent functions that

$$\begin{aligned}
\int_0^\pi |\Psi_1(\vartheta)|^2\,\mathrm{d}\vartheta \;&=\; \int_0^{\frac{\pi}{2}} |\Psi_1(\vartheta)|^2\,\mathrm{d}\vartheta + \int_{\frac{\pi}{2}}^\pi |\Psi_1(\vartheta)|^2\,\mathrm{d}\vartheta \\
&=\; \int_0^{\frac{\pi}{2}} (\cos\tfrac{\vartheta}{2})^{-(2k+1)}(\sin\tfrac{\vartheta}{2})^{2k+1}\,\mathrm{d}\vartheta + \int_{\frac{\pi}{2}}^\pi \left(\tan\tfrac{\vartheta}{2}\right)^{2k+1}\mathrm{d}\vartheta \\
&\leq\; \pi^{-(2k+1)}\cos(0)\int_0^{\frac{\pi}{2}} \vartheta^{2k+1}\,\mathrm{d}\vartheta + \tan\frac{\pi}{4}\int_{\frac{\pi}{2}}^\pi \mathrm{d}\vartheta \\
&=\; \frac{1}{\pi^{2k+1}(2k+2)}\left[\vartheta^{2k+2}\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} \;<\; \infty.
\end{aligned}$$

It is clear that for nonreal k the operator $\mathcal{A}_u$ is not even symmetric, hence it cannot be essentially selfadjoint. $\square$

Note that the proof of lemma 2.8 does not rely on k being integer though in the following k is always assumed to have this property.

We can also find an explicit representation of the domain of $\mathcal{A}_u$. To this end we introduce the so-called *maximal operator* associated with $\mathfrak{A}_u$ by

$$\begin{aligned} \mathcal{D}(\mathcal{A}_u^{\max}) &:= \left\{ \Psi \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2 \ : \ \Psi \text{ is absolutely continuous}, \mathfrak{A}_u\Psi \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2 \right\}, \\ \mathcal{A}_u^{\max}\Psi &:= \mathfrak{A}_u\Psi. \end{aligned} \tag{2.17}$$

By [Wei87, Theorem 3.9] we have $\mathcal{A}_u^* = \mathcal{A}_u^{\max}$. Since for $k \in \mathbb{R}\backslash(-1,0)$ the operator $\mathcal{A}_u$ is selfadjoint by lemma 2.8, it follows that

$$\mathcal{A}_u \;=\; \mathcal{A}_u^{\max}.$$

Although $\mathcal{A}_u$ is not essentially selfadjoint for $k \in (-1,0)$, it is still symmetric and all selfadjoint extensions are given as restrictions of the maximal operator associated with $\mathfrak{A}_u$ in terms of boundary conditions.

The next theorem contains the main result of this section.

Theorem 2.9. *The angular operator*

$$\begin{aligned} \mathcal{D}(\mathcal{A}) \;&:=\; \left\{ \Psi \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2 \ : \ \Psi \textit{ is absolutely continuous}, \mathfrak{A}\Psi \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2 \right\}, \\ \mathcal{A}\Psi \;&:=\; \mathfrak{A}\Psi \;=\; \begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix}\Psi \end{aligned} \tag{2.18}$$

is selfadjoint if and only if $k \in \mathbb{R}\setminus(-1,0)$. In this case, $\mathcal{A}$ is the closure of the minimal operator $\mathcal{A}^{\min}$, defined by $\mathcal{D}(\mathcal{A}^{\min}) := \mathcal{C}_0^\infty(0,\pi)^2$, $\mathcal{A}^{\min}\Psi := \mathfrak{A}\Psi$.

Proof. First note that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_u^{\max})$. Let $\mathcal{A}_b$ be the operator maximal associated with the formal multiplication operator $\mathfrak{A}_b$, i.e., $\mathcal{D}(\mathcal{A}_b) = \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$, $\mathcal{A}_b\Psi = \mathfrak{A}_b\Psi$. The operator $\mathcal{A}_b$ is symmetric and bounded in the Hilbert space $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2$. Hence the stability theorem for selfadjoint operators [Kat80, chap. V, theorem 4.10] shows that $\mathcal{A} = \mathcal{A}_u + \mathcal{A}_b$ with domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_u)$ is selfadjoint if and only if $\mathcal{A}_u$ is selfadjoint. The assertion follows now from lemma 2.8. $\square$

Recall that an operator matrix $\mathcal{T} = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$ on a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ has a block operator matrix representation if its domain can be written as $\mathcal{D}(\mathcal{T}) = \mathcal{D}_1 \oplus \mathcal{D}_2$ with suitable linear manifolds $\mathcal{D}_j \subseteq \mathcal{H}_j$, $j = 1, 2$.

Remark 2.10. The angular operator $\mathcal{A}$ has a block operator matrix representation.

Proof. Let $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_u)$. We have to show that $\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ g \end{smallmatrix}\right)$ lie in the domain of $\mathcal{A}$. Since $\mathcal{A}_u$ is the closure of $\mathcal{A}_u^{\min}$, there is a sequence $\left(\left(\begin{smallmatrix} f_n \\ g_n \end{smallmatrix}\right)\right)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathcal{A}_u^{\min}) = \mathcal{C}_0^\infty(0,\pi)^2$ such that

$$\lim_{n\to\infty} \begin{pmatrix} f_n \\ g_n \end{pmatrix} \;=\; \begin{pmatrix} f \\ g \end{pmatrix}$$

and

$$\lim_{n\to\infty} \mathcal{A}_u^{\min} \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta}\right) g_n \\ \left(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta}\right) f_n \end{pmatrix} = \mathcal{A}_u \begin{pmatrix} f \\ g \end{pmatrix}.$$

This shows that both sequences $\left(\left(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta}\right) f_n\right)_{n\in\mathbb{N}}$ and $\left(\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta}\right) g_n\right)_{n\in\mathbb{N}}$ converge. Hence also $\lim\limits_{n\to\infty} \left(\begin{smallmatrix} f_n \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right)$ and $\lim\limits_{n\to\infty} \left(\begin{smallmatrix} 0 \\ g_n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ g \end{smallmatrix}\right)$ lie in the domain of $\mathcal{A}_u$. $\square$

Remark 2.11. Consider the angular operator in the special case $m = 0$ and define the formal differential expression

$$\mathfrak{B} := \begin{pmatrix} 0 & \mathfrak{B}_+ \\ \mathfrak{B}_- & 0 \end{pmatrix} := \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & 0 \end{pmatrix}.$$

It follows from theorem 2.9 that for $k \in \mathbb{R} \setminus (-1, 0)$ the operator

$$\begin{aligned} \mathcal{D}(\mathcal{B}) &:= \mathcal{D}(\mathcal{A}) = \{\Psi \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2 : \Psi \text{ is absolutely continuous, } \mathfrak{B}\Psi \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2\}, \\ \mathcal{B}\Psi &:= \mathfrak{B}\Psi \end{aligned}$$

is selfadjoint and that it is the closure of the minimal operator $\mathcal{B}^{\min}$, given by $\mathcal{D}(\mathcal{B}^{\min}) := \mathcal{C}_0^\infty(0,\pi)^2$, $\mathcal{B}^{\min}\Psi := \mathfrak{B}\Psi$. This implies that the operators

$$\begin{aligned} \mathcal{D}(B) &:= \{\Psi_2 \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta) : \Psi_2 \text{ is absolutely continuous, } \mathfrak{B}_+\Psi_2 \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)\}, \\ B\Psi_2 &:= \mathfrak{B}_+\Psi_2, \\ \mathcal{D}(B_-) &:= \{\Psi_1 \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta) : \Psi_1 \text{ is absolutely continuous, } \mathfrak{B}_-\Psi_1 \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)\}, \\ B_-\Psi_1 &:= \mathfrak{B}_-\Psi_1 \end{aligned}$$

are adjoint to each other, so that we have $\mathcal{B} = \left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$. Moreover, the operators B and $B^* = B_-$ are the closures of

$$\begin{aligned} \mathcal{D}(B^{\min}) &:= \mathcal{C}_0^\infty(0,\pi), & B^{\min}\Psi_2 &:= \mathfrak{B}_+\Psi_2, \\ \mathcal{D}(B_-^{\min}) &:= \mathcal{C}_0^\infty(0,\pi), & B_-^{\min}\Psi_1 &:= \mathfrak{B}_-\Psi_1, \end{aligned}$$

respectively. $\lozenge$

2.2 Spectrum of $\mathcal{A}$

Since $\mathcal{A}$ is a selfadjoint operator, its spectrum $\sigma(\mathcal{A})$ is real. In order to determine the essential spectrum of $\mathcal{A}$ we use the so-called decomposition method. The idea is to find a symmetric operator $\mathcal{T}$ such that $\mathcal{A}$ is a finite dimensional extension of the closure of $\mathcal{T}$. Since the essential spectra of all finite dimensional selfadjoint extensions of $\mathcal{T}$ coincide (see, e.g., [Wei80, theorem 8.17]), it suffices to determine the essential spectrum of one particular finite dimensional selfadjoint extension of $\mathcal{T}$.

Let $c \in (0,\pi)$ be arbitrary and let $\mathfrak{A}^0$ and $\mathfrak{A}^\pi$ be the restrictions of $\mathfrak{A}$ to $(0,c)$ and (c,π), respectively. With these formal differential expressions we associate the minimal operators

$$\begin{aligned} \mathcal{D}(\mathcal{A}^{0\min}) &:= \mathcal{C}_0^\infty(0,c)^2, & \mathcal{A}^{0\min}\Psi &:= \mathfrak{A}^0\Psi, \\ \mathcal{D}(\mathcal{A}^{\pi\min}) &:= \mathcal{C}_0^\infty(c,\pi)^2, & \mathcal{A}^{\pi\min}\Psi &:= \mathfrak{A}^\pi\Psi. \end{aligned}$$

From the general theory of differential operators we know that these operators are closable and symmetric. In the proof of lemma 2.8 we have seen that $\mathfrak{A}^0$ is in the limit point case at 0 and that $\mathfrak{A}^\pi$ is in the limit point case at π. Both operators are in the limit circle case at c because they are regular at the point c. Hence the operators $\mathcal{A}^{0\,\mathrm{min}}$ and $\mathcal{A}^{\pi\,\mathrm{min}}$ are not essentially selfadjoint, but we can construct selfadjoint extensions if we restrict the corresponding maximal operators $\mathcal{A}^{0\,\mathrm{max}}$ and $\mathcal{A}^{\pi\,\mathrm{max}}$ in terms of boundary conditions at the regular point c; more precisely:

Lemma 2.12. *Let v^0 be an arbitrary non-trivial real solution of $\mathfrak{A}^0 v = 0$ and v^π be an arbitrary non-trivial real solution of $\mathfrak{A}^\pi v = 0$. Then selfadjoint extensions $\mathcal{A}^0$ and $\mathcal{A}^\pi$ of $\mathcal{A}^{0\,\mathrm{min}}$ and $\mathcal{A}^{\pi\,\mathrm{min}}$ are given by*

$$\mathcal{D}(\mathcal{A}^0) := \left\{\Psi \in \mathscr{L}^2((0,c),\mathrm{d}\vartheta)^2 : \Psi \textit{ is absolutely continuous, } \mathfrak{A}^0\Psi \in \mathscr{L}^2((0,c),\mathrm{d}\vartheta),\ [v^0,\Psi]_c = 0\right\},$$
$$\mathcal{A}^0\Psi := \mathfrak{A}^0\Psi,$$
$$\mathcal{D}(\mathcal{A}^\pi) := \left\{\Psi \in \mathscr{L}^2((c,\pi),\mathrm{d}\vartheta)^2 : \Psi \textit{ is absolutely continuous, } \mathfrak{A}^\pi\Psi \in \mathscr{L}^2((c,\pi),\mathrm{d}\vartheta),\ [v^\pi,\Psi]_c = 0\right\},$$
$$\mathcal{A}^\pi\Psi := \mathfrak{A}^\pi\Psi,$$

where
$$[v,\Psi]_c \ := \ \langle (q_0 - q_0^*)v(c), \Psi(c)\rangle = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v(c), \Psi(c) \right\rangle.$$

Proof. This is an application of [Wei87, theorem 5.8.iii]. □

Since both differential expressions $\mathfrak{A}^0$ and $\mathfrak{A}^\pi$ are regular at c, there are solutions v^0 and v^π such that $v^0(c) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $v^\pi = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. With these functions we obtain the particular selfadjoint extensions

$$\mathcal{D}(\mathcal{A}^0) \ := \ \left\{\Psi \in \mathscr{L}^2((0,c),\mathrm{d}\vartheta)^2 \ : \ \Psi \text{ is absolutely continuous, } \mathfrak{A}^0\Psi \in \mathscr{L}^2((0,c)\mathrm{d}\vartheta,,)\ \Psi_1(c) = 0\right\},$$
$$\mathcal{A}^0\Psi \ := \ \mathfrak{A}^0\Psi,$$
$$\mathcal{D}(\mathcal{A}^\pi) \ := \ \left\{\Psi \in \mathscr{L}^2((c,\pi),\mathrm{d}\vartheta)^2 \ : \ \Psi \text{ is absolutely continuous, } \mathfrak{A}^\pi\Psi \in \mathscr{L}^2((c,\pi),\mathrm{d}\vartheta),\ \Psi_2(c) = 0\right\},$$
$$\mathcal{A}^\pi\Psi := \mathfrak{A}^\pi\Psi.$$

It is clear that the operator $\mathcal{A}^0 \oplus \mathcal{A}^\pi$ with domain $\mathcal{D}(\mathcal{A}^0) \oplus \mathcal{D}(\mathcal{A}^\pi)$ is selfadjoint and we have $\sigma_{ess}(\mathcal{A}^0 \oplus \mathcal{A}^\pi) = \sigma_{ess}(\mathcal{A}^0) \cup \sigma_{ess}(\mathcal{A}^\pi)$.

Lemma 2.13. *$\sigma_{ess}(\mathcal{A}) = \sigma_{ess}(\mathcal{A}^0) \cup \sigma_{ess}(\mathcal{A}^\pi)$.*

Proof. Consider the operator $\mathcal{T}$ given by

$$\mathcal{D}(\mathcal{T}) := \{\Psi \in \mathcal{D}(\mathcal{A}) : \Psi(c) = 0\}, \qquad \mathcal{T}\Psi := \mathcal{A}\Psi.$$

Obviously, both $\mathcal{A}^0 \oplus \mathcal{A}^\pi$ and $\mathcal{A}$ are finite dimensional selfadjoint extensions of the closed symmetric operator $\mathcal{T}$, so by [Wei80, theorem 8.17] their essential spectra are equal, i.e.,

$$\sigma_{ess}(\mathcal{A}) = \sigma_{ess}(\mathcal{A}^0 \oplus \mathcal{A}^\pi) = \sigma_{ess}(\mathcal{A}^0) \cup \sigma_{ess}(\mathcal{A}^\pi). \qquad \square$$

Using oscillation theory for Dirac operators we can prove the following theorem.

Theorem 2.14. $\sigma_{ess}(\mathcal{A}) = \emptyset$.

Proof. Using the preceding lemma it suffices to show that $\sigma_{ess}(\mathcal{A}^0)$ and $\sigma_{ess}(\mathcal{A}^\pi)$ are empty. First we consider $\mathcal{A}^\pi$. If we apply the unitary transformation

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix},$$

we obtain the formal differential expression

$$\mathfrak{A}_U^\pi \;:=\; U\mathfrak{A}^\pi U^{-1} \;=\; \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} & 0 \end{pmatrix} + \begin{pmatrix} \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \\ am\cos\vartheta & -\left(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta\right) \end{pmatrix}$$

and the operator $\mathcal{A}_U^\pi := U\mathcal{A}^\pi U^{-1}$, $\mathcal{D}(\mathcal{A}_U^\pi) := U\mathcal{D}(\mathcal{A}^\pi)$. Since U is unitary and $\mathcal{A}^\pi$ is selfadjoint, $\mathcal{A}_U^\pi$ is also selfadjoint and $\sigma_{ess}(\mathcal{A}) = \sigma_{ess}(\mathcal{A}_U)$.
For real λ and real solutions $\Psi = \binom{\psi_1}{\psi_2}$ of $(\mathfrak{A}_U - \lambda)\Psi = 0$ we apply the transformation

$$\Psi(\vartheta) \;=\; \rho(\vartheta) \begin{pmatrix} \cos\delta(\vartheta) \\ \sin\delta(\vartheta) \end{pmatrix},$$

where $\quad \rho(\vartheta) \;:=\; \sqrt{\psi_1^2(\vartheta) + \psi_2(\vartheta)^2} \quad$ and $\quad \delta(\vartheta) \;:=\; \begin{cases} \operatorname{arctan}\frac{\psi_2(\vartheta)}{\psi_1(\vartheta)} & \text{if } \psi_1(\vartheta) \neq 0, \\ \operatorname{arccot}\frac{\psi_1(\vartheta)}{\psi_2(\vartheta)} & \text{if } \psi_2(\vartheta) \neq 0. \end{cases}$

This transformation is known as Prüfer's transformation, see also section 4.2.2. By the requirement $\tan(\delta(\vartheta)) = \frac{\psi_2(\vartheta)}{\psi_1(\vartheta)}$ and $\cot(\delta(\vartheta)) = \frac{\psi_1(\vartheta)}{\psi_2(\vartheta)}$, respectively, the function δ is determined modulo 2π only, but it is possible to choose δ such that it is continuous. According to [Wei87, chap. 16], the function δ fulfils the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\delta(\vartheta) \;=\; \left(G(\vartheta) \begin{pmatrix} \cos\delta(\vartheta) \\ \sin\delta(\vartheta) \end{pmatrix}, \begin{pmatrix} \cos\delta(\vartheta) \\ \sin\delta(\vartheta) \end{pmatrix} \right), \qquad \vartheta \in (0,\pi), \tag{2.19}$$

with

$$G(\vartheta) \;:=\; \lambda - \begin{pmatrix} \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \\ am\cos\vartheta & -\left(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta\right) \end{pmatrix}.$$

To express the fact that δ depends also on λ via the function Ψ, we frequently write $\delta(\vartheta, \lambda)$.

As already mentioned, the phase function δ is determined a priori modulo 2π only. In the following we choose δ such that $\delta(c,\lambda) \in [0, 2\pi)$. Let E be the spectral resolution of $\mathcal{A}_U^\pi$. For all $\lambda_1 < \lambda_2$ we define

$$\begin{aligned} n_+(\lambda_1,\lambda_2) &:= \frac{1}{\pi} \liminf_{\vartheta \nearrow \pi} \left(\delta(\vartheta,\lambda_2) - \delta(\vartheta,\lambda_1)\right), \\ n_-(\lambda_1,\lambda_2) &:= \frac{1}{\pi} \limsup_{\vartheta \nearrow \pi} \left(\delta(\vartheta,\lambda_2) - \delta(\vartheta,\lambda_1)\right), \\ M(\lambda_1,\lambda_2) &:= \dim(E(\lambda_2) - E(\lambda_1)). \end{aligned}$$

According to [Wei87, theorem 16.4] we have the inequalities

$$n_-(\lambda_1,\lambda_2) - 2 \ \leq \ M(\lambda_1,\lambda_2) \ \leq \ n_+(\lambda_1,\lambda_2) + 2. \tag{2.20}$$

For arbitrary $N \in \mathbb{N}$ we show that for all $\lambda \in (-N, N)$ the function $\delta(\cdot,\lambda)$ is bounded. By (2.19) it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\,\delta(\vartheta) \ &= \ -\left(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta - \lambda\right)\cos^2\delta(\vartheta) + \left(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta + \lambda\right)\sin^2\delta(\vartheta) \\ &\quad\ - 2am\cos\vartheta\sin\delta(\vartheta)\cos\delta(\vartheta) \\ &= \ \lambda - 2am\cos\vartheta\sin\delta(\vartheta)\cos\delta(\vartheta) + \left(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta\right)\left(\sin^2\delta(\vartheta) - \cos^2\delta(\vartheta)\right). \end{aligned} \tag{2.21}$$

Let $|\lambda| < N$ and assume that $\delta(\cdot,\lambda)$ is unbounded from above or from below. Furthermore we assume $k + \frac{1}{2} > 0$. Since the function $\vartheta \mapsto \frac{k+\frac{1}{2}}{\sin\vartheta}$, $\vartheta \in (\frac{\pi}{2}, \pi)$, is strictly increasing and unbounded, we can choose $\vartheta_0 \in (c, \pi)$ such that

$$\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta > |\lambda| + 2|am|, \qquad \vartheta \in (\vartheta_0, \pi). \tag{2.22}$$

By assumption, δ is continuous and unbounded, hence there exists either $\nu_+ \in \mathbb{Z}$, $\vartheta_+ \in (\vartheta_0, \pi)$, or $\nu_- \in \mathbb{Z}$, $\vartheta_- \in (\vartheta_0, \pi)$, with the properties

$$\begin{aligned} \delta(\vartheta_+,\lambda) &= \nu_+\pi, & \delta(\vartheta,\lambda) &> \delta(\vartheta_+,\lambda) & &\text{in a right neighbourhood of } \vartheta_+, \\ \delta(\vartheta_-,\lambda) &= (\nu_- + \tfrac{1}{2})\pi, & \delta(\vartheta,\lambda) &< \delta(\vartheta_-,\lambda) & &\text{in a right neighbourhood of } \vartheta_-. \end{aligned} \tag{2.23}$$

On the other hand, (2.21) shows that the phase δ is monotonously decreasing in a neighbourhood of ϑ_+ and monotonously increasing in a neighbourhood of ϑ_- in contradiction to (2.23). For $k+\frac{1}{2} < 0$ the proof is similar.
Thus for all $\lambda \in (-N, N)$ the function $\delta(\cdot,\lambda)$ is bounded. Hence also $n_\pm$ and, consequently, $M(\lambda_1,\lambda_2)$ are bounded for all $\lambda_1, \lambda_2 \in (-N, N)$. By definition of $M(\lambda_1,\lambda_2)$ it follows that

$$\sigma_{ess}(\mathcal{A}_U^\pi) \cap (-N, N) = \sigma_{ess}(\mathcal{A}^\pi) \cap (-N, N) = \emptyset.$$

Since this result is valid for all $N \in \mathbb{N}$, it follows that $\sigma_{ess}(\mathcal{A}^\pi) = \emptyset$.
Analogously we can show $\sigma_{ess}(\mathcal{A}^0) = \emptyset$. □

As a corollary we obtain the following theorem.

Theorem 2.15. *The spectrum of $\mathcal{A}$ consists of isolated eigenvalues only which accumulate at most at $+\infty$ or $-\infty$.*

In fact, since $\mathcal{A}$ is unbounded and selfadjoint, the spectrum of $\mathcal{A}$ is unbounded. Hence at least one of the points $\pm\infty$ must be an accumulation point. Later, in section 3.2, we show that the spectrum of $\mathcal{A}$ is neither bounded from below nor from above.

Remark 2.16. In theorem 3.23 we show that $\mathcal{A}$ has compact resolvent which also implies that the essential spectrum of $\mathcal{A}$ empty.

2.3 Symmetries of the angular operator $\mathcal{A}$

In this section we establish some symmetry properties of the formal differential expression $\mathfrak{A}$ with respect to the physical parameters k, a, m and ω. Recall that the formal angular operator is given by

$$\mathfrak{A} = \begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} \qquad \text{on} \quad (0,\pi).$$

Since in the following we consider different values of the variables a, k, m and ω, we often write $\mathfrak{A}(k,a,m,\omega)$ and $\mathcal{A}(k,a,m,\omega)$ to indicate the dependence of $\mathfrak{A}$ and $\mathcal{A}$ on these variables explicitly. If no confusion arises, we omit some or all of them.
For fixed $\lambda \in \mathbb{R}$, the equation $(\mathfrak{A}-\lambda)u = 0$ is a linear system of two differential equations on the interval $(0,\pi)$, hence it has two linearly independent solutions. Using the symmetry properties of $\mathfrak{A}$ given in lemma 2.17, we can, for instance, construct a second solution of $(\mathfrak{A}-\lambda)u = 0$ if one solution is already known.
Here we are only interested in formal solutions, i.e., in solutions that need not be square integrable; therefore we work with the formal differential expression $\mathfrak{A}$ rather than with the operator $\mathcal{A}$.

Lemma 2.17. *Fix $k \in \mathbb{Z}$ and $a, m, \omega, \lambda \in \mathbb{R}$. Further, let $\Psi = \left(\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right)$ be a formal solution of*

$$(\mathfrak{A}(k,a,m,\omega)-\lambda)u = 0.$$

Then the following holds:

(i) *Ψ is also a solution of*

$$(\mathfrak{A}(k,-a,-m,-\omega)-\lambda)u \;=\; 0.$$

(ii) *The function $X(\vartheta) := \Psi(\pi-\vartheta)$, $\vartheta \in (0,\pi)$, is a formal solution of*

$$(\mathfrak{A}(-(k+1),a,m,-\omega)+\lambda)u \;=\; 0.$$

(iii) *The function $\Phi(\vartheta) := \left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)\Psi(\pi-\vartheta) = \left(\begin{smallmatrix}\psi_2(\pi-\vartheta)\\ \psi_1(\pi-\vartheta)\end{smallmatrix}\right)$, $\vartheta \in (0,\pi)$, is also a formal solution of*

$$(\mathfrak{A}(k,a,m,\omega)-\lambda)u \;=\; 0.$$

(iv) *The function $Z(\vartheta) := \left(\begin{smallmatrix}-I & 0\\ 0 & I\end{smallmatrix}\right)\Psi(\vartheta) = \left(\begin{smallmatrix}-\psi_1(\vartheta)\\ \psi_2(\vartheta)\end{smallmatrix}\right)$, $\vartheta \in (0,\pi)$, is a formal solution of*

$$(\mathfrak{A}(k,a,-m,\omega)+\lambda)u \;=\; 0.$$

Recall that Ψ is an eigenfunction of $\mathcal{A}(k,a,m,\omega)$ with eigenvalue λ if and only if Ψ is a solution of the differential equation $(\mathfrak{A}(k,a,m,\omega)-\lambda)u = 0$ and both Ψ and $\mathfrak{A}(k,a,m,\omega)\Psi$ are elements of $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2$. Hence Ψ is an eigenfunction of $\mathcal{A}(k,a,m,\omega)$ if and only if the functions X, Φ and Z are also eigenfunctions of the corresponding operators.

Proof of lemma 2.17. (i) The assertion follows from $\mathfrak{A}(k,-a,-m,-\omega) = \mathfrak{A}(k,a,m,\omega)$.

(ii) Let X be defined by $X(\vartheta) := \Psi(\pi-\vartheta)$ for all $\vartheta \in (0,\pi)$. If we apply the coordinate transformation $\vartheta \mapsto \pi-\vartheta$, the eigenvalue equation $(\mathfrak{A}(k,a,\omega,m)-\lambda)\Psi = 0$ becomes $(\mathfrak{A}^-(k,a,m,\omega)-\lambda)X = 0$ with the formal differential expression $\mathfrak{A}^-$ given by

$$\mathfrak{A}^-(k,a,m,\omega) = \begin{pmatrix} am\cos\vartheta & -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & -am\cos\vartheta \end{pmatrix}.$$

Now we have the following equivalent equalities:

$$\begin{aligned} & 0 = (\mathfrak{A}^-(k,a,m,\omega)-\lambda)X \\ \iff\quad & 0 = (-\mathfrak{A}^-(k,a,m,\omega)+\lambda)X \\ \iff\quad & 0 = \left(\begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{-(k+1)+\frac{1}{2}}{\sin\vartheta} - a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{-(k+1)+\frac{1}{2}}{\sin\vartheta} - a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} + \lambda\right) X, \end{aligned}$$

and the last line is exactly the assertion.

(iii) For X and $\mathfrak{A}^-(k,a,m,\omega)$ as above we know that $(\mathfrak{A}^-(k,a,m,\omega)-\lambda)X = 0$. Since the matrix $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ is invertible, we have the following equivalences:

$$\begin{aligned} & 0 = (\mathfrak{A}^-(k,a,m,\omega)-\lambda)X \\ \iff\quad & 0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \left(\mathfrak{A}^-(k,a,m,\omega)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X \\ \iff\quad & 0 = \left(\begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} - \lambda\right)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X. \end{aligned}$$

The last line is the same as $(\mathfrak{A}(k,a,m,\omega)-\lambda)\Phi = 0$ and the assertion is proved.

(iv) Since the matrix $\left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)$ is invertible and self-inverse, we have the following equivalences:

$$\begin{aligned} & 0 = (\mathfrak{A}(k,a,m,\omega)-\lambda)\Psi \\ \iff\quad & 0 = \left(\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\mathfrak{A}(k,a,m,\omega)\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} - \lambda\right)\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\Psi \\ \iff\quad & 0 = \left(\begin{pmatrix} am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & -am\cos\vartheta \end{pmatrix} + \lambda\right) Z \\ \iff\quad & 0 = (\mathfrak{A}(k,a,-m,\omega)+\lambda)Z. \end{aligned}$$

□

The next corollary follows immediately from lemma 2.17 (iv).

Corollary 2.18. *If either $a = 0$ or $m = 0$, then the point spectrum of $\mathcal{A}$ is symmetric with respect to 0.*

Lemma 2.17 allows us to draw some conclusions about the value of an eigenfunction at $\vartheta = \frac{\pi}{2}$.

Corollary 2.19. *Let* $\Psi = \binom{\psi_1}{\psi_2}$ *be an eigenfunction of* $\mathcal{A}$ *with eigenvalue* λ. *Then lemma 2.17 (iii) implies that also* $\Phi := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \Psi(\pi - \cdot)$ *is an eigenfunction of* $\mathcal{A}$. *Moreover, there is a* $\gamma \in \mathbb{C}$ *with* $|\gamma| = 1$ *such that* $\Phi = \gamma\Psi$ *holds. With this* γ *we obtain*

$$\Psi(\tfrac{\pi}{2}) \;=\; \psi_1(\tfrac{\pi}{2}) \begin{pmatrix} 1 \\ \gamma \end{pmatrix}, \qquad \Psi'(\tfrac{\pi}{2}) \;=\; \psi_1'(\tfrac{\pi}{2}) \begin{pmatrix} 1 \\ -\gamma \end{pmatrix}. \tag{2.24}$$

In particular, none of the components of Ψ *vanishes identically.*

Proof. Let Ψ and λ be as in the assertion. Furthermore, let X be another solution of $(\mathcal{A} - \lambda)u = 0$ such that Ψ, X form a fundamental system of $(\mathcal{A} - \lambda)u = 0$. Since $\mathfrak{A} - \lambda$ is in the limit point case both at 0 and at π and since Ψ is square integrable by assumption, the function X lies neither left nor right in $\mathscr{L}^2((0, \pi), \mathrm{d}\vartheta)^2$. The preceding lemma shows that also $\Phi := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \Psi(\pi - \cdot)$ is an eigenfunction of $(\mathcal{A} - \lambda)u = 0$, hence there exist $\gamma, \delta \in \mathbb{C}$ such that $\Phi = \gamma\Psi + \delta X$. Since both Ψ and Φ are square integrable on $(0, \pi)$, but X is not, it follows that δ must be zero. Furthermore, the equality $\|\Psi\| = \|\Phi\|$ implies $|\gamma| = 1$. Comparing Ψ and Φ leads to

$$\gamma \begin{pmatrix} \psi_1(\vartheta) \\ \psi_2(\vartheta) \end{pmatrix} \;=\; \gamma\Psi(\vartheta) \;=\; \Phi(\vartheta) \;=\; \begin{pmatrix} \psi_2(\pi - \vartheta) \\ \psi_1(\pi - \vartheta) \end{pmatrix}, \qquad \vartheta \in (0, \pi),$$

and

$$\gamma \begin{pmatrix} \psi_1'(\vartheta) \\ \psi_2(\vartheta) \end{pmatrix} \;=\; \gamma\Psi'(\vartheta) \;=\; \Phi'(\vartheta) \;=\; \begin{pmatrix} -\psi_2'(\pi - \vartheta) \\ -\psi_1'(\pi - \vartheta) \end{pmatrix}, \qquad \vartheta \in (0, \pi).$$

In the special case $\vartheta = \frac{\pi}{2}$ these equations show that

$$\psi_1(\tfrac{\pi}{2}) \;=\; \gamma\psi_2(\tfrac{\pi}{2}), \qquad\qquad \psi_1'(\tfrac{\pi}{2}) \;=\; -\gamma\psi_2'(\tfrac{\pi}{2}),$$

which proves equation (2.24). □

Chapter 3

Three different lower bounds for the modulus of the eigenvalues of $\mathcal{A}$

In theorem 2.9 we have shown that the angular operator

$$\mathcal{A} = \begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} =: \begin{pmatrix} -D & B \\ B^* & D \end{pmatrix} \tag{3.1}$$

in the Hilbert space $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2$ with domain

$$\mathcal{D}(\mathcal{A}) \;=\; \mathcal{D}(B^*)\oplus\mathcal{D}(B) \;=\; \left\{\Psi\in\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2 : \Psi \text{ is abs. cont., } \mathfrak{A}\Psi\in\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)^2\right\}$$

is selfadjoint and has purely discrete point spectrum. All eigenvalues of $\mathcal{A}$ are simple because $\mathcal{A}$ is a selfadjoint linear differential operator of first order.

The operator D with domain $\mathcal{D}(D) = \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ is a bounded multiplication operator in the Hilbert space $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$. In remark 2.11 we have seen that the operator B is the closure of $B^{\min}$, defined by $B^{\min}f := \mathfrak{B}_+f$, $f\in\mathcal{D}(B^{\min}) := \mathcal{C}_0^\infty(0,\pi)$. Where necessary, we express the dependence of B on the wave number k by writing B_k instead of B, otherwise we suppress the subscript k in order to keep the notation as simple as possible.

The aim of this chapter is to establish lower bounds for the modulus of eigenvalues of $\mathcal{A}$ by applying an off-diagonalisation method. In order to express these lower bounds in terms of the physical parameters k, a, m and ω, we need an explicit upper bound for $\|B^{-1}\|$. Since we know the form of B^{-1} as an integral operator explicitly, we can derive various upper bounds, depending on how the integral kernel of B^{-1} is estimated, see lemmata 3.30 and 3.34. The lower bounds for the modulus of eigenvalues of $\mathcal{A}$ resulting from the off-diagonalisation method are established in theorem 3.35. Other lower bounds for the modulus of the eigenvalues of $\mathcal{A}$ are obtained in the following chapter where we use Sturm's comparison theorem to obtain bounds for the eigenvalues of BB^*. Both the off-diagonalisation method presented in this chapter and the variational principle of the next chapter basically treat the bounded operators on the diagonal of $\mathcal{A}$ as a perturbation. A completely different approach to obtain lower bounds for the modulus of eigenvalues of $\mathcal{A}$ is given in chapter 5.

In the first section of this chapter, we consider the angular operator for $a = 0$. In this case, the eigenfunctions and eigenvalues of $\mathcal{A}$ are explicitly known, see lemma 3.3. Since for $a = 0$ the spectrum is unbounded both from below and from above, it follows from standard perturbation theory that also in the case $a \neq 0$ the set of eigenvalues of $\mathcal{A}$ is neither bounded from below nor from above. The knowledge of the eigenfunctions of $\mathcal{A}$ for $a = 0$ allows us to give first order approximations of the eigenvalues $\lambda_{\pm1}$ (the first positive and the first negative eigenvalue of $\mathcal{A}$) with respect to a for small a.

In the following we always assume that $k\in\mathbb{R}\setminus(-1,0)$.

3.1 The special case $a = 0$

The aim of this section is twofold. Firstly, if we know the eigenvalues of $\mathcal{A}$ in the case $a = 0$, then we can use analytic perturbation theory to derive estimates for the eigenvalues in the case of small $|a|$. On the other hand, a comparison of the angular operator (3.1) in the case $a = 0$ with the angular part of the usual Dirac operator in flat spacetime as given, e.g., in [Gre87] provides us with a physical interpretation of the eigenvalue λ of the angular operator $\mathcal{A}$.

3.1.1 The Dirac operator in flat spacetime

The usual Dirac equation without potential in flat spacetime is a linear system of four coupled partial differential equations given by

$$\Big(-\mathrm{i}\frac{\partial}{\partial t} + H_D\Big)\widehat{\Phi} \;=\; 0 \qquad \text{with} \quad H_D = \vec{\alpha}\cdot\vec{p} + \beta m, \tag{3.2}$$

where $\beta = \left(\begin{smallmatrix} I_2 & 0 \\ 0 & -I_2 \end{smallmatrix}\right)$ and $\vec{\alpha} = \left(\begin{smallmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{smallmatrix}\right)$ with $\vec{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$, consisting of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The quantum mechanical momentum $\vec{p}$ is given by the formal differential operator

$$\vec{p} \;=\; -\mathrm{i}\,\nabla \;=\; -\mathrm{i}\Big(\frac{\partial}{\partial x},\, \frac{\partial}{\partial y},\, \frac{\partial}{\partial z}\Big).$$

Note that throughout the text we follow the standard convention $\hbar = c = 1$.

In most textbooks on relativistic quantum mechanics, the Dirac equation is separated by applying a suitable ansatz for the angular part of the eigenfunctions, see for example [Lan96] or [Sch99]. However, in order to see that the Dirac equation (3.2) and the Dirac equation for a fermion in the Kerr-Newman background as given in (2.4) are equivalent in the special case $a = 0,\ M = 0,\ Q = 0$, we carry out the separation process explicitly, see [Tha92]. In analogy to the ansatz (2.7) for the solution of the Dirac equation in the case of the Kerr-Newman metric, we use the ansatz $\widehat{\Phi} = \mathrm{e}^{-\mathrm{i}\omega t}\widetilde{\Phi}$ for solutions of (3.2) so that the derivative with respect to t can be substituted by $-\mathrm{i}\omega$. Further, we use polar coordinates (r, ϑ, φ), with the normalised basis vectors

$$\vec{\mathrm{e}}_r = (\sin\vartheta\cos\varphi,\, \sin\vartheta\sin\varphi,\, \cos\vartheta),\ \ \vec{\mathrm{e}}_\vartheta = (\cos\vartheta\cos\varphi,\, \cos\vartheta\sin\varphi,\, -\sin\vartheta),\ \ \vec{\mathrm{e}}_\varphi = (-\sin\varphi,\, \cos\varphi,\, 0).$$

In polar coordinates, the formal differentiation operator $-\mathrm{i}\nabla$ and the *angular momentum operator* $\vec{L} := \vec{r}\times\vec{p} = -\mathrm{i}\,\vec{r}\times\nabla$ have the form

$$-\mathrm{i}\,\nabla \;=\; -\mathrm{i}\,\vec{\mathrm{e}}_r\frac{\partial}{\partial r} - \frac{1}{r}(\vec{\mathrm{e}}_r\times\vec{L}), \qquad\qquad \vec{L} \;=\; \mathrm{i}\,\vec{\mathrm{e}}_\vartheta\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \mathrm{i}\,\vec{\mathrm{e}}_\varphi\frac{\partial}{\partial\vartheta}. \tag{3.3}$$

Obviously, we have $\vec{\mathrm{e}}_r\cdot\vec{L} = 0$.
Since in the following it is always clear on what spaces ∇ and the angular momentum operator $\vec{L}$ act, we do not use different notations for these operators with respect to the dimension of the $\mathscr{L}^2$-spaces they are acting on; for example, $\vec{L}$ has to be understood to be the block matrix $\left(\begin{smallmatrix} \vec{L} & 0 \\ 0 & \vec{L} \end{smallmatrix}\right)$ when applied to functions with values in $\mathbb{C}^2$.

It is convenient to introduce the so-called *spin operator*

$$\vec{S} \;:=\; \tfrac{1}{2}\,\gamma_5\,\vec{\alpha} \;=\; \tfrac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

with the 4×4-matrix $\gamma_5 = \left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$. It is well known that for arbitrary vectors $\vec{A}$ and $\vec{B}$ the relation

$$(\vec{\alpha}\cdot\vec{A})(\vec{\alpha}\cdot\vec{B}) \;=\; \vec{A}\cdot\vec{B} + 2\mathrm{i}\,\vec{S}\cdot(\vec{A}\times\vec{B})$$

holds. In particular, using $\vec{\alpha} = 2\gamma_5\vec{S}$, we obtain for $\vec{A} = \vec{\mathrm{e}}_r$ and $\vec{B} = \vec{L}$ that

$$\vec{\alpha}\cdot(\vec{\mathrm{e}}_r\times\vec{L}) \;=\; 2\,\gamma_5\,\vec{S}\cdot(\vec{\mathrm{e}}_r\times\vec{L}) \;=\; \mathrm{i}\,\gamma_5\,(\vec{\mathrm{e}}_r\cdot\vec{L}) - \mathrm{i}\,\gamma_5\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,(\vec{\alpha}\cdot\vec{L}) \;=\; -\mathrm{i}\,\gamma_5\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,(\vec{\alpha}\cdot\vec{L})\;.$$

With this relation, the equality $\gamma_5\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r) = (\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\gamma_5$ and the representation (3.3) of ∇, it is easy to verify that in polar coordinates the Dirac operator has the form

$$\begin{aligned} H_D \;&=\; -\mathrm{i}\,\vec{\alpha}\cdot\nabla + \beta m \;=\; -\mathrm{i}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\frac{\partial}{\partial r} - \frac{1}{r}\,\vec{\alpha}\cdot(\vec{\mathrm{e}}_r\times\vec{L}) + \beta m \\ &=\; -\mathrm{i}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\gamma_5\,(\vec{\alpha}\cdot\vec{L}) + \beta m \\ &=\; -\mathrm{i}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\frac{\partial}{\partial r} + 2\,\frac{\mathrm{i}}{r}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)(\vec{S}\cdot\vec{L}) + \beta m. \end{aligned}$$

If we introduce the operators $\mathfrak{K} := \vec{\sigma}\cdot\vec{L}+1$ acting on functions with values in $\mathbb{C}^2$ and the *spin-orbit operator* $\widehat{\mathfrak{K}} := \beta\,(2\vec{S}\cdot\vec{L}+1) = \left(\begin{smallmatrix} \mathfrak{K} & 0 \\ 0 & -\mathfrak{K} \end{smallmatrix}\right)$ acting on functions with values in $\mathbb{C}^4$, the Dirac equation becomes

$$0 \;=\; (H_D-\omega)\widetilde{\Phi} \;=\; \Big(-\frac{\mathrm{i}}{r}(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\Big(r\frac{\partial}{\partial r} + 1 - \beta\,\widehat{\mathfrak{K}}\Big) + \beta m - \omega\Big)\widetilde{\Phi}. \tag{3.4}$$

It is easy to see that

$$\vec{\alpha}\cdot\vec{\mathrm{e}}_r \;=\; \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\mathrm{e}}_r \\ \vec{\sigma}\cdot\vec{\mathrm{e}}_r & 0 \end{pmatrix} \qquad \text{with} \qquad \vec{\sigma}\cdot\vec{\mathrm{e}}_r \;=\; \begin{pmatrix} \cos\vartheta & \mathrm{e}^{-\mathrm{i}\varphi}\sin\vartheta \\ \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta & -\cos\vartheta \end{pmatrix},$$

and by straightforward calculations it can be shown that $(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)$ and $\mathfrak{K}$ anticommute, and that $\widehat{\mathfrak{K}}$ commutes with the Dirac operator, i.e., $(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\mathfrak{K} + \mathfrak{K}(\vec{\sigma}\cdot\vec{\mathrm{e}}_r) = 0$ and $\widehat{\mathfrak{K}}H_D - H_D\widehat{\mathfrak{K}} = 0$.
Furthermore, it should be mentioned that

$$\widehat{\mathfrak{K}} \;=\; \beta\,(\mathfrak{J}^2 - L^2 - S^2 + 1) \;=\; \beta\,(\mathfrak{J}^2 - L^2 + \tfrac{1}{4}), \tag{3.5}$$

where $\vec{\mathfrak{J}} := \vec{L}+\vec{S}$ is the *total angular momentum* of the Dirac particle From the formula above, it follows that $\widehat{\mathfrak{K}}$ commutes also with $\mathfrak{J}^2$ and $\mathfrak{J}_z$.
Next we transform the Dirac equation with the unitary matrix

$$\widetilde{U} = \begin{pmatrix} U & 0 \\ 0 & \mathrm{i}\,U \end{pmatrix}, \qquad \text{with} \qquad U = \frac{1}{\sqrt{2(1-\cos\vartheta)}} \begin{pmatrix} -\mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi}(1-\cos\vartheta) & \mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi}\sin\vartheta \\ \mathrm{e}^{\frac{\mathrm{i}}{2}\varphi}\sin\vartheta & \mathrm{e}^{\frac{\mathrm{i}}{2}\varphi}(1-\cos\vartheta) \end{pmatrix}.$$

Observe that

$$U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U \;=\; \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

$$\begin{aligned} U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)(\vec{\sigma}\cdot\vec{L})\,U \;&=\; -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\left(\frac{\partial}{\partial\vartheta} + \frac{\cot\vartheta}{2}\right) - \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\frac{\mathrm{i}}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \\ &=\; \begin{pmatrix} 0 & -\mathfrak{L}_-^{\varphi} \\ -\mathfrak{L}_+^{\varphi} & 0 \end{pmatrix} - \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

with the differential expressions $\mathfrak{L}_\pm^\varphi = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{\cot\vartheta}{2} \mp \big(a\omega\sin\vartheta + \frac{\mathrm{i}}{\sin\vartheta}\frac{\partial}{\partial\varphi}\big)$, cf. (2.10). Hence we obtain for the Dirac equation in the form (3.4)

$$\begin{aligned}
0 &= (\widetilde{U}^{-1}(H_D-\omega)\widetilde{U})\,(\widetilde{U}^{-1}\widetilde{\Phi}) \\
&= \Big(-\mathrm{i}\,\widetilde{U}^{-1}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\widetilde{U}\,\Big(\frac{\partial}{\partial r}+\frac{1}{r}\Big)+\frac{\mathrm{i}}{r}\,\widetilde{U}^{-1}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\beta\,\widehat{\mathfrak{K}}\,\widetilde{U}+\beta m-\omega\Big)\,(\widetilde{U}^{-1}\widetilde{\Phi}) \\
&= \Big(-\mathrm{i}\,\widetilde{U}^{-1}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\widetilde{U}\,\Big(\frac{\partial}{\partial r}+\frac{1}{r}\Big)+\frac{\mathrm{i}}{r}\,\widetilde{U}^{-1}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)(2\vec{S}\cdot\vec{L}+1)\,\widetilde{U}+\beta m-\omega\Big)(\widetilde{U}^{-1}\widetilde{\Phi}) \\
&= \Bigg(-\mathrm{i}\begin{pmatrix} 0 & \mathrm{i}\,U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U \\ -\mathrm{i}\,U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U & 0\end{pmatrix}\Big(\frac{\partial}{\partial r}+\frac{1}{r}\Big) \\
&\qquad +\frac{\mathrm{i}}{r}\begin{pmatrix} 0 & \mathrm{i}\,U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)(\vec{\sigma}\cdot\vec{L}+1)\,U \\ -\mathrm{i}\,U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)(\vec{\sigma}\cdot\vec{L}+1)\,U & 0\end{pmatrix}+\beta m-\omega\Bigg)\,(\widetilde{U}^{-1}\widetilde{\Phi}) \\
&= \left(\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -I & 0 \\ & & 0 & I \\ \hline I & 0 & \multicolumn{2}{c}{0} \\ 0 & -I & & \end{array}\right)\Big(\frac{\partial}{\partial r}+\frac{1}{r}\Big)+\frac{1}{r}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & \mathfrak{L}_-^\varphi \\ & & \mathfrak{L}_+^\varphi & 0 \\ \hline 0 & -\mathfrak{L}_-^\varphi & \multicolumn{2}{c}{0} \\ -\mathfrak{L}_+^\varphi & 0 & & \end{array}\right)+\beta m-\omega\right)(\widetilde{U}^{-1}\widetilde{\Phi}).
\end{aligned}$$

Next we transform with the self-inverse, unitary matrix $U_0 := \left(\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I\end{smallmatrix}\right)$ and apply the ansatz $\widetilde{\Phi} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}(k+\frac{1}{2})\varphi}\Phi$ so that we have $(\frac{\partial}{\partial r}+\frac{1}{r})\widetilde{\Phi} = \frac{1}{r}\frac{\partial}{\partial r}\Phi$ and $\mathrm{i}\frac{\partial}{\partial\varphi}\widetilde{\Phi} = (k+\frac{1}{2})\widetilde{\Phi}$. We finally obtain that the Dirac equation (3.2) is equivalent to

$$\begin{aligned}
0 &= U_0\widetilde{U}^{-1}(H_D-\omega)\widetilde{U}U_0\;(U_0\widetilde{U}^{-1}\widetilde{\Phi}) \\
&= \left(\begin{pmatrix} m-\omega & 0 & -\frac{\partial}{\partial r} & 0 \\ 0 & m-\omega & 0 & -\frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & 0 & -(m+\omega) & 0 \\ 0 & \frac{\partial}{\partial r} & 0 & -(m+\omega)\end{pmatrix}+\frac{1}{r}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & \mathfrak{L}_- \\ & & -\mathfrak{L}_+ & 0 \\ \hline 0 & \mathfrak{L}_- & \multicolumn{2}{c}{0} \\ -\mathfrak{L}_+ & 0 & & \end{array}\right)\right)(U_0\widetilde{U}^{-1}\Phi).
\end{aligned}$$

This is exactly equation (2.11) with $a = 0$, $M = 0$, $Q = 0$, since in this case $\sqrt{\Delta(r)} = r$ and $\Omega(r) = \omega$.

Note that the ansatz $\widetilde{\Phi} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}(k+\frac{1}{2})\varphi}\Phi$ is natural in the sense that, by physical reasoning, $\widetilde{\Phi}$ is supposed to be square integrable on $(0,\infty)$ with respect to $r^2\sin\vartheta\,\mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi$, cf. also section 2.1.1.

The calculations above show that in flat spacetime, that is, $a = 0$, $M = 0$ and $Q = 0$, the angular operator $\mathcal{A}$ is similar to the spin-orbit operator $\mathfrak{K}$. In fact, it follows from the calculation above that

$$\begin{aligned}
\begin{pmatrix} 0 & \mathfrak{A}^{(d)} \\ \mathfrak{A}^{(d)} & 0\end{pmatrix} &= \mathrm{i}\,U_0\,\widetilde{U}^{-1}\,(\vec{\alpha}\cdot\vec{\mathrm{e}}_r)\,\beta\,\widehat{\mathfrak{K}}\,\widetilde{U}U_0 \\
&= U_0\begin{pmatrix} 0 & -U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,\mathfrak{K}\,U \\ U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,\mathfrak{K}\,U & 0\end{pmatrix}U_0 \\
&= \begin{pmatrix} 0 & -\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}U^{-1}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,\mathfrak{K}\,U \\ -U^{-1}\,\mathfrak{K}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix} & 0\end{pmatrix}.
\end{aligned}$$

Because of

$$-\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} U^{-1}(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,\mathfrak{K}\,U \;=\; -\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} U^{-1}(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U\,U^{-1}\,\mathfrak{K}\,U \;=\; U^{-1}\,\mathfrak{K}\,U$$

and

$$-U^{-1}\,\mathfrak{K}\,(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \;=\; -U^{-1}\,\mathfrak{K}\,U\,U^{-1}(\vec{\sigma}\cdot\vec{\mathrm{e}}_r)\,U\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \;=\; U^{-1}\,\mathfrak{K}\,U$$

we obtain

$$\mathfrak{A}^{(d)} \;=\; U^{-1}\,\mathfrak{K}\,U. \tag{3.6}$$

Remark 3.1. Let $a=0$. Then $\widehat{\Psi}$ is a solution of the Dirac equation $(\widehat{\mathfrak{R}}+\widehat{\mathfrak{A}})\widehat{\Psi}=0$ in the form of (2.4) if and only if $\widehat{\Phi}=\frac{1}{r}\,\widetilde{U}U_0^{-1}V\widehat{\Psi}$ is a solution of the Dirac equation (3.2).

We conclude this section with some remarks on the spectrum of the angular part of the Dirac operator in flat spacetime.

It is well known that neither the spin operator $\vec{S}$ nor the angular momentum operator $\vec{L}$ commute with the Dirac operator H_D, see for example [BD64] or [Lan96]; but the total angular momentum $\vec{\mathfrak{J}}=\vec{L}+\vec{S}$ and the *parity operator* P commute with H_D and with each other. Furthermore, $\mathfrak{J}^2$ and $\mathfrak{J}_z$ have purely discrete point spectrum and there is a basis $\{\Psi^{\pm}_{j,\,m_j} : j,\, m_j\in\mathbb{N}+\frac{1}{2},\ |m_j|\le j\}$ of simultaneous eigenfunctions of $\mathfrak{J}^2$, $\mathfrak{J}_z$ and the parity operator P, with

$$\mathfrak{J}^2\Psi^{\pm}_{j,m_j}=j(j+1)\Psi^{\pm}_{j,m_j},\qquad \mathfrak{J}_z\Psi^{\pm}_{j,m_j}=m_j\Psi^{\pm}_{j,m_j}\qquad\text{and}\qquad P\Psi^{\pm}_{j,m_j}=\pm\Psi^{\pm}_{j,m_j}.$$

It is possible to choose these eigenfunctions $\Psi^{\pm}_{j,m_j}$ such that $\Psi^{\pm}_{j,m_j}=\begin{pmatrix}\psi^{l}_{j,m_j}\\ \psi^{l'}_{j,m_j}\end{pmatrix}$ where the two-spinors ψ^{l}_{j,m_j} and $\psi^{l'}_{j,m_j}$ are eigenfunctions of $\vec{L}$ with eigenvalues l and l' respectively, with $|l-l'\,|=1$. From the angular operator algebra it follows that $|j-l\,|=\frac{1}{2}$ and $|j-l'\,|=\frac{1}{2}$.
The functions $\Psi^{\pm}_{j,m_j}$ are also eigenfunctions of the spin-orbit operator $\widehat{\mathfrak{K}}$. As already mentioned, $\widehat{\mathfrak{K}}$ commutes with $\mathfrak{J}^2$, $\mathfrak{J}_z$ and the Dirac operator H_D. From (3.5) it follows that $\widehat{\mathfrak{K}}$ has purely discrete point spectrum, and that the $\Psi^{\pm}_{j,m_j}$ are eigenstates of $\widehat{\mathfrak{K}}$ with eigenvalue $\widetilde{\kappa}$, where

$$\widetilde{\kappa} \;=\; \begin{cases} \quad j+\frac{1}{2} \;=\; l+1 & \text{if } j=l+\frac{1}{2},\\ -(j+\frac{1}{2}) \;=\; -l & \text{if } j=l-\frac{1}{2}. \end{cases} \tag{3.7}$$

Thus, instead of classifying the eigenstates according to their parity, we can classify the eigenfunctions according to the eigenvalues of $\widehat{\mathfrak{K}}$.
If we fix an $m_j\in\mathbb{N}+\frac{1}{2}$, then the eigenvalues of $\mathfrak{J}^2$ are $j_n(j_n+1)$ with $j_n=|m_j|+n$, $n\in\mathbb{N}_0$. Hence the eigenvalues $\widetilde{\kappa}$ of $\widehat{\mathfrak{K}}$ are given by

$$\begin{aligned}\{\widetilde{\kappa}_n=\pm(j_n+\tfrac{1}{2})\,:\,n\in\mathbb{N}_0\} \;&=\; \{\widetilde{\kappa}_n=\pm(|m_j|+\tfrac{1}{2}+n)\,:\,n\in\mathbb{N}_0\}\\ &=\; \{\widetilde{\kappa}_n=\operatorname{sign}(n)(|m_j|-\tfrac{1}{2}+|n|)\,:\,n\in\mathbb{Z}\setminus\{0\}\}.\end{aligned} \tag{3.8}$$

3.1.2 Eigenvalues in the case $a = 0$

In the previous section we have seen that in the case $a = 0$ the angular operator $\mathcal{A}$ is similar to the spin-orbit operator $\mathfrak{K}$, hence their spectra coincide. Although the spectrum of $\mathfrak{K}$ is known, see (3.8), we present a direct calculation of the spectrum of $\mathcal{A}$ that also provides the eigenfunctions explicitly. For $a = 0$ the angular operator reduces to

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} & 0 \end{pmatrix}.$$

Recall that $\Psi = \binom{S_-}{S_+}$ is an eigenfunction of $\mathcal{A}$ with eigenvalue λ if and only if Ψ does not vanish identically on $(0, \pi)$ and satisfies the differential equation

$$(\mathcal{A} - \lambda)\Psi = 0 \tag{3.9}$$

and the integrability condition

$$\|\Psi\|_2^2 \;=\; \int_0^\pi \langle \Psi(\vartheta), \Psi(\vartheta)\rangle \,\mathrm{d}\vartheta \;<\; \infty. \tag{3.10}$$

As an abbreviation we set $\kappa := k + \frac{1}{2}$. Recall that $\mathfrak{A}$ is in the limit point case at both endpoints of the interval $(0, \pi)$ if and only if $|\kappa| \geq \frac{1}{2}$, otherwise it is in the limit circle case at both endpoints. In the rest part of this chapter we assume $|\kappa| \geq \frac{1}{2}$.
The differential equation (3.9) is equal to the coupled system

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{\kappa}{\sin\vartheta}\right) S_+ - \lambda S_- = 0, \qquad \vartheta \in (0,\pi), \\ \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} - \frac{\kappa}{\sin\vartheta}\right) S_- + \lambda S_+ = 0, \qquad \vartheta \in (0,\pi), \end{aligned} \tag{3.11}$$

of differential equations for the components S_- and S_+ of Ψ.

Lemma 3.2. *Let $a = 0$. If $\Psi = \binom{S_-}{S_+}$ is an eigenfunction of $\mathcal{A}$, then neither of its components S_- nor S_+ vanishes identically. Further, $\lambda = 0$ is not an eigenvalue.*

Proof. Assume that one of the components $S_\pm$ vanishes identically. Then, by corollary 2.19, we have $|S_-(\frac{\pi}{2})| = |S_+(\frac{\pi}{2})| = 0$. The uniqueness theorem for solutions of linear differential operators implies that Ψ vanishes identically, in contradiction to our assumptions.
From the form of the fundamental system (2.16) of $\mathfrak{A}\Psi = 0$ it is clear that $\lambda = 0$ cannot be an eigenvalue of $\mathcal{A}$. $\square$

Note that $0 \notin \sigma(\mathcal{A})$ is also a direct consequence of the fact that the differential operators $\frac{\mathrm{d}}{\mathrm{d}\vartheta} \pm \frac{\kappa}{\sin\vartheta}$ are boundedly invertible. These differential operators are discussed in more detail in section 3.3.2.

Solving the first equation for S_- and inserting into the second equation yields

$$\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} - \frac{\kappa}{\sin\vartheta}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{\kappa}{\sin\vartheta}\right) S_+ + \lambda^2 S_+ = 0. \tag{3.12}$$

Evaluating this product and applying similar calculations for the function S_+ we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} + \frac{1}{\sin^2\vartheta}\left(-\kappa\cos\vartheta - \kappa^2\right) + \lambda^2\right] S_+ \;=\; 0, \tag{3.13 a}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} + \frac{1}{\sin^2\vartheta}\left(\kappa\cos\vartheta - \kappa^2\right) + \lambda^2\right] S_- \;=\; 0. \tag{3.13 b}$$

Note that the second differential equation can be obtained from the first differential equation if we substitute κ by $\kappa' = -\kappa$ (or, equivalently, k by $k' = -k-1$).

Lemma 3.3. *If $a = 0$ and $k \in \mathbb{R} \setminus (-1, 0)$, then the spectrum of $\mathcal{A}$ is given by*

$$\sigma_p(\mathcal{A}) \;=\; \left\{\lambda_n = \operatorname{sign}(n)\left((|k+\tfrac{1}{2}|) - \tfrac{1}{2} + |n|\right) \;:\; n \in \mathbb{Z} \setminus \{0\}\right\}. \tag{3.14}$$

The corresponding eigenfunctions are $\Psi_n := \begin{pmatrix} S_{-,n} \\ S_{+,n} \end{pmatrix}$ with

$$S_{-,n}(\vartheta) \;=\; s_{-,n}(1+\cos\vartheta)^{\beta}(1-\cos\vartheta)^{\alpha}\, F(-(|n|-1), 2(\alpha+\beta)+|n|-1;\, 2\beta+\tfrac{1}{2};\, \tfrac{1}{2}(1+\cos\vartheta)), \tag{3.15}$$

$$S_{+,n}(\vartheta) \;=\; s_{+,n}(1+\cos\vartheta)^{\alpha}(1-\cos\vartheta)^{\beta}\, F(-(|n|-1), 2(\alpha+\beta)+|n|-1;\, 2\alpha+\tfrac{1}{2};\, \tfrac{1}{2}(1+\cos\vartheta)), \tag{3.16}$$

with $\alpha = \frac{1}{2}|k| + \frac{1}{4}$ and $\beta = \frac{1}{2}|k+1| + \frac{1}{4}$ and the hypergeometric functions $F(\widetilde{a}, \widetilde{b}; \widetilde{c}; x)$, see also remark 3.4. Furthermore, we have

$$\frac{s_{+,n}}{s_{-,n}} = \operatorname{sign}(k+\tfrac{1}{2})\left(\frac{\lambda_n}{|k+\frac{1}{2}|+\frac{1}{2}}\right)^{\operatorname{sign}(k+\frac{1}{2})}.$$

Proof. Since $\kappa = k + \frac{1}{2}$ the condition $k \in \mathbb{R} \setminus (-1,\, 0)$ is equivalent to $|\kappa| \geq \frac{1}{2}$. To find eigenfunctions of $\mathcal{A}$, we must solve the system of differential equations (3.13 a) and (3.13 b) and then check that the solution also satisfies the coupling condition (3.11) and the integrability condition (3.10). We transform the independent variable ϑ according to

$$x \;=\; \tfrac{1}{2}(1+\cos\vartheta). \tag{3.17}$$

Short calculations show

$$\cos\vartheta \;=\; 2x-1, \qquad \sin\vartheta \;=\; 2\sqrt{x(1-x)},$$
$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} \;=\; -\sqrt{x(1-x)}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} \;=\; x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{2}(2x-1)\frac{\mathrm{d}}{\mathrm{d}x}.$$

Inserting into the differential equations (3.13 a) and (3.13 b) yields

$$\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{2}(2x-1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{4x(1-x)}\left(-2\kappa x - (\kappa-\tfrac{1}{2})^2 + \frac{1}{4}\right) + \lambda^2\right]\widetilde{f} \;=\; 0, \tag{3.18 a}$$

$$\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{2}(2x-1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{4x(1-x)}\left(2\kappa x - (\kappa+\tfrac{1}{2})^2 + \frac{1}{4}\right) + \lambda^2\right]\widetilde{g} \;=\; 0, \tag{3.18 b}$$

where $\widetilde{f}(x) := S_+(\vartheta(x))$ and $\widetilde{g}(x) := S_-(\vartheta(x))$ for all $x \in (0,1)$. It is easy to see that $\begin{pmatrix} S_- \\ S_+ \end{pmatrix}$ is an eigenfunction of $\mathcal{A}$ if and only if $(\widetilde{f}, \widetilde{g})$ are solutions of (3.18 a) and (3.18 b), coupled by

$$\begin{aligned} &\left(-\sqrt{x(1-x)}\,\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\kappa}{2\sqrt{x(1-x)}}\right)\widetilde{f}(x) - \lambda\,\widetilde{g}(x) = 0, \qquad x \in (0,1), \\ &\left(-\sqrt{x(1-x)}\,\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\kappa}{2\sqrt{x(1-x)}}\right)\widetilde{g}(x) + \lambda\,\widetilde{f}(x) = 0, \qquad x \in (0,1), \end{aligned} \tag{3.19}$$

and satisfy the integrability condition

$$\int_0^1 (|\widetilde{f}(x)|^2 + |\widetilde{g}(x)|^2)\sqrt{x(1-x)}^{\,-1}\mathrm{d}x < \infty. \tag{3.20}$$

If we also transform the dependent variables according to

$$\widetilde{f}(x) \;=:\; x^{\alpha}(1-x)^{\beta} f(x), \qquad \widetilde{g}(x) \;=:\; x^{\beta}(1-x)^{\alpha} g(x)$$

with

$$\alpha \;:=\; \tfrac{1}{2}|k| + \tfrac{1}{4} \;=\; \tfrac{1}{2}|\kappa - \tfrac{1}{2}| + \tfrac{1}{4}, \qquad \beta \;:=\; \tfrac{1}{2}|k+1| + \tfrac{1}{4} \;=\; \tfrac{1}{2}|\kappa + \tfrac{1}{2}| + \tfrac{1}{4}$$

we obtain the differential equations

$$\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2\alpha + \tfrac{1}{2} - (1+2\alpha+2\beta)x)\frac{\mathrm{d}}{\mathrm{d}x} + \lambda^2 - (\alpha+\beta)^2\right] f(x) \;=\; 0, \tag{3.21 a}$$

$$\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2\beta + \tfrac{1}{2} - (1+2\alpha+2\beta)x)\frac{\mathrm{d}}{\mathrm{d}x} + \lambda^2 - (\alpha+\beta)^2\right] g(x) \;=\; 0. \tag{3.21 b}$$

These are hypergeometric differential equations. Recall that the general hypergeometric differential equation is given by

$$\left\{x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\widetilde{c} - (1+\widetilde{a}+\widetilde{b})x)\frac{\mathrm{d}}{\mathrm{d}x} - \widetilde{a}\widetilde{b}\right\} w(x) \;=\; 0. \tag{3.22}$$

Comparison of equation (3.22) with (3.21 a) and (3.21 b), respectively, yields

$$\widetilde{a} + \widetilde{b} \;=\; 2(\alpha+\beta), \qquad \widetilde{c} \;=\; \begin{cases} \widetilde{c}_f \;=\; 2\alpha + \tfrac{1}{2} = |k| + 1 & \text{for (3.21 a)}, \\ \widetilde{c}_g \;=\; 2\beta + \tfrac{1}{2} = |k+1| + 1 & \text{for (3.21 b)}. \end{cases} \tag{3.23}$$

$$\lambda^2 \;=\; (\alpha+\beta)^2 - \widetilde{a}\widetilde{b}. \tag{3.24}$$

In particular,

$$|\lambda| \;=\; \tfrac{1}{2}|\widetilde{a} - \widetilde{b}|$$

and $\widetilde{c}_f = |k| + 1 \geq 1$ and $\widetilde{c}_g = |k+1| + 1 \geq 1$.
If the parameter $\widetilde{c}$ in (3.22) the not a negative integer, then the hypergeometric function

$$F(\widetilde{a}, \widetilde{b}; \widetilde{c}; x) = \sum_{n=0}^{\infty} \frac{(\widetilde{a})_n (\widetilde{b})_n}{(\widetilde{c})_n} \frac{x^n}{n!},$$

with Pochhammer's symbol

$$(r)_0 := 1, \qquad (r)_n := r(r+1)\ldots(r+n-1) \quad \text{for } n \in \mathbb{N} \text{ and } r \in \mathbb{R}$$

converges for $|z| < 1$ and is a solution of the differential equation (3.22). Its behaviour at the point 1 depends on $\delta := \mathrm{Re}(\widetilde{c} - \widetilde{a} - \widetilde{b})$; obviously, $F(\widetilde{a}, \widetilde{b}; \widetilde{c}; 0) = 1$, see [Ste84, sec 15.1].
Since in our case both $\widetilde{c}_f$ and $\widetilde{c}_g$ are positive numbers, the functions $f(x) := \eta_f F(\widetilde{a}, \widetilde{b}; \widetilde{c}_f; x)$ and $g(x) := \eta_g F(\widetilde{a}, \widetilde{b}; \widetilde{c}_g; x)$ with constants η_f and η_g are solutions of the equations (3.18 a) and (3.18 b); moreover, we show at the end of the proof that the numbers η_f and η_g can be chosen such that f and g also satisfy the coupling condition (3.19). Since $\alpha \geq \frac{1}{4}$ and $\beta \geq \frac{1}{4}$, it is follows that the corresponding functions $\widetilde{f}$ and $\widetilde{g}$ lie left in $\mathscr{L}^2((0,1), \sqrt{x(1-x)}^{-1}\mathrm{d}x)$, thus the corresponding wave function Ψ lies left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$.
Since the differential equations (3.21 a) and (3.21 b) are of second order, there are solutions $f_{(2)}$, $g_{(2)}$ independent of the hypergeometric functions, that might also lead to an eigenfunction $\Psi_{(2)}$ of the original eigenvalue equation $(\mathcal{A} - \lambda)\Psi = 0$. The wave function $\Psi_{(2)}$ is linearly independent of

Ψ (constructed from the hypergeometric functions) and therefore cannot lie left in $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ because $\mathcal{A}$ is in the limit point case at 0.
Thus we have shown that if f and g solve the differential equation (3.21 a) and (3.21 b) such that also the integrability condition for the corresponding $\widetilde{f}$ and $\widetilde{g}$ is satisfied, they must be proportional to hypergeometric functions.
Since we already know that none of the functions f and g vanishes identically, it follows that $\eta_f \neq 0$ and $\eta_g \neq 0$. Now we have to distinguish several cases.

Case 1. $\widetilde{a} = 0$.
In this case, solutions of the differential equations (3.21 a) and (3.21 b) are constant functions and (3.20) is satisfied. Hence,

$$|\lambda| \;=\; \tfrac{1}{2}|\widetilde{b}| \;=\; \alpha+\beta \;=\; \tfrac{1}{2}(|\kappa-\tfrac{1}{2}|+|\kappa+\tfrac{1}{2}|+1) \;=\; \begin{cases} \kappa+\frac{1}{2} & \text{if } \kappa \geq \frac{1}{2}, \\ -\kappa+\frac{1}{2} & \text{if } \kappa \leq -\frac{1}{2} \end{cases}$$

is an eigenvalue of $\mathcal{A}$.

Case 2. $\widetilde{a} = -n$ *with* $n \in \{1,2,3,\dots\}$.
In this case, the hypergeometric function $F(\widetilde{a},\widetilde{b};\,\widetilde{c};\,x)$ reduces to a polynomial and therefore is finite at $x=1$. From (3.23) it follows that $\widetilde{b} = 2(\alpha+\beta)+n$ and hence

$$|\lambda| \;=\; \alpha+\beta+n \;=\; \tfrac{1}{2}(|\kappa-\tfrac{1}{2}|+|\kappa+\tfrac{1}{2}|+1)+n \;=\; \begin{cases} \kappa+\frac{1}{2}+n & \text{if } \kappa \geq \frac{1}{2}, \\ -\kappa+\frac{1}{2}+n & \text{if } \kappa \leq -\frac{1}{2} \end{cases}$$

is an eigenvalue of $\mathcal{A}$.

Case 3. $\widetilde{b} = -n$ *with* $n \in \{0,1,2,\dots\}$.
This is analogous to the previous cases since the differential equation (3.22) is symmetric in $\widetilde{a}$ and $\widetilde{b}$.

It is clear that in the cases 1, 2 and 3 the functions $\widetilde{f}$ and $\widetilde{g}$ fulfil the integrability condition (3.20) because $\alpha,\ \beta \geq \frac{1}{4}$ and the hypergeometric functions are polynomials so that the integrand in (3.20) is bounded.

Case 4. $\widetilde{a},\ \widetilde{b} \notin \{0,-1,-2,\dots\}$.
We show that in this case the vector function $(\widetilde{f},\ \widetilde{g})$ is not an eigenfunction of the system (3.18 a), (3.18 b). As already mentioned, the behaviour of $F(\widetilde{a},\widetilde{b};\,\widetilde{c};\,x)$ at $x=1$ is determined by the value of

$$\delta = \mathrm{Re}(\widetilde{c}-\widetilde{a}-\widetilde{b}) = \mathrm{Re}(\widetilde{c}-2(\alpha+\beta)),$$

see [Ste84, 15.1.1]. If $\delta \leq -1$, then $F(\widetilde{a},\widetilde{b};\,\widetilde{c};\,x)$ diverges at $x=1$. In our case we always have either $\delta_f = \widetilde{c}_f - 2(\alpha+\beta) \leq -1$ or $\delta_g = \widetilde{c}_g - 2(\alpha+\beta) \leq -1$.
Suppose that $\binom{S_-}{S_+} = \binom{\widetilde{g}(x(\cdot))}{\widetilde{f}(x(\cdot))}$ is an eigenfunction of $\mathcal{A}$. Without restriction we may assume that $\widetilde{c}_f - 2(\alpha+\beta) \leq -1$ so that $f = F(\widetilde{a},\widetilde{b};\,\widetilde{c}_f;\,\cdot\,)$ diverges at $x=1$.
By lemma 2.17 (iii), also $\binom{S_+(\pi-\cdot)}{S_-(\pi-\cdot)}$ is an eigenfunction of $\mathcal{A}$ with the same eigenvalue. This implies that there is constant $\gamma \neq 0$ such that $\widetilde{f}(x) \;=\; \gamma\,\widetilde{g}(1-x)$, and hence

$$\eta_f F(\widetilde{a},\widetilde{b};\,\widetilde{c}_f;\,x) \;=\; \gamma\,\eta_g F(\widetilde{a},\widetilde{b};\,\widetilde{c}_g;\,1-x)$$

for all $x \in (0,1)$. For $x \to 1$ the function $F(\widetilde{a}, \widetilde{b};\, \widetilde{c}_g;\, 1-x)$ remains bounded whereas the function $F(\widetilde{a}, \widetilde{b};\, \widetilde{c}_f;\, x)$ diverges. Therefore, $\begin{pmatrix} S_- \\ S_+ \end{pmatrix}$ cannot be an eigenfunction of $\mathcal{A}$.

Summarising the cases 1 to 4, we find that for $k \in \mathbb{R}\setminus(-1,0)$ and $a = 0$, a number λ is an eigenvalue of $\mathcal{A}$ if and only if

$$|\lambda| \;\in\; \{\alpha + \beta + n \,:\, n \in \mathbb{N}_0\} \;=\; \{\tfrac{1}{2}(|\kappa - \tfrac{1}{2}| + |\kappa + \tfrac{1}{2}| + 1) + n \,:\, n \in \mathbb{N}_0\}.$$

Since by assumption $|\kappa| \geq \frac{1}{2}$ we can simplify $|\kappa + \frac{1}{2}| + |\kappa - \frac{1}{2}| = 2|\kappa| = |2k+1|$. From corollary 2.18 we know that the spectrum of $\mathcal{A}$ is symmetric to 0 in the case $a = 0$, so it follows that λ is an eigenvalue of $\mathcal{A}$ if and only if

$$\lambda \;\in\; \{\pm(|k + \tfrac{1}{2}| + \tfrac{1}{2} + n) \,:\, n \in \mathbb{N}_0\} \;=\; \{\operatorname{sign}(n)(|k + \tfrac{1}{2}| - \tfrac{1}{2} + |n|) \,:\, n \in \mathbb{Z} \setminus \{0\}\}.$$

For $n \in \mathbb{Z} \setminus \{0\}$, let $\lambda_n := \operatorname{sign}(n)(|k + \frac{1}{2}| - \frac{1}{2} + |n|)$. It remains to determine the ratio of η_f and η_g for fixed $n \in \mathbb{Z} \setminus \{0\}$. From the differential equations for S_- and S_+ and the ansatz $S_+(\vartheta(x)) = x^\alpha (1-x)^\beta f(x)$ and $S_-(\vartheta(x)) = x^\beta (1-x)^\alpha g(x)$ we obtain

$$\lambda_n\, x^{\beta-\alpha+\frac{1}{2}} (1-x)^{\alpha-\beta+\frac{1}{2}}\, g(x) \;=\; -x(1-x)f'(x) + \big(-\alpha + \tfrac{1}{2}(k + \tfrac{1}{2}) + x(\alpha+\beta)\big)\, f(x).$$

With $f(x) = \eta_f\, F(\widetilde{a}, \widetilde{b};\, \widetilde{c};\, x)$ a straightforward evaluation of the right hand side yields

$$\lambda_n g(x) \;=\; \begin{cases} \eta_f\, \frac{(k+n+1)^2}{k+1}\, F(\widetilde{a}, \widetilde{b};\, \widetilde{c}_g\,;x) & \text{if } k \geq 0, \\ \eta_f\, k\, F(\widetilde{a}, \widetilde{b};\, \widetilde{c}_g;\, x) & \text{if } k \leq -1, \end{cases}$$

hence, using $\lambda_n^2 = (\alpha + \beta + n)^2 = (|k + \frac{1}{2}| + n + \frac{1}{2})^2$, we find in both cases

$$\eta_g = \operatorname{sign}(k + \tfrac{1}{2}) \left(\frac{\lambda_n}{|k + \frac{1}{2}| + \frac{1}{2}} \right)^{\operatorname{sign}(k+\frac{1}{2})} \eta_f.$$

The above calculation also shows that the coupling condition (3.19) is satisfied. $\square$

Remark 3.4. The polynomials $F(\widetilde{a}, \widetilde{b}; \widetilde{c}, x)$ with $\widetilde{a} \in -\mathbb{N}$ are the so-called *Jacobi polynomials.* $\diamondsuit$

A comparison of the sets (3.8) and (3.14) shows that the spectra of $\mathcal{A}$ and $\mathfrak{K}$ coincide, as is clear from the relation $\mathfrak{A}^{(d)} = U^{-1}\,\mathfrak{K}\,U$, see (3.6). The quantity $k + \frac{1}{2}$ originating from the separation ansatz (2.7) can be identified with the z-component of the total angular momentum of the fermion. From (3.7) it follows that if $\lambda = \widetilde{\kappa}$ is positive, then $j = l + \frac{1}{2}$ which implies that the spin and the angular momentum of the upper component of the fermion are parallel; if λ is negative, then we have $j = l - \frac{1}{2}$, i.e., the spin and the angular momentum of the upper component of the fermion are antiparallel.
Another way to see that λ should be interpreted as the parameter describing the spin-orbit coupling, without using the angular equation, is to consider the radial equation. For $a = M = Q = 0$ the radial equation $(\mathfrak{R}^{(d)} - \lambda) \begin{pmatrix} X_- \\ X_+ \end{pmatrix} = 0$, see (2.8), reduces to

$$\begin{pmatrix} -\mathrm{i}\, m - \frac{\lambda}{r} & \frac{\mathrm{d}}{\mathrm{d}r} - \mathrm{i}\,\omega \\ \frac{\mathrm{d}}{\mathrm{d}r} + \mathrm{i}\,\omega & \mathrm{i}\, mr - \frac{\lambda}{r} \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \;=\; 0.$$

A transformation with the unitary matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}$ yields

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}r} - \frac{\lambda}{r} & \omega - m \\ \omega + m & \frac{\mathrm{d}}{\mathrm{d}r} - \frac{\lambda}{r} \end{pmatrix} \begin{pmatrix} X_+ + X_- \\ \mathrm{i}\,(X_+ - X_-) \end{pmatrix} \;=\; 0,$$

which is equivalent to the radial part of the Dirac equation in flat spacetime without potential, where λ is the eigenvalue of the spin-orbit coupling operator $\widehat{\mathfrak{K}}$, see, e.g., [Lan96] or [Sch99].

3.2 Analytic perturbation theory

In the previous section we have computed the spectrum of the angular operator $\mathcal{A}$ in the case $a = 0$. Now we apply analytic perturbation theory to derive some basic estimates for the eigenvalues of $\mathcal{A}$ in the case $a \neq 0$ from the results in the previous section. We also give a first order approximation in terms of the Kerr parameter a for the first positive and first negative eigenvalue in a neighbourhood of $a = 0$.
To indicate the dependence of the angular operator $\mathcal{A}$ and its eigenvalues λ_n on the parameter a, we use the notation $\mathcal{A}(a)$ and $\lambda_n(a)$.

In general, perturbation theory deals with operator valued functions $\mathcal{T} : U \to \mathscr{C}(X, Y)$ defined on an open set $U \subseteq \mathbb{C}$ with values in the set of closed operators from the Banach space X to the Banach space Y. Sometimes, we also use the notation of a family of closed operators $(\mathcal{T}(\zeta))_{\zeta\in U}$. Without loss of generality, we assume that $0 \in U$. Usually the spectrum of $\mathcal{T}(0)$ is known or can be approximated. If the operators $\mathcal{T}(0)$ and $\mathcal{T}(\zeta)$ differ only slightly in an appropriate sense for $|\zeta|$ small enough, then knowledge of the spectrum of the unperturbed operator $\mathcal{T}(0)$ leads to information about the spectrum of $\mathcal{T}(\zeta)$.

For the purpose of this work it is sufficient to consider holomorphic families of operators only; for the following definitions and properties of holomorphic families we refer to [Kat80, chap. VII].

Definition 3.5. Let U be a domain in $\mathbb{C}$ and let X and Y be Banach spaces.

(i) A family of bounded operators $(\mathcal{T}(\zeta))_{\zeta\in U}$ from X to Y is called *(bounded-)holomorphic* if it is holomorphic in norm in U.

(ii) Let $\mathcal{T} = (\mathcal{T}(\zeta))_{\zeta\in U}$ be a family of closed operators from X to Y such that each $\mathcal{T}(\zeta)$ has nonempty resolvent set. Let $\zeta_0 \in U$ and $\lambda \in \rho(\mathcal{T}(\zeta_0))$. Then the family $\mathcal{T}$ is called *holomorphic at* ζ_0 if there is a neighbourhood U_0 of ζ_0 in U such that $\lambda \in \rho(\mathcal{T}(\zeta))$ for all $\zeta \in U_0$ and the family of the resolvents $((\mathcal{T}(\lambda) - \zeta)^{-1})_{\zeta\in U_0}$ is bounded-holomorphic. The family $\mathcal{T}$ is called *holomorphic in* U it it is holomorphic at each $\zeta_0 \in U$.

Definition 3.6. Let $U \subseteq \mathbb{C}$ be some domain. A family of densely defined, closed operators $(T(\zeta))_{\zeta\in U}$ is called a *selfadjoint family* if

$$T(\zeta)^* = T(\overline{\zeta}), \qquad \zeta \in U.$$

Definition 3.7. Let U be a domain in $\mathbb{C}$ and let X and Y be Banach spaces. A family of closed operators $(\mathcal{T}(\zeta))_{\zeta\in U} \subseteq \mathscr{C}(X, Y)$ is called a *holomorphic family of type* (A) if

(i) $\mathcal{D}(\mathcal{T}(\zeta)) = \mathcal{D}$ is independent of ζ,

(ii) for all $u \in U$ the vector valued function $\zeta \mapsto T(\zeta)u$ is holomorphic for $\zeta \in U$.

It can be shown that holomorphic families of type (A) are holomorphic families. Holomorphic families of type (B) are defined in section 4.1.
Obviously, the function

$$\mathbb{C} \longrightarrow \mathscr{C}(\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2), \quad a \mapsto \mathcal{A}(a) = \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} & 0 \end{pmatrix} + a \begin{pmatrix} -m\cos\vartheta & \omega\sin\vartheta \\ \omega\sin\vartheta & m\cos\vartheta \end{pmatrix}$$

defines the holomorphic family of type (A) $(\mathcal{A}(a))_{a\in\mathbb{C}}$. We showed earlier in lemma 3.3 that the spectrum of $\mathcal{A}(0)$ is given by

$$\sigma_p(\mathcal{A}(0)) = \left\{\lambda_n(0) = \operatorname{sign}(n) \left((|k + \tfrac{1}{2}|) - \tfrac{1}{2} + |n|\right) : n \in \mathbb{Z} \setminus \{0\}\right\}.$$

Since for real a the differential expression $\mathfrak{A}(a)$ is in the limit point case at both endpoints of the interval $(0, \pi)$, all eigenvalues of $\mathcal{A}(a)$ are simple. From analytic perturbation theory it follows that the eigenfunctions and eigenvalues of $\mathcal{A}$ depend analytically on a. Consequently, if $\lambda_m(a_0) < \lambda_n(a_0)$ for some $a_0 \in \mathbb{R}$, then also $\lambda_m(a) < \lambda_n(a)$ for all $a \in \mathbb{R}$.

Remark 3.8. The first derivative of $\mathcal{A}$ with respect to a is the bounded operator

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a) = \begin{pmatrix} -m\cos\vartheta & \omega\sin\vartheta \\ \omega\sin\vartheta & m\cos\vartheta \end{pmatrix}, \qquad a \in \mathbb{C}, \tag{3.25}$$

and we have $\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\| = \max\{|m|,\, |\omega|\}$, $a \in \mathbb{C}$.

Proof. Formula (3.25), and consequently the boundedness of $\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\|$ is obvious. The assertion concerning the bound of $\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\|$, we note that for arbitrary $\Psi = \binom{\Psi_1}{\Psi_2} \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2$

$$\begin{aligned} \Big\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\Psi\Big\|_2^2 &= \int_0^\pi \big(m^2\cos^2\vartheta + \omega^2\sin^2\vartheta\big)\big(|\Psi_1(\vartheta)|^2 + |\Psi_2(\vartheta)|^2\big)\,\mathrm{d}\vartheta \\ &\le \max\{m^2\cos^2\vartheta + \omega^2\sin^2\vartheta \,:\, \vartheta \in (0,\pi)\}\,\|\Psi\|_2^2 \;=\; \max\{m^2,\ \omega^2\}\,\|\Psi\|_2^2, \end{aligned} \tag{3.26}$$

hence $\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\| \le \max\{|m|,\, |\omega|\}$. Let T be the multiplication operator $\sqrt{m^2\cos^2\vartheta + \omega^2\sin^2(\vartheta)}$ on the Hilbert space $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. Then (3.26) implies that $\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\| \ge \|T\| = \max\{|m|,\, |\omega|\}$. $\square$

Lemma 3.9. *Let $\lambda_n(0)$, $n \in \mathbb{Z}\setminus\{0\}$, be the nth eigenvalue of $\mathcal{A}(0)$. Then for the eigenvalue $\lambda_n(a)$ of $\mathcal{A}(a)$ the following estimate holds:*

$$|\lambda_n(a) - \lambda_n(0)| \le \max\{|m|,\, |\omega|\}. \tag{3.27}$$

Proof. Let Ψ_n^a be a normalised eigenfunction of $\mathcal{A}(a)$ with eigenvalue $\lambda_n(a)$. The index n enumerates the eigenvalues, the argument a denotes the dependence of the eigenvalue on the parameter a. By [Kat80, chap. VII, §3.4], the derivative of λ_n with respect to a is given by

$$\frac{\mathrm{d}\lambda_n}{\mathrm{d}a}(a) \;=\; \Big(\Psi_n^a,\ \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\,\Psi_n^a\Big). \tag{3.28}$$

Since $\|\Psi_n^a\| = 1$ by assumption, the previous remark yields

$$\Big|\frac{\mathrm{d}\lambda_n}{\mathrm{d}a}(a)\Big| \;\le\; \Big\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\ \Psi_n^a\Big\| \;\le\; \Big\|\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(a)\ \Big\| \;\le\; \max\{|m|,\, |\omega|\}.$$

Application of the mean value theorem to the continuous function λ_n leads to

$$|\lambda_n(0) - \lambda_n(a)| \;\le\; |a|\ \max\Big\{\frac{\mathrm{d}\lambda_n}{\mathrm{d}a}(\tilde{a}) \;:\; 0 \le \tilde{a} \le a\Big\} \;=\; |a|\max\{|m|,\, |\omega|\}. \qquad \square$$

Equivalent to (3.27) is

$$\lambda_n(0) - \max\{|m|,\, |\omega|\} \;\le\; \lambda_n(a) \;\le\; \lambda_n(0) + \max\{|m|,\, |\omega|\},$$

or, using the explicit formula (3.14) for the eigenvalues $\lambda_n(0)$,

$$\begin{aligned} &\operatorname{sign}(n)(|k+\tfrac12| - \tfrac12 + |n|) - |a|\max\{|m|,\, |\omega|\} \\ &\qquad\qquad \le\; \lambda_n(a) \;\le\; \operatorname{sign}(n)(|k+\tfrac12| - \tfrac12 + |n|) + |a|\max\{|m|,\, |\omega|\}, \end{aligned}$$

Corollary 3.10. *The set of eigenvalues of $\mathcal{A}(a)$ is neither bounded from below nor from above.*

Proof. The assertion holds for $\mathcal{A}(0)$ as shown in lemma 3.3. Since the derivative of the eigenvalues with respect to a is bounded, the assertion follows also for all $\mathcal{A}(a)$. □

Now we use equation (3.28) to give a first order approximation of the eigenvalues λ_n in a neighbourhood of $a = 0$. We consider the case $|n| = 1$ only, since this case can be treated fairly well analytically; for higher eigenvalues the computations become rather involved.

Lemma 3.11. *Up to first order in a, the eigenvalues $\lambda_{\pm 1}$ have the asymptotics*

$$\begin{aligned}
\lambda_1(a) &= \ \ |k+\tfrac{1}{2}| + \tfrac{1}{2} + \frac{\operatorname{sign}(k+\frac{1}{2})}{2}\,\frac{|k+\frac{1}{2}|+\frac{1}{2}}{|k+\frac{1}{2}|+1}\left(\ \ 2\omega - \frac{m}{|k+\frac{1}{2}|+\frac{1}{2}}\right) a + \mathcal{O}(a^2),\\
\lambda_{-1}(a) &= -|k+\tfrac{1}{2}| - \tfrac{1}{2} + \frac{\operatorname{sign}(k+\frac{1}{2})}{2}\,\frac{|k+\frac{1}{2}|+\frac{1}{2}}{|k+\frac{1}{2}|+1}\left(-2\omega - \frac{m}{|k+\frac{1}{2}|+\frac{1}{2}}\right) a + \mathcal{O}(a^2),
\end{aligned}$$

where the Landau symbol $\mathcal{O}(a^2)$ denotes a function such that $\lim\limits_{a\to 0}\frac{\mathcal{O}(a^2)}{a^2}$ is bounded.

Proof. It follows from analytic perturbation theory that the eigenvalues $\lambda_{\pm 1}$ are analytic functions with respect to a. We have $\lambda_{\pm 1}(0) = \pm(|k+\frac{1}{2}|+\frac{1}{2})$ by (3.14), so it remains to calculate the first derivative of $\lambda_{\pm 1}(0)$. By lemma 3.3, the eigenfunctions of the angular operator $\mathcal{A}$ for $a = 0$ are $\Psi_n^0 = \binom{S_{-,n}}{S_{+,n}}$ with $S_{\mp,n}$ defined in (3.15) and (3.16). According to (3.28), the derivative of the eigenvalue with respect to a at $a = 0$ is given by

$$\begin{aligned}
\frac{\mathrm{d}\lambda_n}{\mathrm{d}a}(0) &= \left(\Psi_n^0,\ \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(0)\Psi_n^0\right) = \left(\begin{pmatrix} S_{-,n}\\ S_{+,n}\end{pmatrix},\ \begin{pmatrix} -m\cos\vartheta & \omega\sin\vartheta\\ \omega\sin\vartheta & m\cos\vartheta\end{pmatrix}\begin{pmatrix} S_{-,n}\\ S_{+,n}\end{pmatrix}\right)\\
&= m\int_0^\pi \cos\vartheta\,\left(-S_{-,n}(\vartheta)^2 + S_{+,n}(\vartheta)^2\right)\mathrm{d}\vartheta + 2\omega\int_0^\pi \sin\vartheta\, S_{-,n}(\vartheta)\, S_{+,n}(\vartheta)\,\mathrm{d}\vartheta.
\end{aligned}$$

In the special case $|n| = 1$, the hypergeometric functions appearing in the formulae for $S_{\pm,n}$ are constant functions, identical to 1, and we have $s_{-,n}/s_{+,n} = n\operatorname{sign}(k+\frac{1}{2})$, implying $|s_{-,n}| = |s_{-,-n}|$. The numerical value of $|s_{-,1}|$ is derived below. With the transformation $x = \frac{1}{2}(1+\cos\vartheta)$ the first integral becomes

$$\begin{aligned}
&\int_0^\pi \cos\vartheta\,\left(-S_{-,n}(\vartheta)^2 + S_{+,n}(\vartheta)^2\right)\mathrm{d}\vartheta\\
&\qquad = 2^{2\alpha+2\beta}\, s_{-,1}^2 \int_0^1 (2x-1)\left(-x^{2\beta}(1-x)^{2\alpha} + x^{2\alpha}(1-x)^{2\beta}\right)\left(x(1-x)\right)^{-\frac{1}{2}}\,\mathrm{d}x\\
&\qquad = 2^{|2k+1|+1}\, s_{-,1}^2 \int_0^1 (2x-1)\left(-x^{|k+1|}(1-x)^{|k|} + x^{|k|}(1-x)^{|k+1|}\right)\mathrm{d}x.
\end{aligned}$$

For $k \geq 0$, integration by parts yields

$$\int_0^\pi \cos\vartheta \left(-S_{-,n}(\vartheta)^2 + S_{+,n}(\vartheta)^2\right)\mathrm{d}\vartheta \;=\; 2^{|2k+1|+1}\, s_{-,1}^2 \int_0^1 (2x-1)\,(x(1-x))^k\,(1-2x)\mathrm{d}x$$

$$= \frac{2^{|2k+1|+1}\, s_{-,1}^2}{k+1}\int_0^1 (2x-1)\,((x(1-x))^{k+1})'\,\mathrm{d}x \;=\; -\frac{2^{|2k+1|+2}\, s_{-,1}^2}{k+1}\int_0^1 (x(1-x))^{k+1}\,\mathrm{d}x.$$

For $k \leq -1$ we obtain by an analogous calculation

$$\int_0^\pi \cos\vartheta \left(-S_{-,n}(\vartheta)^2 + S_{+,n}(\vartheta)^2\right)\mathrm{d}\vartheta \;=\; -\frac{2^{|2k+1|+2}\, s_{-,1}^2}{|k|}\int_0^1 (x(1-x))^{-k}\,\mathrm{d}x.$$

Using $-k = -(k+\frac{1}{2}) + \frac{1}{2} = |k+\frac{1}{2}| + \frac{1}{2}$ for $k \leq -1$, we can summarise both cases in the formula

$$\int_0^\pi \cos\vartheta \left(-S_{-,n}(\vartheta)^2 + S_{+,n}(\vartheta)^2\right)\mathrm{d}\vartheta \;=\; -\frac{2^{|2k+1|+2}\ \operatorname{sign}(k+\frac{1}{2})\, s_{-,1}^2}{|k+\frac{1}{2}|+\frac{1}{2}}\int_0^1 (x(1-x))^{|k+\frac{1}{2}|+\frac{1}{2}}\,\mathrm{d}x.$$

Similarly, we can compute the second integral; the result is

$$2\int_0^\pi \sin\vartheta\, S_{-,n}(\vartheta)\, S_{+,n}(\vartheta)\mathrm{d}\vartheta \;=\; 2^{|2k+1|+3}\, n\operatorname{sign}(k+\tfrac{1}{2})\, s_{-,1}^2 \int_0^1 x^{\alpha+\beta}(1-x)^{\alpha+\beta}\mathrm{d}x$$

$$= 2^{|2k+1|+3}\, n\operatorname{sign}(k+\tfrac{1}{2})\, s_{-,1}^2 \int_0^1 (x(1-x))^{|k+\frac{1}{2}|+\frac{1}{2}}\,\mathrm{d}x.$$

Hence we obtain for the derivative of λ_n, $n = \pm 1$, at $a = 0$

$$\frac{\mathrm{d}}{\mathrm{d}a}\lambda_n(0) \;=\; \left(\Psi_n^0,\ \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a}(0)\Psi_n^0\right)$$

$$= \left(2n\,\omega - \frac{m}{|k+\frac{1}{2}|+\frac{1}{2}}\right) 2^{|2k+1|+2}\ \operatorname{sign}(k+\tfrac{1}{2})\, s_{-,1}^2 \int_0^1 (x(1-x))^{|k+\frac{1}{2}|+\frac{1}{2}}\,\mathrm{d}x$$

$$= \left(2n\,\omega - \frac{m}{|k+\frac{1}{2}|+\frac{1}{2}}\right) 2^{|2k+1|+2}\ \operatorname{sign}(k+\tfrac{1}{2})\, s_{-,1}^2\, \frac{2^{-2-|2k+1|}\sqrt{\pi}\ \Gamma(|k+\frac{1}{2}|+\frac{3}{2})}{\Gamma(|k+\frac{1}{2}|+2)}.$$

The constant $s_{-,1}$ is defined be the requirement that Ψ_1^0 be normalised, that is,

$$1 \;=\; \int_0^\pi S_{-,1}(\vartheta)^2 + S_{+,1}(\vartheta)^2\,\mathrm{d}\vartheta \;=\; 2^{|2k+1|+1}\, s_{-,1}^2 \int_0^1 (x(1-x))^{|k+\frac{1}{2}|-\frac{1}{2}}\mathrm{d}x$$

$$= 2^{|2k+1|+1}\, s_{-,1}^2\, \frac{2^{-|2k+1|}\sqrt{\pi}\ \Gamma(|k+\frac{1}{2}|+\frac{1}{2})}{\Gamma(|k+\frac{1}{2}|+1)}.$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}a}\lambda_n(0) \;=\; \frac{\operatorname{sign}(k+\frac{1}{2})}{2}\left(2n\,\omega - \frac{m}{|k+\frac{1}{2}|+\frac{1}{2}}\right)\frac{\Gamma(|k+\frac{1}{2}|+1)}{\Gamma(|k+\frac{1}{2}|+\frac{1}{2})}\ \frac{\Gamma(|k+\frac{1}{2}|+\frac{3}{2})}{\Gamma(|k+\frac{1}{2}|+2)}$$

$$= \frac{\operatorname{sign}(k+\frac{1}{2})}{2}\left(2n\,\omega - \frac{m}{|k+\frac{1}{2}|+\frac{1}{2}}\right)\frac{|k+\frac{1}{2}|+\frac{1}{2}}{|k+\frac{1}{2}|+1}. \qquad \square$$

In general, the slope of λ at $a = 0$ is the steeper the larger m or ω are, unless they are somehow balanced, that is, $m \approx \operatorname{sign}(n)(|k+\frac{1}{2}|+\frac{1}{2})\omega$. Further it should be noted that for large values of $|k+\frac{1}{2}|$ the first order behaviour of the eigenvalues is governed by ω.

Remark 3.12. There are other ways to apply analytic perturbation theory to the angular operator $\mathcal{A}$. For example, we can treat m or ω as the perturbation parameter while a is fixed. We can even consider $\mathcal{A}$ a family of operators depending on two parameters, say m and ω.
If, for instance, we want to use m as perturbation parameter, we have to compute or at least approximate the eigenvalues of $\mathcal{B} = \left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$. Since $\mathcal{B}$ is the angular operator in the special case $m = 0$, the spectrum of $\mathcal{B}$ consists of simple isolated eigenvalues only. Let $\ldots < \mu_{-1} < 0 < \mu_1 < \ldots$ be the eigenvalues of $\mathcal{B}$; then perturbation theory yields for the eigenvalues λ_n of $\mathcal{A}$, now depending on m,

$$\mu_n - |am| \;\leq\; \lambda_n(m) \;\leq\; \mu_n + |am|, \qquad n \in \mathbb{Z} \setminus \{0\}.$$

Estimates for the μ_n are derived in theorem 4.39 with the help of Sturm's comparison theorem applied to the operator BB^*. $\lozenge$

3.3 An off-diagonalisation of certain block operator matrices

Our aim is to show that certain 2×2 block operator matrices allow for a factorisation if the off-diagonal elements are closed and boundedly invertible. This factorisation is then applied to the angular operator $\mathcal{A}$ to derive a lower bound for the modulus of its eigenvalues.

3.3.1 The general case

In this section we consider block operator matrices on a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $\mathcal{H}$ is endowed with the usual scalar product induced by the scalar products of $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $B(\mathcal{H}_2 \to \mathcal{H}_1)$ and $C(\mathcal{H}_1 \to \mathcal{H}_2)$ be closed linear operators. Further, assume that $A(\mathcal{H}_1 \to \mathcal{H}_1)$ is a C-bounded and $D(\mathcal{H}_2 \to \mathcal{H}_2)$ is a B-bounded linear operator, that is, we have the inclusions

$$\mathcal{D}(C) \subseteq \mathcal{D}(A), \qquad \mathcal{D}(B) \subseteq \mathcal{D}(D)$$

and there exist real numbers $\alpha, \gamma, \beta, \delta \geq 0$ such that

$$\begin{aligned} \|Ax\| &\leq \alpha\|x\| + \gamma\|Cx\|, \qquad x \in \mathcal{D}(C), \\ \|Dx\| &\leq \delta\|x\| + \beta\|Bx\|, \qquad x \in \mathcal{D}(B). \end{aligned}$$

Then the block operator matrix

$$\mathcal{T} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathcal{D}(\mathcal{T}) = \mathcal{D}(C) \oplus \mathcal{D}(B),$$

is a well defined operator in $\mathcal{H}$. Note, however, that $\mathcal{T}$ is not necessarily closed. But if we strengthen the assumptions on A and D, then the following lemma implies the closedness of $\mathcal{T}$.

Lemma 3.13. *Assume that $B(\mathcal{H}_2 \to \mathcal{H}_1)$ and $C(\mathcal{H}_1 \to \mathcal{H}_2)$ are closed linear operators. Furthermore, let $A(\mathcal{H}_1 \to \mathcal{H}_1)$ and $D(\mathcal{H}_2 \to \mathcal{H}_2)$ be bounded linear operators. Then the operator $\mathcal{T}$ defined above is closed.*

Proof. We can show the assertion directly. Let $\left(\binom{x_n}{y_n}\right)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathcal{T})$ be an $\mathcal{T}$-convergent sequence, that is, there are $\binom{x}{y}, \binom{f}{g} \in \mathcal{H}$ such that $\binom{x_n}{y_n} \to \binom{x}{y}$ and $\mathcal{T}\binom{x_n}{y_n} \to \binom{f}{g}$ for $n \to \infty$. Obviously, we also have $Ax_n \to Ax$ and $Dy_n \to Dy$ because A and D are bounded. Thus we can conclude

$$\begin{matrix} Ax_n + By_n \to f \\ Cx_n + Dy_n \to g \end{matrix} \quad \Longrightarrow \quad \begin{matrix} By_n \to f - Ax \\ Cx_n \to g - Dy \end{matrix} \quad \Longrightarrow \quad \begin{matrix} y \in \mathcal{D}(B),\ By = f - Ax \\ x \in \mathcal{D}(C),\ Cx = g - Dy, \end{matrix}$$

since B and C are closed. Hence $\binom{x}{y} \in \mathcal{D}(\mathcal{T})$ and $\mathcal{T}\binom{x}{y} = \binom{f}{g}$ holds. □

Remark 3.14. In lemma 3.13 it would suffice to assume that only one of the operators A or D is bounded. For example, let A be bounded and assume that D is closed and B-bounded. Then it follows as above that $y \in \mathcal{D}(B) \subseteq \mathcal{D}(D)$. Then, also as above, we have $x \in \mathcal{D}(C)$. ◊

Remark 3.15. A more elegant proof of lemma 3.13 makes use of a stability theorem. If A and D are bounded, then also $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ is bounded. Since $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ is closed, the perturbation theorem [Kat80, chap. IV, theorem 1.1] implies that also their sum is closed. ◊

In the spectral theory of operator matrices the so-called *Schur factorisation* plays an important role (see, e.g., [Nag89], [ALMS94], [ALMS96]). For $\lambda \in \rho(A)$ or $\lambda \in \rho(D)$ we have, at least formally, the following factorisations

$$\mathcal{T} - \lambda = \begin{pmatrix} I & 0 \\ C(A-\lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} A-\lambda & 0 \\ 0 & S_D(\lambda) \end{pmatrix} \begin{pmatrix} I & (A-\lambda)^{-1}B \\ 0 & I \end{pmatrix}, \qquad \lambda \in \rho(A), \tag{3.29}$$

$$\mathcal{T} - \lambda = \begin{pmatrix} I & B(D-\lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_A(\lambda) & 0 \\ 0 & D-\lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (D-\lambda)^{-1}C & I \end{pmatrix}, \qquad \lambda \in \rho(D), \tag{3.30}$$

with the operator valued functions

$$\begin{aligned} S_A(\lambda) &= A - \lambda - B(D-\lambda)^{-1}C, && \lambda \in \rho(D), \\ S_D(\lambda) &= D - \lambda - C(A-\lambda)^{-1}B, && \lambda \in \rho(A). \end{aligned}$$

The functions S_A and S_D are called the *Schur complements* of the matrix $\mathcal{T}$. Usually, the domains of the operators $S_A(\lambda)$ and $S_D(\lambda)$ are taken to be their natural domains, for example, for $\lambda \in \rho(D)$, it is natural to define $\mathcal{D}(S_A(\lambda)) := \{x \in \mathcal{H}_1 : x \in \mathcal{D}(A) \cap \mathcal{D}(C),\ (D-\lambda)^{-1}Cx \in \mathcal{D}(B)\}$; note that in general these domains depend on the parameter λ.
The factorisations (3.29) and (3.30) can be used to characterise the spectrum of $\mathcal{T}$; for instance, $\lambda \in \rho(\mathcal{T}) \cap \rho(D)$ if and only if 0 lies in the resolvent set of $S_A(\lambda)$. Note, however, that the Schur factorisation gives no results for $\lambda \in \sigma(A) \cap \sigma(D)$. Roughly speaking, the Schur factorisation is obtained if the linear systems $(\mathcal{T} - \lambda)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 0$ is decoupled by using the fact that either $A - \lambda$ or $D - \lambda$ is invertible. We will use the Schur complements later to obtain lower and upper bounds for the eigenvalues of the angular operator $\mathcal{A}$, see section 4.2 and appendix B.
On the other hand, if we know that B and C are invertible, then it is also possible to decouple the equation $(\mathcal{T} - \lambda)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 0$ by inverting B and C. This results in the off-diagonalisation stated in the next lemma. Note that we need not assume that both A and D are bounded.

Lemma 3.16. *Let $A(\mathcal{H}_1 \to \mathcal{H}_1)$, $D(\mathcal{H}_2 \to \mathcal{H}_2)$, $B(\mathcal{H}_2 \to \mathcal{H}_1)$ and $C(\mathcal{H}_1 \to \mathcal{H}_2)$ be densely defined linear operators and assume that B and C are surjective and boundedly invertible. Further we assume that A is C-bounded and D is B-bounded.*

(i) *If D is bounded, we define $\mathcal{T}_1$ by*

$$\mathcal{T}_1 - \lambda := \begin{pmatrix} I & (A-\lambda)C^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & T_1(\lambda) \\ C & 0 \end{pmatrix} \begin{pmatrix} I & C^{-1}(D-\lambda) \\ 0 & I \end{pmatrix} \tag{3.31}$$

with

$$T_1(\lambda) = B - (A-\lambda)C^{-1}(D-\lambda), \qquad \mathcal{D}(T_1(\lambda)) = \mathcal{D}(B), \tag{3.32}$$

and its natural domain

$$\mathcal{D}(\mathcal{T}_1) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{D}(T_1) \ :\ x + C^{-1}(D-\lambda)y \in \mathcal{D}(C) \right\}.$$

(ii) *If A is bounded, we define $\mathcal{T}_2$ by*

$$\mathcal{T}_2 - \lambda := \begin{pmatrix} I & 0 \\ (D-\lambda)B^{-1} & I \end{pmatrix} \begin{pmatrix} 0 & B \\ T_2(\lambda) & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ B^{-1}(A-\lambda) & I \end{pmatrix} \tag{3.33}$$

with

$$T_2(\lambda) = C - (D-\lambda)B^{-1}(A-\lambda), \qquad \mathcal{D}(T_2(\lambda)) = \mathcal{D}(C), \tag{3.34}$$

and its natural domain

$$\mathcal{D}(\mathcal{T}_2) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(T_2) \oplus \mathcal{H}_2 \; : \; y + B^{-1}(A-\lambda)x \in \mathcal{D}(B) \right\}.$$

Furthermore, let

$$\mathcal{T} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathcal{D}(\mathcal{T}) = \mathcal{D}(C) \oplus \mathcal{D}(B).$$

Then we have the following factorisations

$$\mathcal{T} - \lambda = \begin{cases} \mathcal{T}_1 - \lambda & \text{if } D \text{ is bounded,} \\ \mathcal{T}_2 - \lambda & \text{if } A \text{ is bounded.} \end{cases}$$

Note that the domains of $T_1(\lambda)$ and $T_2(\lambda)$ do not depend on λ, so we write $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$ instead of $\mathcal{D}(T_1(\lambda))$ and $\mathcal{D}(T_2(\lambda))$.

Proof. We have to show the equalities $\mathcal{D}(\mathcal{T}) = \mathcal{D}(\mathcal{T}_1) = \mathcal{D}(\mathcal{T}_2)$ and that $\mathcal{T}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mathcal{T}_1\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mathcal{T}_2\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$.
In formulae (3.31) and (3.33), the first and the last factor of the first term on the right hand side are bounded and boundedly invertible. For example, in case (i), the operator $C^{-1}(D-\lambda)$ is bounded because both C^{-1} and D are bounded; furthermore, for all $x \in \mathcal{H}_1$ we have

$$\|(A-\lambda)C^{-1}x\| \;\le\; \lambda\|C^{-1}x\| + \alpha\|C^{-1}x\| + \gamma\|x\| \;\le\; (\|C^{-1}\|(\lambda+\alpha)+\gamma)\|x\|$$

so that also $(A-\lambda)C^{-1}$ is bounded. A purely algebraic calculation shows that the operators $\mathcal{T}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ coincide formally, see also the calculation below. To prove $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{T} = \mathcal{T}_2$ it remains to show that the domains of these operators coincide. We consider $\mathcal{T}_1$ only, the proof for $\mathcal{T}_2$ is analogous. For $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$ the component y lies in $\mathcal{D}(B) = \mathcal{D}(T_1)$, and since D is B-bounded, also $y \in \mathcal{D}(D)$ holds. Thus the element $x + C^{-1}(D-\lambda)y$ is well defined and lies in $\mathcal{D}(C)$. This shows that $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T}_1)$ and

$$\begin{aligned} \mathcal{T}_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} I & (A-\lambda)C^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} T_1(\lambda)y \\ C(x + C^{-1}(D-\lambda)y) \end{pmatrix} \\ &= \begin{pmatrix} T_1(\lambda)y + (A-\lambda)C^{-1}(Cx + (D-\lambda)y) \\ Cx + (D-\lambda)y \end{pmatrix} = \mathcal{T}\begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Hence $\mathcal{T} \subseteq \mathcal{T}_1$ is proved. Now consider $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T}_1)$. Since $y \in \mathcal{D}(T_1) = \mathcal{D}(B) \subseteq \mathcal{D}(D)$, it follows from $x \in \{-C^{-1}(D-\lambda)y + x_0 \,:\, x_0 \in \mathcal{D}(C)\} = \mathcal{D}(C)$ that $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ lies also in $\mathcal{D}(\mathcal{T})$, hence the above calculation implies that $\mathcal{T}_1\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mathcal{T}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. □

The previous lemma shows that under the given assumptions the spectrum of $\mathcal{T}$ can be obtained from the spectra of T_1 and T_2, respectively (for the definition of the spectrum of operator valued functions see definition 4.8). As an example we state the following corollary.

Corollary 3.17. *Let $\lambda \in \mathbb{C}$ and assume that the assumptions of one of the cases* (i) *or* (ii) *in lemma 3.16 are satisfied. Then the following equivalences hold.*

(i) *If D is bounded, we have the equivalences*

$$\begin{aligned} \mathcal{T} - \lambda \text{ is bijective} \quad &\Longleftrightarrow \quad T_1 \text{ is bijective,} \\ \mathcal{T} - \lambda \text{ is not injective} \quad &\Longleftrightarrow \quad T_1 \text{ is not injective.} \end{aligned}$$

(ii) *If A is bounded, we have the equivalences*

$$\begin{aligned} \mathcal{T} - \lambda \text{ is bijective} \quad &\iff \quad T_2 \text{ is bijective,} \\ \mathcal{T} - \lambda \text{ is not injective} \quad &\iff \quad T_2 \text{ is not injective.} \end{aligned}$$

Proof. This is a direct consequence of the factorisations (3.31) and (3.33). □

Corollary 3.18. *Let $\mathcal{T}$ be a block operator matrix as in lemma 3.16. In addition assume that D is bounded. Then for every eigenvalue $\lambda \in \mathbb{C}$ of $\mathcal{T}$ the following inequality holds:*

$$|\lambda| \;\geq\; -\tfrac{1}{2}\big(\alpha + \gamma\|C^{-1}\| + \|D\|\big) + \Big(\tfrac{1}{4}\big(\alpha + \gamma\|C^{-1}\| - \|D\|\big)^2 + \big(\|B^{-1}\|\,\|C^{-1}\|\big)^{-1}\Big)^{\frac{1}{2}}. \tag{3.35}$$

If both D and A are bounded, then we obtain

$$|\lambda| \;\geq\; -\tfrac{1}{2}\big(\|A\| + \|D\|\big) + \Big(\tfrac{1}{4}\big(\|A\| - \|D\|\big)^2 + \big(\|B^{-1}\|\,\|C^{-1}\|\big)^{-1}\Big)^{\frac{1}{2}}. \tag{3.36}$$

Proof. Let λ be an eigenvalue of $\mathcal{T}$. Then, by the previous corollary, 0 is an eigenvalue of the operator $T_1(\lambda) = B - (A-\lambda)C^{-1}(D-\lambda)$. For an eigenfunction f of $T_1(\lambda)$ with eigenvalue 0 we have the identity $f = B^{-1}(A-\lambda)C^{-1}(D-\lambda)f$ and hence it follows

$$\begin{aligned} \|f\| \;&\leq\; \|B^{-1}\|\,\|(A-\lambda)C^{-1}(D-\lambda)\|\,\|f\| \\ &\leq\; \|B^{-1}\|\,\big(|\lambda|\,\|C^{-1}\| + \alpha\|C^{-1}\| + \gamma\big)\,\|(D-\lambda)f\| \\ &\leq\; \|B^{-1}\|\,\|C^{-1}\|\,\big(|\lambda| + \alpha + \gamma\,\|C^{-1}\|^{-1}\big)\big(\|D\| + |\lambda|\big)\,\|f\| \\ &=\; \|B^{-1}\|\,\|C^{-1}\|\,\big(|\lambda|^2 + |\lambda|\big(\alpha + \gamma\,\|C^{-1}\|^{-1} + \|D\|\big) + \alpha\|D\| + \gamma\|D\|\|C^{-1}\|\big)\,\|f\| \\ &=\; \|B^{-1}\|\,\|C^{-1}\|\,\Big(\big(|\lambda| + \tfrac{1}{2}\big(\alpha + \gamma\,\|C^{-1}\|^{-1} + \|D\|\big)\big)^2 - \tfrac{1}{4}\big(\alpha + \gamma\,\|C^{-1}\|^{-1} - \|D\|\big)\Big)\,\|f\|. \end{aligned}$$

Dividing by $\|f\|$ yields

$$\big(|\lambda| + \tfrac{1}{2}\big(\alpha + \gamma\,\|C^{-1}\|^{-1} + \|D\|\big)\big)^2 \geq\; \tfrac{1}{4}\big(\alpha + \gamma\,\|C^{-1}\|^{-1} - \|D\|\big) + \big(\|B^{-1}\|\,\|C^{-1}\|\big)^{-1}$$

which implies inequality (3.35). If A is bounded, we can choose $\alpha = \|A\|$ and $\gamma = 0$ which gives inequality (3.36). □

In the special case $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}$ and $A, B, C, D \in \mathbb{C}$ the eigenvalues of the matrix $\mathcal{T}$ are given by $\lambda_{\pm} = \frac{1}{2}(A+D) \pm \sqrt{(A-D)^2 + BC}$. This formula shows that corollary 3.18 gives an optimal result in the case $A = D = 0$.

Before we apply this result to the angular operator, we want to point out the connection of the off-diagonalisation given in lemma 3.16 with the Schur factorisation. Let $\mathcal{T}$ be a block operator matrix as in lemma 3.16 and assume $\mathcal{H}_1 = \mathcal{H}_2$. Instead of $\mathcal{T}$ we consider the block operator matrix

$$\widetilde{\mathcal{T}}(\lambda) \;:=\; \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} (\mathcal{T} - \lambda) \;=\; \begin{pmatrix} C & D-\lambda \\ A-\lambda & B \end{pmatrix}$$

with domain $\mathcal{D}(\widetilde{\mathcal{T}}) = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \mathcal{D}(\mathcal{T}) = \mathcal{D}(B) \oplus \mathcal{D}(C)$. For all $\mu_C \in \rho(C)$ and $\mu_B \in \rho(B)$ we have the Schur factorisations

$$\widetilde{\mathcal{T}}(\lambda) - \mu_C \;=\; \begin{pmatrix} I & 0 \\ (A-\lambda)(C-\mu_C)^{-1} & I \end{pmatrix} \begin{pmatrix} C-\mu_C & 0 \\ 0 & S_B(\mu_C) \end{pmatrix} \begin{pmatrix} I & (C-\mu_C)^{-1}(D-\lambda) \\ 0 & I \end{pmatrix}$$

$$\widetilde{\mathcal{T}}(\lambda) - \mu_B \;=\; \begin{pmatrix} I & (D-\lambda)(B-\mu_B)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_C(\mu_B) & 0 \\ 0 & B-\mu_B \end{pmatrix} \begin{pmatrix} I & 0 \\ (B-\mu_B)^{-1}(A-\lambda) & I \end{pmatrix},$$

where the Schur complements for the block operator matrix $\widetilde{\mathcal{T}}(\lambda)$ are given by

$$\begin{aligned} S_B(\mu_C) &= B - \mu_C - (A-\lambda)(C-\mu_C)^{-1}(D-\lambda), \qquad && \mu_C \in \rho(C), \\ S_C(\mu_B) &= C - \mu_B - (D-\lambda)(B-\mu_B)^{-1}(A-\lambda), \qquad && \mu_B \in \rho(B), \end{aligned}$$

see formulae 3.29 and 3.30.
Since $0 \in \rho(C) \cap \rho(B)$, the Schur complements $S_B(\mu_C)$ and $S_C(\mu_B)$ are well defined for $\mu_C = \mu_B = 0$ and the factorisation yields

$$\widetilde{\mathcal{T}}(\lambda) = \begin{pmatrix} I & 0 \\ (A-\lambda)C^{-1} & I \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & S_B(0) \end{pmatrix} \begin{pmatrix} I & C^{-1}(D-\lambda) \\ 0 & I \end{pmatrix} \tag{3.37}$$

$$= \begin{pmatrix} I & (D-\lambda)B^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_C(0) & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ B^{-1}(A-\lambda) & I \end{pmatrix} \tag{3.38}$$

with

$$\begin{aligned} S_B(0) &= B - (A-\lambda)C^{-1}(D-\lambda) = T_1(\lambda), \\ S_C(0) &= C - (D-\lambda)B^{-1}(A-\lambda) = T_2(\lambda). \end{aligned}$$

From the above factorisation of $\widetilde{\mathcal{T}}$ we recover the off-diagonal factorisation of $\mathcal{T}$ given in lemma 3.16 if we multiply equations (3.37) and (3.38) from the left by $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and insert the factor $I = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ after the first factor on the right hand sides.

3.3.2 Application to the angular operator

In this section we apply the off-diagonalisation presented in the previous section to the angular operator $\mathcal{A} = \left(\begin{smallmatrix} -D & B \\ B^* & D \end{smallmatrix}\right)$, see (3.1). To this end, we have to investigate the operators B and B^* in greater detail. First we verify that the off-diagonal entries B and B^* of $\mathcal{A}$ are boundedly invertible; in fact, we show that $\sigma(B) = \sigma(B^*) = \emptyset$. Then, in lemma 3.30, we derive upper bounds for $\|B^{-1}\|$ and $\|B^{*-1}\|$. In section 3.3.3 these estimates are further improved by an iteration method (see lemma 3.34). Together with corollary 3.18, we obtain lower bounds for the modulus of the eigenvalue of $\mathcal{A}$ with smallest modulus.

For $\mu \in \mathbb{C}$ we introduce the formal differential expression defined by

$$\mathfrak{B}_\mu := \begin{pmatrix} 0 & \mathfrak{B}_+ - \mu \\ \mathfrak{B}_- - \overline{\mu} & 0 \end{pmatrix}.$$

With $\mathfrak{B}_\mu$ we associate the differential operator

$$\mathcal{D}(\mathcal{B}_\mu) = \mathcal{D}(\mathcal{A}), \qquad \mathcal{B}_\mu \Psi := \mathfrak{B}_\mu \Psi.$$

Furthermore, with the notation in remark 2.11, we have

$$\mathcal{B} = \mathcal{B}_0 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

For every $\mu \in \mathbb{C}$, the operator $\mathcal{B}_\mu$ is selfadjoint since $\mathcal{A}$ is selfadjoint and $\mathcal{B}_\mu - \mathcal{A}$ is symmetric and bounded. It can be shown that $\sigma_{ess}(\mathcal{B}_\mu)$ is empty; the proof is analogous to that of theorem 2.14, where we have shown $\sigma_{ess}(\mathcal{A}) = \emptyset$.
A main tool for computing the inverse operators is to consider the selfadjoint operator $\mathcal{B}$ instead of B and B^* separately because to $\mathcal{B}$ we can apply well known results for Dirac operators, see for example [Wei87].
First of all, we show that the point spectrum of B and B^* is empty. Eventually, it turns out that the spectrum of B and B^* is empty.

Lemma 3.19. $\sigma_p(B) = \sigma_p(B^*) = \emptyset$.

Proof. Fix an arbitrary $\mu \in \mathbb{C}$. The number μ lies in $\sigma_p(B) \cup \sigma_p(B^*)$ if and only if at least one of the differential equations

$$\begin{aligned} \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta - \mu\Big)\varphi(\vartheta) &= 0, \\ \Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta - \mu\Big)\psi(\vartheta) &= 0 \end{aligned}$$

has a square integrable solution. The solutions of these differential equations are

$$\begin{aligned} \varphi_{[\mu]}(\vartheta) &= c\,\mathrm{e}^{\mu\vartheta + a\omega\cos\vartheta}\Big(\tan\frac{\vartheta}{2}\Big)^{-(k+\frac{1}{2})}, \\ \psi_{[\mu]}(\vartheta) &= c\,\mathrm{e}^{-\mu\vartheta - a\omega\cos\vartheta}\Big(\tan\frac{\vartheta}{2}\Big)^{k+\frac{1}{2}} = \big(\varphi_{[\mu]}(\vartheta)\big)^{-1}. \end{aligned}$$

The functions $\varphi_{[\mu]}$ and $\psi_{[\mu]}$ are unique up to a constant factor $c \in \mathbb{C}$; without loss of generality we set $c = 1$. The following computation (cf. also lemma 2.8) shows that $\varphi_{[\mu]}$ and $\psi_{[\mu]}$ are not square integrable on the interval $(0, \pi)$:

$$\begin{aligned} \int_0^\pi \varphi_{[\mu]}(\vartheta)^2 \mathrm{d}\vartheta &= \int_0^\pi \mathrm{e}^{2\operatorname{Re}(\mu)\vartheta + 2a\omega\cos\vartheta}\big(\tan\tfrac{\vartheta}{2}\big)^{-(2k+1)}\,\mathrm{d}\vartheta \\ &\geq M\Big(\int_0^{\frac{\pi}{2}}\big(\tan\tfrac{\vartheta}{2}\big)^{-(2k+1)}\,\mathrm{d}\vartheta + \int_{\frac{\pi}{2}}^{\pi}\big(\tan\tfrac{\vartheta}{2}\big)^{-(2k+1)}\,\mathrm{d}\vartheta\Big), \end{aligned}$$

where $M := \inf\{\mathrm{e}^{2\operatorname{Re}(\mu)\vartheta + 2a\omega\cos\vartheta} : \vartheta \in (0, \pi)\} > 0$. For $k \geq 0$ it follows that

$$\int_0^{\frac{\pi}{2}} \big(\underbrace{\tan\tfrac{\vartheta}{2}}_{\leq 1}\big)^{-(2k+1)}\,\mathrm{d}\vartheta \geq \int_0^{\frac{\pi}{2}} \big(\tan\tfrac{\vartheta}{2}\big)^{-1}\,\mathrm{d}\vartheta = 2\ln(\sin\tfrac{\vartheta}{2})\Big|_0^{\frac{\pi}{2}} = \infty. \tag{3.39}$$

For $k \leq -1$ we estimate

$$\int_{\frac{\pi}{2}}^{\pi} \big(\underbrace{\tan\tfrac{\vartheta}{2}}_{\geq 1}\big)^{-(2k+1)}\,\mathrm{d}\vartheta \geq \int_{\frac{\pi}{2}}^{\pi} \tan\tfrac{\vartheta}{2}\,\mathrm{d}\vartheta = -2\ln(\cos\tfrac{\vartheta}{2})\Big|_{\frac{\pi}{2}}^{\pi} = \infty. \tag{3.40}$$

In both cases we find $\varphi_{[\mu]} \notin \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta) \supseteq \mathcal{D}(B)$. Analogously we can show $\psi_{[\mu]} \notin \mathcal{D}(B^*)$. □

Corollary 3.20. *For all $\mu \in \mathbb{C}$ we have $0 \notin \sigma_p(\mathcal{B}_\mu)$.*

Proof. Assume $0 \in \sigma_p(\mathcal{B}_\mu)$ and let Ψ be an eigenfunction of $\mathcal{B}_\mu$ with eigenvalue 0. From

$$0 = \mathcal{B}_\mu\Psi = \mathcal{B}_\mu \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\Psi = \begin{pmatrix} B-\mu & 0 \\ 0 & B^* - \overline{\mu} \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\Psi$$

it follows that either $(B - \mu)$ or $(B^* - \overline{\mu})$ is not injective, in contradiction to $\sigma_p(B) \cup \sigma_p(B^*) = \emptyset$ as shown in lemma 3.19. □

This corollary together with the fact that $\sigma_{ess}(\mathcal{B}_\mu) = \emptyset$ shows that $\mathcal{B}_\mu$ is boundedly invertible. According to the previous corollary, we have $0 \in \mathbb{R} \setminus (\sigma_p(\mathcal{B}_\mu) \cup \sigma_{ess}(\mathcal{B}_\mu)) = \rho(\mathcal{B}_\mu)$. Thus $B - \mu$ and $B^* - \overline{\mu}$ are boundedly invertible and their resolvents and the resolvent of $\mathcal{B}_\mu$ are connected as follows:

$$\begin{pmatrix} (B-\mu)^{-1} & 0 \\ 0 & (B^*-\overline{\mu})^{-1} \end{pmatrix} = \begin{pmatrix} B-\mu & 0 \\ 0 & B^*-\overline{\mu} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathcal{B}_\mu^{-1} \tag{3.41}$$

In particular, we have $\operatorname{rg}(B-\mu) = \operatorname{rg}(B^* - \overline{\mu}) = \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. Thus we have shown

$$\sigma(B) = \sigma(B^*) = \emptyset. \tag{3.42}$$

Lemma 3.21. *Fix $\mu \in \mathbb{C}$ and define, as in the proof of lemma 3.19, the functions*

$$\varphi_{[\mu]}(\vartheta) := \mathrm{e}^{\mu\vartheta + a\omega\cos\vartheta} \left(\tan\tfrac{\vartheta}{2}\right)^{-(k+\frac{1}{2})}, \qquad \psi_{[\mu]}(\vartheta) := \mathrm{e}^{-(\mu\vartheta + a\omega\cos\vartheta)} \left(\tan\tfrac{\vartheta}{2}\right)^{k+\frac{1}{2}}, \qquad \vartheta \in (0,\pi).$$

Then the inverse operators of $B-\mu$ and $B^*-\mu$ map functions $g, h \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ to

$$(B-\mu)^{-1} g(\vartheta) = \frac{1}{\psi_{[\mu]}(\vartheta)} \cdot \begin{cases} \int_0^\vartheta \psi_{[\mu]}(t) g(t)\,\mathrm{d}t & \text{if } k \geq 0, \\ \int_\pi^\vartheta \psi_{[\mu]}(t) g(t)\,\mathrm{d}t & \text{if } k \leq -1, \end{cases} \qquad \vartheta \in (0,\pi), \tag{3.43 a}$$

$$(B^*-\mu)^{-1} h(\vartheta) = \frac{1}{\varphi_{[\mu]}(\vartheta)} \cdot \begin{cases} \int_\vartheta^\pi \varphi_{[\mu]}(t) h(t)\,\mathrm{d}t & \text{if } k \geq 0, \\ \int_\vartheta^0 \varphi_{[\mu]}(t) h(t)\,\mathrm{d}t & \text{if } k \leq -1, \end{cases} \qquad \vartheta \in (0,\pi). \tag{3.43 b}$$

Proof. We know from the proof of lemma 3.19 that $\varphi_{[\mu]}$ is a solution of $(B-\mu)u = 0$ and that $\psi_{[\mu]}$ is a solution of $(B^*-\mu)u = 0$. To show that formulae (3.43 a) and (3.43 b) indeed represent explicit expressions of the resolvents of $B-\mu$ and $B^*-\mu$ we first show that

$$(B-\mu)G(\vartheta) = g(\vartheta), \qquad (B^*-\mu)H(\vartheta) = h(\vartheta), \qquad \vartheta \in (0,\pi),$$

holds formally; here G and H denote the right hand sides of (3.43 a) and (3.43 b), respectively. Assume, for example, $k \geq 0$. Then for $g \in \operatorname{rg}(B-\mu)$ we obtain

$$\begin{aligned}
(B-\mu)G(\vartheta) &= \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta - \mu\right) \left[\varphi_{[\mu]}(\vartheta) \int_0^\vartheta \psi_{[\mu]}(t) g(t)\,\mathrm{d}t\right] \\
&= \left[\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta - \mu\right)\varphi_{[\mu]}(\vartheta)\right] \int_0^\vartheta \psi_{[\mu]}(t) g(t)\,\mathrm{d}t \\
&\qquad + \varphi_{[\mu]}(\vartheta)\, \frac{\mathrm{d}}{\mathrm{d}\vartheta} \int_0^\vartheta \psi_{[\mu]}(t) g(t)\,\mathrm{d}t \\
&= \varphi_{[\mu]}(\vartheta)\, \psi_{[\mu]}(\vartheta)\, g(\vartheta) \;=\; g(\vartheta)
\end{aligned}$$

where we have used that $\varphi_{[\mu]}(\vartheta) = (\psi_{[\mu]}(\vartheta))^{-1}$ and $(\mathfrak{B}_+ - \mu)\varphi_{[\mu]} = 0$. The case $k \leq -1$ and the equation for h can be shown analogously. It remains to prove $G \in \mathcal{D}(B)$ and $H \in \mathcal{D}(B^*)$. Again, we give an explicit proof only for $G \in \mathcal{D}(B)$ in the case $k \geq 0$. The assertion for $k \leq -1$ and the inclusion $H \in \mathcal{D}(B^*)$ follow by analogous calculations.

Recall that the domain of B is given by

$$\mathcal{D}(B) = \left\{g \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta) \,:\, g \text{ is absolutely continuous, } \mathfrak{B}_+ g \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)\right\},$$

so it remains to be shown that G is square integrable on $(0,\pi)$. The proof for that is similar to that of the subsequent lemma 3.30; here we use a simplified calculation (the estimation for the integrand is less accurate). By assumption we have $k \geq 0$, so that

$$\int_0^\pi |G(\vartheta)|^2 \mathrm{d}\vartheta = \int_0^\pi \frac{1}{|\psi_{[\mu]}(\vartheta)|^2} \left| \int_0^\vartheta \psi_{[\mu]}(t) g(t)\, \mathrm{d}t \right|^2 \mathrm{d}\vartheta \;\leq\; \int_0^\pi \left(\int_0^\vartheta \left| \frac{\psi_{[\mu]}(t)}{\psi_{[\mu]}(\vartheta)} \right| |g(t)|\, \mathrm{d}t \right)^2 \mathrm{d}\vartheta. \tag{3.44}$$

Since by assumption $g \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$, we also have $g|_{(0,\vartheta)} \in \mathscr{L}^2((0,\vartheta), \mathrm{d}t)$ for all $\vartheta \in (0,\pi)$. Furthermore, we have the estimate

$$\left| \frac{\psi_{[\mu]}(t)}{\psi_{[\mu]}(\vartheta)} \right|^2 = \mathrm{e}^{-2\operatorname{Re}(\mu)(t-\vartheta) - 2a\omega(\cos t - \cos\vartheta)} \underbrace{\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}} \right)^{2k+1}}_{\leq 1} \;\leq\; \mathrm{e}^{2|\operatorname{Re}(\mu)|\vartheta + 4|a\omega|}, \qquad 0 < t < \vartheta < \pi,$$

thus $\frac{\psi_{[\mu]}}{\psi_{[\mu]}(\vartheta)} \in \mathscr{L}^2((0,\vartheta), \mathrm{d}t)$ for each fixed $\vartheta \in (0,\,\pi)$. Therefore we can apply the Cauchy-Schwarz inequality to estimate the inner integral in (3.44) and obtain

$$\begin{aligned} \left(\int_0^\vartheta \left| \frac{\psi_{[\mu]}(t)}{\psi_{[\mu]}(\vartheta)} \right| |g(t)|\, \mathrm{d}t \right)^2 \;&\leq\; \left(\int_0^\vartheta \left| \frac{\psi_{[\mu]}(t)}{\psi_{[\mu]}(\vartheta)} \right|^2 \mathrm{d}t \right) \left(\int_0^\vartheta |g(t)|^2\, \mathrm{d}t \right) \\ &\leq\; \vartheta\, \mathrm{e}^{2|\operatorname{Re}(\mu)|\pi + 4|a\omega|}\, \|g\|_2^2. \end{aligned}$$

Inserting into (3.44) shows that

$$\int_0^\pi |G(\vartheta)|^2 \mathrm{d}\vartheta \;\leq\; \frac{\pi^2}{2}\, \mathrm{e}^{2|\operatorname{Re}(\mu)|\pi + 4|a\omega|}\, \|g\|_2^2 < \infty. \qquad \square$$

Since we are only interested in the inverses of B and B^*, that is, in the case $\mu = 0$, we omit the subscript μ in the following.
Now that we have obtained an explicit form of $\mathcal{B}^{-1}$, we can show that $\mathcal{B}$, and consequently $\mathcal{A}$, has compact resolvent.

Lemma 3.22. *The operator $\mathcal{B}$ has compact resolvent.*

Proof. To show that the operator $\mathcal{B}^{-1} = \begin{pmatrix} 0 & B^{*-1} \\ B^{-1} & 0 \end{pmatrix}$ is compact, it suffices to show that the operators B^{-1} and B^{*-1} are compact. We prove only that B^{-1} is compact in the case $k \geq 0$; the case $k \leq -1$ and the assertion concerning B^{*-1} follows analogously.
Recall that for $k \geq 0$ and $g \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$

$$B^{-1} g(\vartheta) \;=\; \frac{1}{\psi(\vartheta)} \int_0^\vartheta \psi(t) g(t)\;\, \mathrm{d}t, \qquad \vartheta \in (0,\pi),$$

with $\psi(\vartheta) = \mathrm{e}^{-a\omega\cos\vartheta} \left(\tan\frac{\vartheta}{2}\right)^{k+\frac{1}{2}}$ defined in lemma 3.19. For each $n \in \mathbb{N}$ we define the operators

$$T_n : \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta) \longrightarrow \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta), \quad T_n f(\vartheta) \;=\; \begin{cases} 0, & \vartheta \notin [\frac{1}{n},\ \pi - \frac{1}{n}], \\ \frac{1}{\psi(\vartheta)} \int_{\frac{1}{n}}^{\vartheta} \psi(t) f(t)\, \mathrm{d}t, & \vartheta \in [\frac{1}{n},\ \pi - \frac{1}{n}] \end{cases}$$

and

$$\widehat{T}_n : \mathscr{L}^2([\tfrac{1}{n}, \pi - \tfrac{1}{n}], \mathrm{d}\vartheta) \longrightarrow \mathscr{L}^2([\tfrac{1}{n}, \pi - \tfrac{1}{n}], \mathrm{d}\vartheta), \quad \widehat{T}_n f(\vartheta) \;=\; \tfrac{1}{\psi(\vartheta)} \int_{\frac{1}{n}}^{\vartheta} \psi(t) f(t)\, \mathrm{d}t.$$

These operators are bounded for all $n \in \mathbb{N}$. Moreover, the operators $\widehat{T}_n$ are even compact since the integral kernel is continuous and bounded, see, e.g., [Kat80, chap. III, example 4.1]. For every $f \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ the corresponding function $\widehat{f} := f|_{[\frac{1}{n},\ \pi - \frac{1}{n}]}$ lies in $\mathscr{L}^2([\frac{1}{n},\ \pi - \frac{1}{n}], \mathrm{d}\vartheta)$. It is clear that for any convergent sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}^2([\frac{1}{n},\ \pi - \frac{1}{n}], \mathrm{d}\vartheta)$ also the sequence $(\check{g})_{n\in\mathbb{N}} \subseteq \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ converges where the $\check{g}_n$ are defined by

$$\check{g}_n(\vartheta) := \begin{cases} 0, & \vartheta \notin [\frac{1}{n},\ \pi - \frac{1}{n}], \\ g(\vartheta), & \vartheta \in [\frac{1}{n},\ \pi - \frac{1}{n}] \end{cases}$$

for all $n \in \mathbb{N}$. Now let $(f_m)_{m\in\mathbb{N}}$ be a bounded sequence in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. Then $(\widehat{f}_m)_{m\in\mathbb{N}}$ is a bounded sequence in $\mathscr{L}^2([\frac{1}{n},\ \pi - \frac{1}{n}], \mathrm{d}\vartheta)$. Hence for every $n \in \mathbb{N}$ the sequence $(\widehat{T}_n \widehat{f}_m)_{m\in\mathbb{N}}$ contains a convergent subsequence. Consequently, also $(T_n f_m)_{m\in\mathbb{N}}$ contains a convergent subsequence since $(\widehat{T}_n \widehat{f}_m)\check{} = (T_n f_m)$. This shows that the operators $T_n,\ n \in \mathbb{N}$, are also compact. If we have shown that $\lim\limits_{n\to\infty} T_n = B^{-1}$ in the operator norm, that is, that $\|T_n - B^{-1}\| \to \infty,\ n \to \infty$, then the lemma is proved. To see that, we note that for all $f \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$

$$\begin{aligned} \|(T_n - B^{-1})f\|_2^2 \;&=\; \int_0^{\pi} |(T_n - B^{-1}) f(\vartheta)|^2 \mathrm{d}\vartheta \;=\; \int_0^{\frac{1}{n}} |B^{-1} f(\vartheta)|^2 \mathrm{d}\vartheta \;+\; \int_{\pi - \frac{1}{n}}^{\pi} |B^{-1} f(\vartheta)|^2 \mathrm{d}\vartheta \\ &\le\; \frac{2\mathrm{e}^{4|a\omega|}\pi}{n}\, \|f\|_2^2 \end{aligned}$$

holds where we have used that for all $(a,b) \subseteq (0,\pi)$

$$\begin{aligned} \int_a^b |B^{-1} f(\vartheta)|^2 \mathrm{d}\vartheta \;&=\; \int_a^b \Big(\int_0^{\vartheta} \underbrace{\frac{\psi(t)}{\psi(\vartheta)}}_{\le \mathrm{e}^{2|a\omega|}} |f(t)|\, \mathrm{d}t \Big)^2 \mathrm{d}\vartheta \;\le\; \mathrm{e}^{4|a\omega|} \int_0^{\pi} \Big(\int_0^{\pi} |f(t)|^2\ \mathrm{d}t \Big) \Big(\int_0^{\pi} 1\, \mathrm{d}t \Big) \mathrm{d}\vartheta \\ &=\; \mathrm{e}^{4|a\omega|} \int_a^b \pi\, \|f\|_2^2\ \mathrm{d}\vartheta \;=\; \mathrm{e}^{4|a\omega|}\, \pi\, (b-a)\, \|f\|_2^2. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \|T_n - B^{-1}\| \;\le\; \lim_{n\to\infty} \frac{2\pi}{n} \;= 0. \qquad \square$$

Theorem 3.23. *The angular operator $\mathcal{A}$ has compact resolvent.*

Proof. We know that both $\mathcal{A}$ and $\mathcal{B}$ are selfadjoint, hence their spectra are real. For any $\mu \in \rho(\mathcal{A})$ and $\nu \in \rho(\mathcal{B})$ the second resolvent equation yields

$$(\mathcal{A} - \mu)^{-1} - (\mathcal{B} - \nu)^{-1} \;=\; (\mathcal{A} - \mu)^{-1} (\mathcal{B} - \mathcal{A} + \mu - \nu)(\mathcal{B} - \nu)^{-1}.$$

Since $(\mathcal{B} - \nu)^{-1}$ is compact and $(\mathcal{A} - \mu)^{-1}$ and $(\mathcal{B} - \mathcal{A} + \mu - \nu)$ are bounded, the operator on the right hand side is compact. Thus also $(\mathcal{A} - \mu)^{-1}$ has to be compact. $\square$

We want to add two remarks concerning this theorem. First, the theorem follows also from the fact that $(\mathcal{A}(a))_a$ is a holomorphic family of type (A), and $\mathcal{A}(0) = \mathcal{B}$ is compact, see [Kat80]. Secondly, we observe that from theorem 3.23 it follows immediately that the spectrum of $\mathcal{A}$ consists of isolated eigenvalues with no accumulation points in $(-\infty, \infty)$, see [Kat80, chap. III, theorem 6.29]. Recall that this has already been proved in section 2.2 with methods of oscillation theory for Dirac operators.

In the remainder of this section we derive estimates for $\|B^{-1}\|$ and $\|B^{*-1}\|$ and apply corollary 3.18 to find a lower bound for the absolute value of eigenvalues of the angular operator $\mathcal{A}$. The next lemmata provide some rather technical estimates used for this task.

Lemma 3.24. *For $0 < x < y < \pi$ we have the following inequalities:*

$$\frac{\tan\frac{x}{2}}{\tan\frac{y}{2}} < \frac{x}{y}, \tag{3.45 a}$$

$$\frac{\tan\frac{x}{2}}{\tan\frac{y}{2}} < \frac{\pi - y}{\pi - x}, \tag{3.45 b}$$

$$\frac{\tan\frac{x}{2}}{\tan\frac{y}{2}} < \frac{\mathrm{e}^x}{\mathrm{e}^y} = \frac{\mathrm{e}^{\pi-y}}{\mathrm{e}^{\pi-x}}. \tag{3.45 c}$$

Proof. (i) Since inequality (3.45 a) is equivalent to the inequality $\frac{\tan\frac{x}{2}}{x} < \frac{\tan\frac{y}{2}}{y}$, we consider the function $f : (0, \pi) \to \mathbb{R}$, $f(x) = \frac{\tan\frac{x}{2}}{x}$. Obviously, f is continuously differentiable and inequality (3.45 a) is equivalent to $f(x) < f(y)$ for $0 < x < y < \pi$. Thus it suffices to show that f is a monotonously increasing function of x. An easy calculation shows that

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\tan\frac{x}{2}}{x} = \frac{1}{2x\cos^2\frac{x}{2}}\Big(1 - \underbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}_{<1}\,\underbrace{\cos\frac{x}{2}}_{<1}\Big) > 0,$$

hence the assertion is proved.

(ii) With the trigonometric identity $\tan(\frac{\pi}{2} - \alpha) = \frac{1}{\tan\alpha}$ we find

$$\frac{\tan\frac{x}{2}}{\tan\frac{y}{2}} = \frac{\tan(\frac{\pi}{2} - \frac{\pi-x}{2})}{\tan(\frac{\pi}{2} - \frac{\pi-y}{2})} = \frac{\tan\frac{\pi-y}{2}}{\tan\frac{\pi-x}{2}} < \frac{\pi - y}{\pi - x},$$

where we have used $0 < \frac{\pi-y}{2} < \frac{\pi-x}{2} < \pi$ and inequality (3.45 a).

(iii) To prove (3.45 c) fix again $y \in (0, \pi)$ and define the function $g : (0, \pi) \to \mathbb{R}, x \mapsto \frac{\tan\frac{x}{2}}{\mathrm{e}^x}$. This function is continuously differentiable, so as before it suffices to show that g is monotonously increasing because obviously inequality (3.45 c) is equivalent to $g(x) < g(y)$. Thus the assertion follows from

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\tan\frac{x}{2}}{\mathrm{e}^x} = \frac{1}{2\,\mathrm{e}^x\cos^2\frac{x}{2}}\Big(\sin\frac{x}{2} - \cos\frac{x}{2}\Big)^2 > 0. \qquad \square$$

Lemma 3.25. *There exists a unique number $\rho_0 \in (0, \pi)$ such that*

$$\rho_0 \sin\rho_0 + \cos\rho_0 - 1 = 0. \tag{3.46}$$

Further, with $\nu := -\sin\rho_0$, $c_- := 1$ and $c_+ := -1 + \pi\sin\rho_0$, the functions

$$g_- : [0, \pi] \longrightarrow \mathbb{R}, \qquad g_-(\vartheta) := \nu\vartheta + c_-,$$
$$g_+ : [0, \pi] \longrightarrow \mathbb{R}, \qquad g_+(\vartheta) := \nu\vartheta + c_+$$

satisfy the inequalities

$$g_-(\vartheta) \;\le\; \cos\vartheta \;\le\; g_+(\vartheta), \qquad \vartheta \in [0,\pi].$$

Corollary 3.26. *For $a\omega \in \mathbb{R}$ the inequalities*

$$a\omega(\nu\vartheta + \gamma_-) \;\le\; a\omega\cos\vartheta \;\le\; a\omega(\nu\vartheta + \gamma_+), \qquad \vartheta \in (0,\pi),$$

and hence

$$\mathrm{e}^{a\omega(\nu\vartheta+\gamma_-)} \;\le\; \mathrm{e}^{a\omega\cos\vartheta} \;\le\; \mathrm{e}^{a\omega(\nu\vartheta+\gamma_+)} \tag{3.47}$$

hold with $\gamma_\pm := \begin{cases} c_\pm & \text{if } a\omega \ge 0, \\ c_\mp & \text{if } a\omega < 0. \end{cases}$ □

Note that the definition of γ_+ and γ_- implies $a\omega(\gamma_+ - \gamma_-) = |a\omega(c_+ - c_-)|$ for all $a, \omega \in \mathbb{R}$.

Proof of lemma 3.25. First, we define the auxiliary functions

$$f : [0,\,\pi] \longrightarrow \mathbb{R}, \qquad f(\vartheta) \;:=\; \vartheta\sin\vartheta + \cos\vartheta - 1$$
$$g_0 : [0,\,\pi] \longrightarrow \mathbb{R}, \qquad g_0(\vartheta) \;:=\; \cos\vartheta.$$

The existence of ρ_0 as in the assertion follows because the function f is continuously differentiable and therefore must have at least one zero ρ_0 because of

$$f(\pi) \;=-2 \;<\; 0 \;<\; \frac{\pi}{2} - 1 \;=\; f(\tfrac{\pi}{2}).$$

On the other hand, it is easy to see that $\rho = \frac{\pi}{2}$ is the only extremal point of f in $(0,\pi)$ and that f is zero for $\rho = 0$, thus ρ_0 is uniquely determined and $\rho_0 \in (\frac{\pi}{2}, \pi)$.
The derivatives of $g_0 - g_-$ and $g_0 - g_+$ are equal and given by

$$(g_0 - g_-)'(\vartheta) = (g_0 - g_+)'(\vartheta) = -\sin\vartheta + \sin\rho_0, \qquad \vartheta \in [0,\,\pi].$$

This shows that $(g_0 - g_-)$ and $(g_0 - g_+)$ are increasing in $[0, \pi - \rho_0) \cup (\rho_0, \pi]$ and decreasing in $(\pi - \rho_0, \rho_0)$. Since $(g_0 - g_-)(0) = (g_0 - g_+)(\pi) = 0$, it follows that

$$\begin{aligned}
&(g_0 - g_-)(\vartheta) > (g_0 - g_-)(\rho_0) = 0, && \vartheta \in (\pi - \rho_0,\, \pi),\\
&(g_0 - g_-)(\vartheta) > (g_0 - g_-)(0) = 0, && \vartheta \in (0,\, \pi - \rho_0)\\
&(g_0 - g_+)(\vartheta) < (g_0 - g_+)(\pi - \rho_0) = 0, && \vartheta \in (0,\, \rho_0),\\
&(g_0 - g_+)(\vartheta) < (g_0 - g_+)(\pi) = 0, && \vartheta \in (\rho_0,\, \pi).
\end{aligned}$$

□

The value ν gives the slope of the linear function by which the cosine is estimated. In the case of the lemma, the numerical values of the constants ρ_0, ν and $\gamma_+ - \gamma_-$ are given by

$$\rho_0 \approx 2.331122370, \qquad \nu \approx -0.7246113541, \qquad \gamma_+ - \gamma_- \approx \operatorname{sign}(a\omega)\cdot 0.276433707.$$

Instead of the number ν of lemma 3.25 we can choose any real value for ν; but then also the numbers $c_\pm$ have to be adapted, see figure 3.1. For example, if we choose $\nu = -1$, we have $c_- = 1$ and $c_+ = \pi - 1$. Hence the exponential function $C(\omega)$ (see lemma 3.27), which is important for the estimation of the eigenvalues of $\mathcal{A}$, has larger values, and in general the estimates will be weaker.

Now let $\alpha \ge 0$. With the help of the previous lemmata we can estimate the following double integrals.

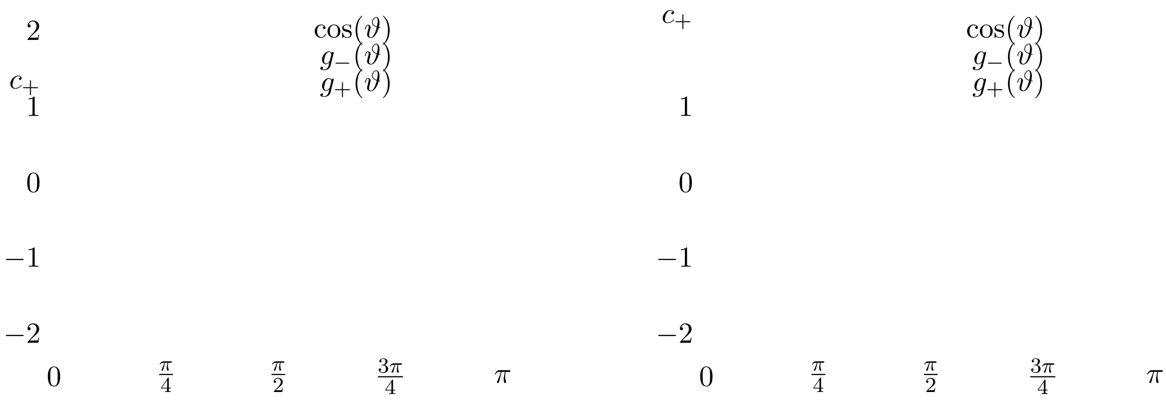

Figure 3.1. The left graph shows the estimates for the cosine with ν from lemma 3.25, the right graph shows the estimates with $\nu = -1$.

Lemma 3.27. *For all $\alpha \geq 0$ we have*

$$\int_0^\pi \int_0^\vartheta \left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^\alpha \mathrm{e}^{2a\omega(\cos\vartheta-\cos t)}\,\mathrm{d}t\,\mathrm{d}\vartheta = \int_0^\pi \int_\vartheta^\pi \left(\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\right)^\alpha \mathrm{e}^{2a\omega(\cos t-\cos\vartheta)}\,\mathrm{d}t\,\mathrm{d}\vartheta \tag{3.48}$$

$$\leq C(\omega)^2\,\delta(\tfrac{1}{2}(\alpha-1),\omega)^2 \tag{3.49}$$

where

$$C(\omega) := \mathrm{e}^{a\omega(\gamma_+-\gamma_-)}, \qquad \delta(\tfrac{1}{2}(\alpha-1),\omega) := \begin{cases} \frac{\mathrm{e}^{\pi(a\omega\nu-\frac{\alpha}{2})}}{(2a\omega\nu-\alpha)} & \textit{if } \frac{\alpha}{2}-a\omega\nu<0,\\ \sqrt{\frac{\pi}{2}\,|a\omega\nu-\frac{\alpha}{2}|^{-1}} & \textit{if } \frac{\alpha}{2}-a\omega\nu>0,\\ \frac{\pi}{\sqrt{2}} & \textit{if } \frac{\alpha}{2}-a\omega\nu=0. \end{cases}$$

Since in the following the function δ is always applied to $\alpha = 2k+1$ as first argument, we have defined it in the seemingly awkward way above. In lemma 3.30 we also admit arbitrary negative values as first argument of δ.

Proof. First we show inequality (3.49). If we use (3.47) to estimate the exponential function containing the cosine and equation (3.45 c) to estimate the quotient of the tangent functions we obtain

$$\int_0^\pi \int_0^\vartheta \left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^\alpha \mathrm{e}^{2a\omega(\cos\vartheta-\cos t)}\,\mathrm{d}t\,\mathrm{d}\vartheta \leq \int_0^\pi \int_0^\vartheta \mathrm{e}^{\alpha(t-\vartheta)}\mathrm{e}^{2a\omega\nu(\vartheta-t)+2a\omega(\gamma_+-\gamma_-)}\,\mathrm{d}t\,\mathrm{d}\vartheta$$

$$= \mathrm{e}^{2|a\omega(c_+-c_-)|}\int_0^\pi \int_0^\vartheta \mathrm{e}^{\vartheta(2a\omega\nu-\alpha)}\mathrm{e}^{t(-2a\omega\nu+\alpha)}\,\mathrm{d}t\,\mathrm{d}\vartheta.$$

For $2a\omega\nu-\alpha=0$ the assertion is now obvious. For $a\omega\nu-\alpha\neq 0$ integration with respect to t yields

$$\int_0^\pi \int_0^\vartheta \left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^\alpha \mathrm{e}^{2a\omega(\cos\vartheta-\cos t)}\,\mathrm{d}t\,\mathrm{d}\vartheta \leq \frac{\mathrm{e}^{2|a\omega(c_+-c_-)|}}{2a\omega\nu-\alpha}\int_0^\pi \mathrm{e}^{\vartheta(2a\omega\nu-\alpha)}-1\,\mathrm{d}\vartheta. \tag{3.50}$$

For $2a\omega\nu - \alpha > 0$ we use

$$0 \leq \int_0^\pi e^{\vartheta(2a\omega\nu-\alpha)} - 1 \, d\vartheta \leq \int_0^\pi e^{\vartheta(2a\omega\nu-\alpha)} \, d\vartheta \leq \frac{e^{\pi(2a\omega\nu-\alpha)}}{2a\omega\nu - \alpha} \tag{3.51}$$

to obtain the assertion. If $2a\omega\nu - \alpha < 0$, then

$$0 \geq \int_0^\pi e^{\vartheta(2a\omega\nu-\alpha)} - 1 \, d\vartheta \geq \int_0^\pi -1 \, d\vartheta = -\pi \tag{3.52}$$

yields the assertion.
Now we have to show the equality in (3.48). To this end, we apply the substitutions $s := \pi - t$ and $\sigma := \pi - \vartheta$. Using the trigonometric identities $\cos(\pi - t) = -\cos t$ and $\tan\frac{\pi-t}{2} = (\tan\frac{t}{2})^{-1}$ we obtain

$$\int_0^\pi \int_\vartheta^\pi \left(\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\right)^\alpha e^{2a\omega(\cos t - \cos\vartheta)} \, dt \, d\vartheta = \int_\pi^0 \int_\sigma^0 \left(\frac{\tan\frac{s}{2}}{\tan\frac{\sigma}{2}}\right)^\alpha e^{2a\omega(\cos\sigma - \cos s)} \, ds \, d\sigma. \qquad \square$$

For $0 < |2a\omega\nu - \alpha| < 1$ we can improve estimates (3.51) and (3.52). Assume that $0 < 2a\omega\nu - \alpha < 1$ holds. Then

$$\begin{aligned} \int_0^\pi e^{\vartheta(2a\omega\nu-\alpha)} - 1 \, d\vartheta &= \frac{1}{2a\omega\nu - \alpha}\Big(e^{(2a\omega\nu-\alpha)\pi} - 1 - \pi(2a\omega\nu - \alpha)\Big) \\ &\leq (2a\omega\nu - \alpha)\pi^2 e^{(2a\omega\nu-\alpha)\pi} \end{aligned} \tag{3.51'}$$

instead of equation (3.51) might provide a better result. For $-1 < 2a\omega\nu - \alpha < 0$ the estimate

$$\int_0^\pi e^{\vartheta(2a\omega\nu-\alpha)} - 1 \, d\vartheta \geq e^{\pi(2a\omega\nu-\alpha)} \int_0^\pi 1 - e^{-\pi(2a\omega\nu-\alpha)} d\vartheta = -\pi(1 - e^{\pi(2a\omega\nu-\alpha)}) \tag{3.52'}$$

yields a better result than the estimate (3.52).

Remark 3.28. There are other estimates for these integrals which are in general weaker, but have a simpler form than the estimate given in the previous lemma. Under the assumptions of lemma 3.27 the following inequalities hold:

$$\int_0^\pi \int_0^\vartheta \left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^\alpha e^{2a\omega(\cos\vartheta - \cos t)} \, dt d\vartheta \leq \frac{\pi^2 \, \Gamma(\omega)^2}{2(\alpha+1)}, \tag{3.53}$$

$$\int_0^\pi \int_0^\vartheta \left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^\alpha e^{2a\omega(\cos\vartheta - \cos t)} \, dt d\vartheta \leq \frac{\pi \, \Gamma(\omega)^2}{\alpha}, \tag{3.54}$$

where $\Gamma(\omega) := \sup\{e^{a\omega(\cos\vartheta - \cos t)} \; : \; 0 < t \leq \vartheta < \pi\}$.

Proof. To prove the inequalities we estimate the exponential functions by $\Gamma(\omega)$ and then use the estimates (3.45 a) and (3.45 c) to estimate the remaining integrand. $\square$

With the help of these rather technical lemmata we are now able to derive upper bounds for the norms of B^{-1} and B^{*-1}. Since the estimates depend on the wave number k, we add a subscript k to the operators B and B^* to emphasise their dependence on k. Likewise, we attach a subscript k to the eigenfunctions φ and ψ of B and B^* to indicate to which wave number k they belong (as mentioned after the proof of lemma 3.21, we omit the subscript $[\mu]$).

Remark 3.29. Let $\widetilde{B}_k^{-1}$ be the operator obtained from B_k^{-1} if we substitute $a\omega$ by $-a\omega$; the same notation applies to the adjoint operator and the eigenfunctions φ_k and ψ_k. Further, define the self-inverse, unitary map

$$R : \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta) \longrightarrow \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta), \qquad h \;\mapsto\; Rh := h(\pi - \cdot).$$

Then we for all $k \in \mathbb{R} \setminus (0,\pi)$ we have

$$\psi_k(\vartheta) = \widetilde{\psi}_{-k-1}(\pi - \vartheta) = \widetilde{\varphi}_{-k-1}(\vartheta), \tag{3.55}$$
$$\varphi_k(\vartheta) = \widetilde{\varphi}_{-k-1}(\pi - \vartheta) = \widetilde{\psi}_{-k-1}(\vartheta), \tag{3.56}$$

and for the operators B_k and B_k^* the following symmetry properties hold:

$$B_k = -\widetilde{B}^*_{-(k+1)}, \tag{3.57}$$
$$B_k = -R\,\widetilde{B}_{-(k+1)}R. \tag{3.58}$$

In particular, we have $\|B_k^{-1}\| = \|\widetilde{B}^{*-1}_{-(k+1)}\| = \|\widetilde{B}^{-1}_{-(k+1)}\|$.

Proof. The assertions concerning R and the eigenfunctions φ_k and ψ_k are clear if we recall that they are given by $\varphi_k(\vartheta) = \mathrm{e}^{a\omega\cos\vartheta}(\tan\frac{\vartheta}{2})^{-(k+\frac{1}{2})}$ and $\psi_k(\vartheta) = \mathrm{e}^{-a\omega\cos\vartheta}(\tan\frac{\vartheta}{2})^{k+\frac{1}{2}}$, see lemma 3.19. Equality (3.57) follows from

$$B_k = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta = -\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{-(k+1)+\frac{1}{2}}{\sin\vartheta} + (-a\omega)\sin\vartheta\Big) = -\widetilde{B}^*_{-(k+1)}.$$

Observing $R\frac{\mathrm{d}}{\mathrm{d}\vartheta}R = -\frac{\mathrm{d}}{\mathrm{d}\vartheta}$, we find

$$-R\widetilde{B}_{-(k+1)}R = -R\Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{-(k+1)+\frac{1}{2}}{\sin\vartheta} + (-a\omega)\sin\vartheta\Big)R = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta = B_k.$$

□

Lemma 3.30. *The following inequalities hold:*

$$\|B_k^{-1}\| = \|B_k^{*-1}\| \le C(\omega)\,\delta(k,\omega), \tag{3.59}$$
$$\|B_k^{-1}\| = \|B_k^{*-1}\| \le \frac{\pi}{2}\,\frac{\Gamma(k,\omega)}{\sqrt{|k+\frac{1}{2}|+\frac{1}{2}}}, \tag{3.60}$$
$$\|B_k^{-1}\| = \|B_k^{*-1}\| \le \Gamma(k,\omega)\sqrt{\frac{\pi}{2|k+\frac{1}{2}|}}, \tag{3.61}$$

where $C(\omega) = \mathrm{e}^{|a\omega(c_+-c_-)|}$ is defined in lemma 3.27,

$$\Gamma(k,\omega) := \begin{cases} \mathrm{e}^{2|a\omega|} & \textit{if } \operatorname{sign}(k+\frac{1}{2})\,a\omega \le 0, \\ 1 & \textit{if } \operatorname{sign}(k+\frac{1}{2})\,a\omega \ge 0, \end{cases}$$

is a generalisation of $\Gamma(\omega)$ *(see remark 3.28) and* δ *is the following extension of the function* δ *in lemma 3.27*

$$\delta(k,\omega) \;=\; \begin{cases} \frac{\mathrm{e}^{\pi|a\omega\nu-(k+\frac{1}{2})|}}{|a\omega\nu-(k+\frac{1}{2})|} & \textit{if} \quad \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)<0,\\ \sqrt{\frac{\pi}{2}\,|a\omega\nu-(k+\frac{1}{2})|^{-1}} & \textit{if} \quad \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)>0,\\ \frac{\pi}{\sqrt{2}} & \textit{if} \quad k+\frac{1}{2}-a\omega\nu=0. \end{cases}$$

A better estimate for $\|B_k^{-1}\|$, involving $|k+\frac{1}{2}-a\omega\nu|$ instead of its square root, is obtained in lemma 3.34.

Proof of lemma 3.30. It is well known that $\|B_k^{-1}\| = \|B_k^{*-1}\|$. Let $k\geq 0$. For every function $g\in\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ and every $\vartheta\in(0,\pi)$, its restriction $g|_{(0,\vartheta)}$ lies in $\mathscr{L}^2((0,\vartheta),\mathrm{d}\vartheta)$. Since also the function $t\mapsto \mathrm{e}^{a\omega(\cos\vartheta-\cos t)}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{k+\frac{1}{2}}$ is square integrable on $(0,\vartheta)$, we obtain by the Cauchy-Schwarz inequality, applied to the inner integral,

$$\begin{aligned} \|B_k^{-1}g\|_2^2 &= \int_0^\pi \left|\frac{1}{\psi(\vartheta)}\int_0^\vartheta \psi(t)g(t)\,\mathrm{d}t\right|^2\mathrm{d}\vartheta\\ &= \int_0^\pi\left|\int_0^\vartheta \mathrm{e}^{a\omega(\cos\vartheta-\cos t)}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{k+\frac{1}{2}} g(t)\,\mathrm{d}t\right|^2\mathrm{d}\vartheta\\ &\leq \int_0^\pi\left(\int_0^\vartheta \mathrm{e}^{2a\omega(\cos\vartheta-\cos t)}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{2k+1}\mathrm{d}t\right)\left(\int_0^\vartheta |g(t)|^2\,\mathrm{d}t\right)\mathrm{d}\vartheta\\ &\leq \|g\|_2^2\int_0^\pi\int_0^\vartheta \mathrm{e}^{2a\omega(\cos\vartheta-\cos t)}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{2k+1}\mathrm{d}t\,\mathrm{d}\vartheta. \end{aligned} \tag{3.62}$$

Now inequalities (3.49) with $\alpha=2k+1$, (3.53) and (3.54) together with the fact that equation (3.62) holds for all $g\in\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ show the assertion.
Now, let $k\leq -1$. The assertions can either be shown by an analogous computation or by a symmetry argument. To this end, we recall that $\|B_k^{-1}\| = \|\widetilde{B}_{-(k+1)}^{*-1}\|$ by remark 3.29, where $\widetilde{B}_k$ is obtained from B_k by substituting $a\omega$ with $-a\omega$. Note that $-(k-1)$ is nonnegative for $k\leq -1$. From

$$\begin{aligned} \operatorname{sign}(k+\tfrac{1}{2})\big(a\omega\nu-(k+\tfrac{1}{2})\big) &= -\operatorname{sign}(k+\tfrac{1}{2})\big((-a\omega)\nu-(-k-\tfrac{1}{2})\big)\\ &= \operatorname{sign}(-(k+1)+\tfrac{1}{2})\big((-a\omega)\nu-(-(k+1)+\tfrac{1}{2})\big),\\ |-a\omega\nu-(-(k+1)+\tfrac{1}{2})| &= |a\omega\nu-(k+\tfrac{1}{2})| \end{aligned}$$

for $k\in\mathbb{R}\setminus(-1,0)$ it follows that $\delta(k,\omega)=\delta(-(k+1),-\omega)$. Thus, by

$$\|B_k^{-1}\| \;=\; \|\widetilde{B}_{-(k+1)}^{-1}\| \;\leq\; C(-\omega)\,\delta(-(k+1),-\omega) \;=\; C(\omega)\,\delta(k,\omega),$$

assertion (3.59) is proved also in the case $k\leq -1$. Assertions (3.60) and (3.61) follow if we observe that $\Gamma(k,\omega)=\Gamma(-(k+1),-\omega)$. □

After this preparatory work we are now able to establish lower bounds for the modulus of the eigenvalues of the angular operator $\mathcal{A}$. These estimates are rather rough; we improve them in the next section.

Lemma 3.31. *Let λ be an eigenvalue of the angular operator $\mathcal{A}$. Then we have the following estimates:*

$$|\lambda| \geq -|am| + \frac{1}{C(\omega)\,\delta(k,\omega)}\ , \tag{3.63}$$

$$|\lambda| \geq -|am| + \frac{2}{\pi}\frac{\sqrt{|k+\frac{1}{2}|+\frac{1}{2}}}{\Gamma(k,\omega)}\ , \tag{3.64}$$

$$|\lambda| \geq -|am| + \frac{1}{\Gamma(k,\omega)}\sqrt{\frac{2|k+\frac{1}{2}|}{\pi}}\ , \tag{3.65}$$

with $C(\omega)$, $\delta(k,\omega)$ and $\Gamma(k,\omega)$ as defined in lemma 3.30.

Proof. Let λ be an eigenvalue of $\mathcal{A}$. Since we have $A = -D$ and $\|B^{-1}\| = \|B^{*-1}\|$, corollary 3.18 yields

$$|\lambda| \ \geq\ -\|D\| + \|B^{-1}\|^{-1}. \tag{3.66}$$

If we observe that $\|D\| = |am|$ and insert the estimates (3.59) through (3.61) into the formula above, we obtain all assertions. □

Remark 3.32. If we apply analytic perturbation theory to the angular operator and use m as the perturbation parameter (cf. section 3.2), then we obtain

$$\mu_n - |am| \ \leq\ \lambda_n \ \leq\ \mu_n + |am|,$$

where μ_n is the nth eigenvalue of $\mathcal{B} = \left(\begin{smallmatrix} 0 & B \\ B^* & 0\end{smallmatrix}\right)$, enumerated such that $\mu_n = \lambda_n$ for $a = 0$. Since B and B^* are boundedly invertible and the spectrum of BB^* consists of eigenvalues only, it follows that $\mu_1 = \|B^{-1}\|^{-1}$. Hence, if (3.66) yields a positive lower bound then $\|B^{-1}\|^{-1} > |am|$ and λ_1 is the first positive eigenvalue of $\mathcal{A}$, hence the estimate (3.63) can also be obtained from standard perturbation theory. ◊

We want to add some comments on the estimates in lemma 3.31. It is not hard to see that for large $|k|$ the bound (3.65) is larger, i.e., sharper, than bound (3.64). However, in section 5.1.1, the method by which the latter estimate was obtained (namely estimation of the tangent functions with a rational function) turns out to be useful when the behaviour of elements $f \in \mathcal{D}(B^*)$ and $g \in \mathcal{D}(B)$ in a neighbourhood of 0 and π is investigated. It is rather hard to compare in general these two estimates with the first one, (3.63), since the exponential functions involved in the expressions differ. However, in the case $\operatorname{sign}(k+\frac{1}{2})(a\omega\nu - (k+\frac{1}{2})) < 0$ the first expression should yield better estimates than the other two if a is considered the perturbation parameter while all other quantities are fixed. In the following, we always work with estimate (3.63).

3.3.3 Estimates of $\|B^{-1}\|$ by an iteration method

The first rough estimates for the eigenvalues λ obtained in lemma 3.31 are only of order $\sqrt{k}$ whereas a bound of order k could be expected from the case $a = 0$. Indeed, an estimate of order k can be achieved if we improve the estimate for the norm of B^{-1}. For this purpose, we estimate the norms $\|(B^{-1}B^{*-1})^n\|$, $n \in \mathbb{N}$, from which we then derive estimates for $\|B^{-1}\|$. By corollary 3.18, all eigenvalues λ of $\mathcal{A}$ satisfy $|\lambda| \geq -|am| + \|B^{-1}\|^{-1}$. Thus a bound for $|\lambda|$ resulting from an estimate $\|B^{-1}\| \leq b$ is the larger, and therefore the better, the smaller the bound b is. In lemma 3.34 such bounds b are established by using the explicit formulae for B^{-1} and B^{*-1} given in lemma 3.21.

We obtain various results, according to which of the estimates provided in lemma 3.24 we use to estimate the quotients of type $\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}$ appearing in the formulae for B^{-1} and B^{*-1} and how the exponential functions are treated. A priori it is not clear which kind of estimate of $\|B^{-1}\|$ yields the largest lower bound for the modulus of the eigenvalues of the angular operator $\mathcal{A}$. Sample plots of the bounds for $\|B^{-1}\|^{-1}$ are given in figures 3.2 and 3.3 at the end of this section. Again, we attach a subscript k to the operators B and B^* to indicate to which wave number k they belong.

The next technical lemma is used in the proof of lemma 3.34.

Lemma 3.33. *Let $k \geq 0$. For every $n \in \mathbb{N}$, $\eta \in \mathbb{R}$ and $s_0 \in (0, \pi)$ the following estimates hold.*

(i)
$$\int_{s_0}^{\pi}\int_{0}^{t_1}\int_{s_1}^{\pi}\cdots\int_{s_{n-1}}^{\pi}\prod_{j=1}^{n}\Big(\frac{\pi-t_j}{\pi-s_{j-1}}\Big)^{k+\frac12}\prod_{j=1}^{n-1}\Big(\frac{s_j}{t_j}\Big)^{k+\frac12}\Bigg(\int_0^{t_n}\Big(\frac{s_n}{t_n}\Big)^{2k+1}\mathrm{d}s_n\Bigg)^{\frac12}\mathrm{d}t_n\,\mathrm{d}s_{n-1}\ldots\mathrm{d}s_1\,\mathrm{d}t_1$$
$$\leq \frac{1}{\sqrt{\pi(2k+2)}}\Bigg(\frac{\pi}{k+\frac32}\Bigg)^{2n-1}(\pi-s_0)$$

(ii)
$$\int_{s_0}^{\pi}\int_{0}^{t_1}\int_{s_1}^{\pi}\cdots\int_{s_{n-1}}^{\pi}\Bigg(\prod_{j=1}^{n}\mathrm{e}^{-2\eta t_j}\Bigg)\Bigg(\prod_{j=1}^{n-1}\mathrm{e}^{2\eta s_j}\Bigg)\Bigg(\int_0^{t_n}\mathrm{e}^{2\eta s_n}\,\mathrm{d}s_n\Bigg)^{\frac12}\mathrm{d}t_n\,\mathrm{d}s_{n-1}\ldots\mathrm{d}s_1\,\mathrm{d}t_1$$
$$\leq \begin{cases}\eta^{-(2n-1)}(2\eta)^{-\frac12}\mathrm{e}^{-\eta s_0} & \text{if } \eta>0,\\ \frac{\mathrm{e}^{-2n\eta\pi}}{|2\eta|^{2n-1}\sqrt{|2\eta|}} & \text{if } \eta<0,\\ \frac{4}{3\sqrt{\pi}}\left(\frac12\pi^2\right)^n & \text{if } \eta=0.\end{cases}$$

Proof. Both Assertions are proved by induction with respect to n.

(i) For $n=1$ we have

$$\int_{s_0}^{\pi}\Big(\frac{\pi-t_1}{\pi-s_0}\Big)^{k+\frac12}\Bigg(\int_0^{t_1}\Big(\frac{s_1}{t_1}\Big)^{2k+1}\mathrm{d}s_1\Bigg)^{\frac12}\mathrm{d}t_1 = \frac{1}{\sqrt{2k+2}}\int_{s_0}^{\pi}\Big(\frac{\pi-t_1}{\pi-s_0}\Big)^{k+\frac12}t_1^{\frac12}\,\mathrm{d}t_1$$
$$\leq \frac{\sqrt{\pi}}{\sqrt{2k+2}}\int_{s_0}^{\pi}\Big(\frac{\pi-t_1}{\pi-s_0}\Big)^{k+\frac12}\mathrm{d}t_1 = \frac{1}{\sqrt{\pi(2k+2)}}\,\frac{\pi}{k+\frac32}(\pi-s_0).$$

Now assume that the assertion holds for an $n \in \mathbb{N}$. Then we have

$$\int_{s_0}^{\pi}\int_{0}^{t_1}\cdots\int_{0}^{t_n}\int_{s_n}^{\pi}\prod_{j=1}^{n+1}\Big(\frac{\pi-t_j}{\pi-s_{j-1}}\Big)^{k+\frac12}\ \prod_{j=1}^{n}\Big(\frac{s_j}{t_j}\Big)^{k+\frac12}\Bigg(\int_0^{t_{n+1}}\Big(\frac{s_{n+1}}{t_{n+1}}\Big)^{2k+1}\,\mathrm{d}s_{n+1}\Bigg)^{\frac12}\ \mathrm{d}t_{n+1}\,\mathrm{d}s_n\ldots\,\mathrm{d}s_1\,\mathrm{d}t_1$$

$$\le\ \frac{1}{\sqrt{\pi(2k+2)}}\Big(\frac{\pi}{k+\frac32}\Big)^{2n-1}\int_{s_0}^{\pi}\int_0^{t_1}\Big(\frac{\pi-t_1}{\pi-s_0}\Big)^{k+\frac12}\ \Big(\frac{s_1}{t_1}\Big)^{k+\frac12}\ \underbrace{(\pi-s_1)}_{\le\pi}\,\mathrm{d}s_1\,\mathrm{d}t_1$$

$$\le\ \frac{1}{\sqrt{\pi(2k+2)}}\Big(\frac{\pi}{k+\frac32}\Big)^{2n}\int_{s_0}^{\pi}\Big(\frac{\pi-t_1}{\pi-s_0}\Big)^{k+\frac12}\ \underbrace{t_1}_{\le\pi}\,\mathrm{d}t_1$$

$$\le\ \frac{1}{\sqrt{\pi(2k+2)}}\Big(\frac{\pi}{k+\frac32}\Big)^{2n+1}(\pi-s_0)$$

which proves the claim for $n+1$.

(ii) We start with the case $\eta>0$. In the proof we repeatedly use the relations

$$0\ \le\ \int_0^t \mathrm{e}^{\eta s}\,\mathrm{d}s=\frac{1}{\eta}\,(\mathrm{e}^{\eta t}-1)\ \le\ \frac{1}{\eta}\,\mathrm{e}^{\eta t},\qquad t\in(0,\pi),$$
$$0\ \le\ \int_s^\pi \mathrm{e}^{-\eta t}\,\mathrm{d}t=\frac{1}{\eta}\,(\mathrm{e}^{-\eta s}-\mathrm{e}^{-\eta\pi})\ \le\ \frac{1}{\eta}\,\mathrm{e}^{-\eta s},\qquad s\in(0,\pi).$$

For $n=1$ it is easy to see that

$$\int_{s_0}^{\pi}\mathrm{e}^{-2\eta t_1}\Bigg(\int_0^{t_1}\mathrm{e}^{2\eta s_1}\,\mathrm{d}s_1\Bigg)^{\frac12}\mathrm{d}t_1\ \le\ \frac{1}{\sqrt{2\eta}}\int_{s_0}^{\pi}\mathrm{e}^{-\eta t_1}\,\mathrm{d}t_1\ \le\ \frac{1}{\eta\sqrt{2\eta}}\,\mathrm{e}^{-\eta s_0},$$

hence the assertion holds. Assume the assertion to be valid for some $n\in\mathbb{N}$. Then

$$\int_{s_0}^{\pi}\int_0^{t_1}\cdots\int_{s_{n-1}}^{\pi}\int_0^{t_n}\int_{s_n}^{\pi}\Bigg(\prod_{j=1}^{n+1}\mathrm{e}^{-2\eta t_j}\Bigg)\Bigg(\prod_{j=1}^{n}\mathrm{e}^{2\eta s_j}\Bigg)\Bigg(\int_0^{t_{n+1}}\mathrm{e}^{2\eta s_{n+1}}\,\mathrm{d}s_{n+1}\Bigg)^{\frac12}\mathrm{d}t_{n+1}\,\mathrm{d}s_n\ldots\,\mathrm{d}s_1\,\mathrm{d}t_1$$
$$\le\ \int_{s_0}^{\pi}\int_0^{t_1}\mathrm{e}^{-2\eta t_1}\mathrm{e}^{2\eta s_1}\eta^{-(2n-1)}(2\eta)^{-\frac12}\,\mathrm{e}^{-\eta s_1}\,\mathrm{d}s_1\,\mathrm{d}t_1$$
$$\le\ \eta^{-2n}\frac{1}{\sqrt{2\eta}}\int_{s_0}^{\pi}\mathrm{e}^{-\eta t_1}\,\mathrm{d}t_1\ \le\ \eta^{-(2n+1)}(2\eta)^{-\frac12}\mathrm{e}^{-\eta s_0}$$

shows the assertion also for $n+1$. Now we show the assertion for $\eta<0$. If we use

$$0\ \le\ \int_0^t \mathrm{e}^{2\eta s}\,\mathrm{d}s\ =\ \frac{1}{2\eta}(\mathrm{e}^{2\eta t}-1)\ =\frac{1}{2|\eta|}(1-\mathrm{e}^{2\eta t})\ \le\ \frac{1}{2|\eta|},\qquad t\in(0,\pi)$$
$$0\ \le\ \int_s^\pi \mathrm{e}^{-2\eta t}\,\mathrm{d}s\ =\ \frac{1}{2\eta}(\mathrm{e}^{-2\eta s}-\mathrm{e}^{-2\eta\pi})\ =\frac{1}{2|\eta|}(\mathrm{e}^{-2\eta\pi}-\mathrm{e}^{-2\eta s})\ \le\ \frac{1}{2|\eta|}\,\mathrm{e}^{-2\eta\pi}\qquad s\in(0,\pi),$$

we find for $n = 1$

$$\int_{s_0}^{\pi} \mathrm{e}^{-2\eta t_1} \left(\int_0^{t_1} \mathrm{e}^{2\eta s_1}\, \mathrm{d}s_1 \right)^{\frac{1}{2}} \mathrm{d}t_1 \;\le\; \frac{1}{\sqrt{|2\eta|}} \int_{s_0}^{\pi} \mathrm{e}^{-2\eta t_1}\, \mathrm{d}t_1 \;\le\; \frac{\mathrm{e}^{-2\eta\pi}}{|2\eta|\sqrt{|2\eta|}}.$$

Assuming that the assertion holds for some $n \in \mathbb{N}$ we find for $n+1$

$$\begin{aligned} &\int_{s_0}^{\pi}\int_0^{t_1} \cdots \int_{s_{n-1}}^{\pi}\int_0^{t_n}\int_{s_n}^{\pi} \left(\prod_{j=1}^{n+1} \mathrm{e}^{-2\eta t_j}\right)\left(\prod_{j=1}^{n} \mathrm{e}^{2\eta s_j}\right)\left(\int_0^{t_{n+1}} \mathrm{e}^{2\eta s_{n+1}}\, \mathrm{d}s_{n+1}\right)^{\frac{1}{2}} \mathrm{d}t_{n+1} r d s_n \ldots \mathrm{d}s_1\, \mathrm{d}t_1 \\ &\qquad\le\; \frac{\mathrm{e}^{-2n\eta\pi}}{|2\eta|^{2n-1}\sqrt{|2\eta|}} \int_{s_0}^{\pi}\int_0^{t_1} \mathrm{e}^{-2\eta t_1}\mathrm{e}^{2\eta s_1}\, \mathrm{d}s_1\, \mathrm{d}t_1 \\ &\qquad\le\; \frac{\mathrm{e}^{-2n\eta\pi}}{|2\eta|^{2n}\sqrt{|2\eta|}} \int_{s_0}^{\pi} \mathrm{e}^{-2\eta t_1}\, \mathrm{d}t_1 \;\le\; \frac{\mathrm{e}^{-2(n+1)\eta\pi}}{|2\eta|^{2n+1}\sqrt{|2\eta|}}. \end{aligned}$$

It remains to show the assertion for the case $\eta = 0$. Using the estimates

$$\int_{s_{n-1}}^{\pi} \left(\int_0^{t_n} \mathrm{d}s_n \right)^{\frac{1}{2}} \mathrm{d}t_n \;=\; \int_{s_{n-1}}^{\pi} t_n^{\frac{1}{2}}\, \mathrm{d}t_n \;=\; \tfrac{2}{3}\,(\pi^{\frac{3}{2}} - s_{n-1}^{\frac{3}{2}}) \;\le\; \tfrac{2}{3}\,\pi^{\frac{3}{2}}, \qquad s_{n-1} \in (0,\pi),$$

$$\int_{s}^{\pi}\int_0^{t} \mathrm{d}s'\mathrm{d}t \;=\; \int_{s}^{\pi} t\, \mathrm{d}t \;=\; \tfrac{1}{2}\,(\pi^2 - s^2) \;\le\; \tfrac{1}{2}\,\pi^2 \qquad s \in (0,\pi),$$

we find

$$\begin{aligned} &\int_{s_0}^{\pi}\int_0^{t_1}\int_{s_1}^{\pi} \cdots \int_{s_{n-1}}^{\pi} \left(\int_0^{t_n} \mathrm{d}s_n \right)^{\frac{1}{2}} \mathrm{d}t_n\, \mathrm{d}s_{n-1} \ldots \mathrm{d}s_1\, \mathrm{d}t_1 \\ &\le\; \tfrac{2}{3}\pi^{\frac{3}{2}} \int_{s_0}^{\pi}\int_0^{t_1}\int_{s_1}^{\pi} \cdots \int_{s_{n-2}}^{\pi}\int_0^{t_{n-1}} \mathrm{d}s_{n-1}\, \mathrm{d}t_{n-1} \ldots \mathrm{d}s_1\, \mathrm{d}t_1 \;\le\; \tfrac{2}{3}\,\pi^{\frac{3}{2}}\,(\tfrac{1}{2}\pi^2)^{n-1} \;=\; \tfrac{4}{3\sqrt{\pi}}\,(\tfrac{1}{2}\,\pi^2)^{n}. \quad \square \end{aligned}$$

Lemma 3.34. *For every $n \in \mathbb{N}$ and $k \in \mathbb{R} \setminus (-1,0)$, we have $\|B_k^{-1}B_k^{*-1}\| = \|B_k^{*-1}B_k^{-1}\|$ and the following estimates hold*

$$\|(B_k^{-1}B_k^{*-1})^n\| \;\le\; \Gamma(k,\omega)^{2n}\, \frac{\pi^2}{2\sqrt{2\pi(|k+\frac{1}{2}|+1)}} \left(\frac{\pi}{|k+\frac{1}{2}|+1} \right)^{2n-1}, \tag{3.67}$$

$$\|(B_k^{-1}B_k^{*-1})^n\| \;\le\; \sqrt{\frac{|k+\frac{1}{2}|}{2}} \left(\frac{\Gamma(k,\omega)}{|k+\frac{1}{2}|} \right)^{2n} \tag{3.68}$$

with $\Gamma(k,\omega) = \begin{cases} \mathrm{e}^{2|a\omega|} & \text{if } \operatorname{sign}(k+\frac{1}{2})\, a\omega \le 0, \\ 1 & \text{if } \operatorname{sign}(k+\frac{1}{2})\, a\omega \ge 0 \end{cases}$ *as defined in lemma 3.30.*

In addition, we have the estimates

$$
\begin{aligned}
&\|(B_k^{-1}B_k^{*-1})^n\| \\
&\leq \begin{cases}
\dfrac{C(\omega)^{2n}}{|a\omega\nu-(k+\frac{1}{2})|^{2n}}\left(\dfrac{\mathrm{e}^{\pi|a\omega\nu-(k+\frac{1}{2})|}}{2}\right)^{2n} & \text{if } \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)<0, \quad (3.69\,\mathrm{a})\\
\dfrac{C(\omega)^{2n}}{|a\omega\nu-(k+\frac{1}{2})|^{2n-\frac{1}{2}}}\sqrt{\dfrac{\pi}{2}} & \text{if } \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)>0, \quad (3.69\,\mathrm{b})\\
\dfrac{4}{3}\,C(\omega)^{2n}\left(\dfrac{1}{2}\,\pi^2\right)^n & \text{if } (k+\frac{1}{2})-a\omega\nu=0, \quad (3.69\,\mathrm{c})
\end{cases}
\end{aligned}
$$

with $C(\omega)=\mathrm{e}^{|a\omega(c_+-c_-)|}$ *as defined in lemma 3.30. Furthermore, we have the following upper bounds for the norm of* B_k^{-1}*:*

$$\|B_k^{-1}\| \;=\; \|B_k^{*-1}\| \;\leq\; \frac{\pi\,\Gamma(k,\omega)}{|k+\frac{1}{2}|+1}, \tag{3.70}$$

$$\|B_k^{-1}\| \;=\; \|B_k^{*-1}\| \;\leq\; \frac{\Gamma(k,\omega)}{|k+\frac{1}{2}|} \tag{3.71}$$

and

$$
\begin{aligned}
&\|B_k^{-1}\| \;=\; \|B_k^{*-1}\| \\
&\leq \begin{cases}
\dfrac{C(\omega)}{|a\omega\nu-(k+\frac{1}{2})|}\;\dfrac{\mathrm{e}^{\pi|a\omega\nu-(k+\frac{1}{2})|}}{2} & \text{if } \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)<0, \quad (3.72\,\mathrm{a})\\
\dfrac{C(\omega)}{|a\omega\nu-(k+\frac{1}{2})|} & \text{if } \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)>0, \quad (3.72\,\mathrm{b})\\
\dfrac{\pi}{\sqrt{2}}\,C(\omega) & \text{if } (k+\frac{1}{2})-a\omega\nu=0. \quad (3.72\,\mathrm{c})
\end{cases}
\end{aligned}
$$

Proof. The equality $\|(B_k^{-1}B_k^{*-1})^n\|=\|(B_k^{*-1}B_k^{-1})^n\|$ can either be seen by exploiting the symmetry properties of B_k^{-1} stated in remark 3.29 (a straightforward calculation shows $B_k^{*-1}B_k^{-1}=RB_k^{-1}B_k^{*-1}R$); or we use that

$$
\begin{aligned}
\|(B_k^{-1}B_k^{*-1})^n\| \;&=\; \|B_k^{-1}B_k^{*-1}\|^n \;=\; \Big(\sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}}|(B_k^{-1}B_k^{*-1}x,\,x)|\Big)^n \\
&=\; \Big(\sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}}|(B_k^{*-1}x,\,B_k^{*-1}x)|\Big)^n \;=\; \|B_k^{*-1}\|^{2n},
\end{aligned} \tag{3.73}
$$

where the first two equalities hold because $B_k^{-1}B_k^{*-1}$ is a bounded selfadjoint operator on the Hilbert space $\mathcal{H}=\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$. Analogously, we obtain $\|(B_k^{*-1}B_k)^n\|=\|B_k^{-1}\|^{2n}$, thus equality $\|(B_k^{-1}B_k^{*-1})^n\|=\|(B_k^{*-1}B_k^{-1})^n\|$ follows now from $\|B^{-1}\|=\|B^{*-1}\|$.
Furthermore, (3.73) shows that formulae (3.70), (3.71) and (3.72 a)–(3.72 c) are direct consequences of (3.67), (3.68) and (3.69 a)–(3.69 c), respectively, if in (3.73) we solve for $\|B_k^{*-1}\|$ and then let $n\to\infty$.

In the case $k \leq -1$, all assertions of the lemma can either be shown by calculations analogous to those carried out below for the case $k \geq 0$; or they can be derived from the results in the case $k \geq 0$ by symmetry arguments similar to those in the proof of lemma 3.30. Thus it remains to prove (3.67), (3.68) and (3.69 a)–(3.69 c) for $k \geq 0$. So fix $k \geq 0$. Then for every $f \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ and every $s_0 \in (0,\pi)$ we obtain by lemma 3.21, with the functions $\varphi := \varphi_{[0]}$ and $\psi := \psi_{[0]}$ defined in that lemma, that

$$|(B_k^{*-1}B_k^{-1})^n f(s_0)| = \left| \int_{s_0}^{\pi}\int_0^{t_1} \ldots \int_{s_{n-1}}^{\pi}\int_0^{t_n} \left(\prod_{j=1}^{n} \frac{\varphi(t_j)}{\varphi(s_{j-1})} \frac{\psi(s_j)}{\psi(t_j)} \right) f(s_n)\,\mathrm{d}s_n\,\mathrm{d}t_n \ldots \mathrm{d}s_1\,\mathrm{d}t_1 \right|$$

$$\leq \int_{s_0}^{\pi}\int_0^{t_1} \ldots \int_{s_{n-1}}^{\pi} \left(\prod_{j=1}^{n} \frac{\varphi(t_j)}{\varphi(s_{j-1})} \right) \left(\prod_{j=1}^{n-1} \frac{\psi(t_j)}{\psi(s_{j-1})} \right) \left(\int_0^{t_n} \frac{\psi(s_n)}{\psi(t_n)} \, |f(s_n)|\,\mathrm{d}s_n \right) \mathrm{d}t_n \ldots \mathrm{d}s_1\,\mathrm{d}t_1$$

$$\leq \|f\|_2 \int_{s_0}^{\pi}\int_0^{t_1} \ldots \int_{s_{n-1}}^{\pi} \left(\prod_{j=1}^{n} \frac{\varphi(t_j)}{\varphi(s_{j-1})} \right) \left(\prod_{j=1}^{n-1} \frac{\psi(t_j)}{\psi(s_{j-1})} \right) \left(\int_0^{t_n} \frac{\psi(s_n)}{\psi(t_n)} \right)^{\frac{1}{2}} \mathrm{d}t_n \ldots \mathrm{d}s_1\,\mathrm{d}t_1 \qquad (3.74)$$

where in the last step we have used that for all $t_n \in (0,\pi)$ the restricted function $|\,f|_{(0,t_n)}|$ lies in $\mathscr{L}^2((0,t_n), \mathrm{d}s_n)$ and that $(0,t_n) \to \mathbb{R}$, $s_n \mapsto \frac{\psi(s_n)}{\psi(t_n)}$ is bounded, so that the Cauchy-Schwarz inequality applied to the innermost integral yields

$$\int_0^{t_n} \frac{\psi(s_n)}{\psi(t_n)} \, |f(s_n)|\,\mathrm{d}s_n \leq \left(\int_0^{t_n} |f(s_n)|^2\,\mathrm{d}s_n \right)^{\frac{1}{2}} \left(\int_0^{t_n} \left(\frac{\psi(s_n)}{\psi(t_n)} \right)^2 \mathrm{d}s_n \right)^{\frac{1}{2}}$$
$$\leq \|f\|_2 \left(\int_0^{t_n} \left(\frac{\psi(s_n)}{\psi(t_n)} \right)^2 \mathrm{d}s_n \right)^{\frac{1}{2}}.$$

Observe that in (3.74) we have $0 < s_{j-1} \leq t_j < \pi$ and $0 < s_j \leq t_j < \pi$ for each $j \in \{1, \ldots, n\}$. Hence we can apply lemma 3.24 and corollary 3.26 to estimate the expression (3.74).
First, we prove (3.67) with the help of (3.45 a) and (3.45 b). For $j \in \{1, \ldots, n\}$ we have

$$\frac{\varphi(t_j)}{\varphi(s_{j-1})} = \left(\frac{\tan \frac{s_{j-1}}{2}}{\tan \frac{t_j}{2}} \right)^{k+\frac{1}{2}} \mathrm{e}^{a\omega(\cos t_j - \cos s_{j-1})} \leq \left(\frac{\pi - t_j}{\pi - s_{j-1}} \right)^{k+\frac{1}{2}} \Gamma(k,\omega),$$
$$\frac{\psi(s_j)}{\psi(t_j)} = \left(\frac{\tan \frac{s_j}{2}}{\tan \frac{t_j}{2}} \right)^{k+\frac{1}{2}} \mathrm{e}^{a\omega(\cos t_j - \cos s_j)} \leq \left(\frac{s_j}{t_j} \right)^{k+\frac{1}{2}} \Gamma(k,\omega).$$

With these estimates and lemma 3.33(i) it follows from (3.74) that

$$|(B_k^{*-1}B_k^{-1})^n f(s_0)| \leq \|f\|_2\,\Gamma(k,\omega)^{2n} \frac{1}{\sqrt{\pi(2k+2)}} \left(\frac{\pi}{k+\frac{3}{2}} \right)^{2n-1} (\pi - s_0).$$

Taking the $\mathscr{L}^2$-norm on both sides gives

$$\|(B_k^{*-1}B_k^{-1})^n f\|_2 \leq \|f\|_2\,\Gamma(k,\omega)^{2n} \frac{\pi^2}{2\sqrt{\pi(2k+2)}} \left(\frac{\pi}{k+\frac{3}{2}} \right)^{2n-1},$$

thus (3.67) is proved.

For the proof of (3.69 a)–(3.69 c) we use estimates (3.45 c) and (3.47) to obtain

$$\frac{\varphi(t_j)}{\varphi(s_{j-1})} = \left(\frac{\tan\frac{s_{j-1}}{2}}{\tan\frac{t_j}{2}}\right)^{k+\frac{1}{2}} \mathrm{e}^{a\omega(\cos t_j-\cos s_{j-1})} \le \mathrm{e}^{s_{j-1}(k+\frac{1}{2}-a\omega\nu)}\mathrm{e}^{t_j(-(k+\frac{1}{2})+a\omega\nu)}\mathrm{e}^{|a\omega(c_+-c_-)|},$$

$$\frac{\psi(s_j)}{\psi(t_j)} = \left(\frac{\tan\frac{s_j}{2}}{\tan\frac{t_j}{2}}\right)^{k+\frac{1}{2}} \mathrm{e}^{a\omega(\cos t_j-\cos s_j)} \le \mathrm{e}^{s_j(k+\frac{1}{2}-a\omega\nu)}\mathrm{e}^{t_j(-(k+\frac{1}{2})+a\omega\nu)}\mathrm{e}^{|a\omega(c_+-c_-)|}.$$

Thus we have

$$|(B_k^{*-1}B_k^{-1})^n f(s_0)| \le \|f\|_2\,\mathrm{e}^{2n|a\omega(c_+-c_-)|}\,\mathrm{e}^{s_0(k+\frac{1}{2}-a\omega\nu)} \times$$
$$\int_{s_0}^{\pi}\int_0^{t_1}\cdots\int_{s_n}^{\pi}\left(\prod_{j=1}^{n}\mathrm{e}^{-2t_j(k+\frac{1}{2}-a\omega\nu)}\right)\left(\prod_{j=1}^{n-1}\mathrm{e}^{2s_j(k+\frac{1}{2}-a\omega\nu)}\right)\left(\int_0^{t_n}\mathrm{e}^{2s_n(k+\frac{1}{2}-a\omega\nu)}\,\mathrm{d}s_n\right)^{\frac{1}{2}}\mathrm{d}t_n\ldots\mathrm{d}s_1\,\mathrm{d}t_1$$

to which we can apply lemma 3.33(ii) with $\eta=\operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)$. Hence, for $s_0\in(0,\pi)$ we have in the case $\operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)<0$

$$|(B_k^{*-1}B_k^{-1})^n f(s_0)| \le \|f\|_2\,\mathrm{e}^{2n|a\omega(c_+-c_-)|}\,\mathrm{e}^{s_0(k+\frac{1}{2}-a\omega\nu)}\,\frac{\mathrm{e}^{-2n(k+\frac{1}{2}-a\omega\nu)\pi}}{(2|k+\frac{1}{2}-a\omega\nu|)^{2n-\frac{1}{2}}}\,; \tag{3.69 a$'$}$$

in the case $\operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)>0$ we have

$$|(B_k^{*-1}B_k^{-1})^n f(s_0)| \le \frac{1}{\sqrt{2}}\|f\|_2\,\mathrm{e}^{2n|a\omega(c_+-c_-)|}\,(k+\tfrac{1}{2}-a\omega\nu)^{-2n+\frac{1}{2}}\,; \tag{3.69 b$'$}$$

and for $k+\frac{1}{2}-a\omega\nu=0$ we have

$$|(B_k^{*-1}B_k^{-1})^n f(s_0)| \le \|f\|_2\,\mathrm{e}^{2n|a\omega(c_+-c_-)|}\,\frac{4}{3\sqrt{\pi}}\left(\frac{1}{2}\,\pi^2\right)^n. \tag{3.69 c$'$}$$

Taking the $\mathscr{L}^2$-norm on both sides of (3.69 a′)–(3.69 c′) shows (3.69 a)–(3.69 c).
Estimate (3.68) is obtained from (3.69 b) if we set $\nu=0$ and substitute $C(\omega)$ by $\Gamma(k,\omega)$. □

Note that both the estimates (3.71) and (3.72 a)–(3.72 c) are obtained by estimating the quotient of tangent function by exponential functions. Although it seems that often the estimates (3.72 a)–(3.72 c) provide stronger lower bounds than (3.71), the advantage of the latter is that it exhibits no exponential decay with respect to $a\omega$ for $\operatorname{sign}(k+\frac{1}{2})(a\omega\nu-(k+\frac{1}{2}))<0$.
It is not easy to decide in general which of the estimates (3.70), (3.71) and (3.72 a)–(3.72 c) yields the best lower bound for the modulus of eigenvalues of the angular operator. It seems that for $\operatorname{sign}(k+\frac{1}{2})a\omega\ge 0$ estimates (3.70) and (3.71) provide better results since there is no exponential decay with respect to a. On the other hand, figure 3.2 shows that, for small $|a|$, also in this case the estimates (3.72 a)–(3.72 c) provide larger lower bounds for $\|B^{-1}\|^{-1}$ than the exponentially nondecaying solutions. An other sample plot is given in figure 3.3.

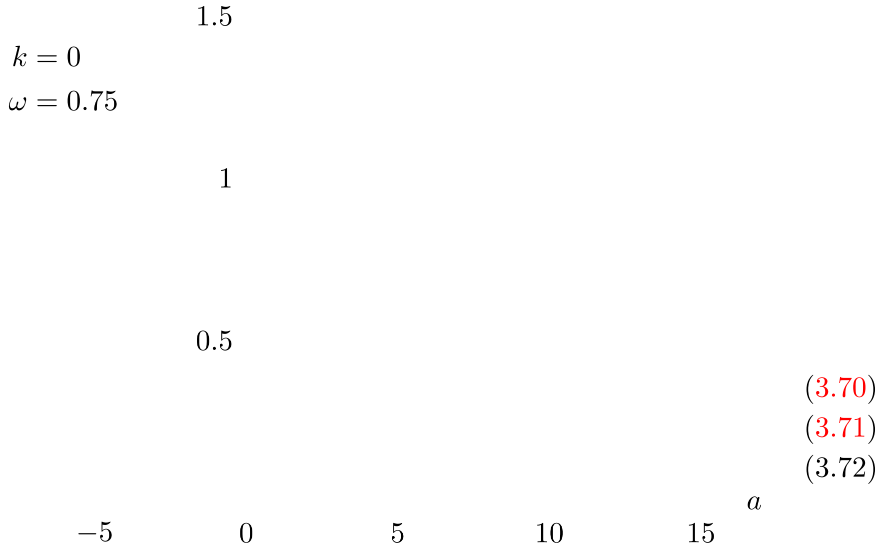

Figure 3.2. The plots show the estimates for $\|B^{-1}\|^{-1}$ given by (3.70), (3.71) and (3.72 a)–(3.72 c) in lemma 3.34 for $\omega = 0.75$ and the wave number $k = 0$.

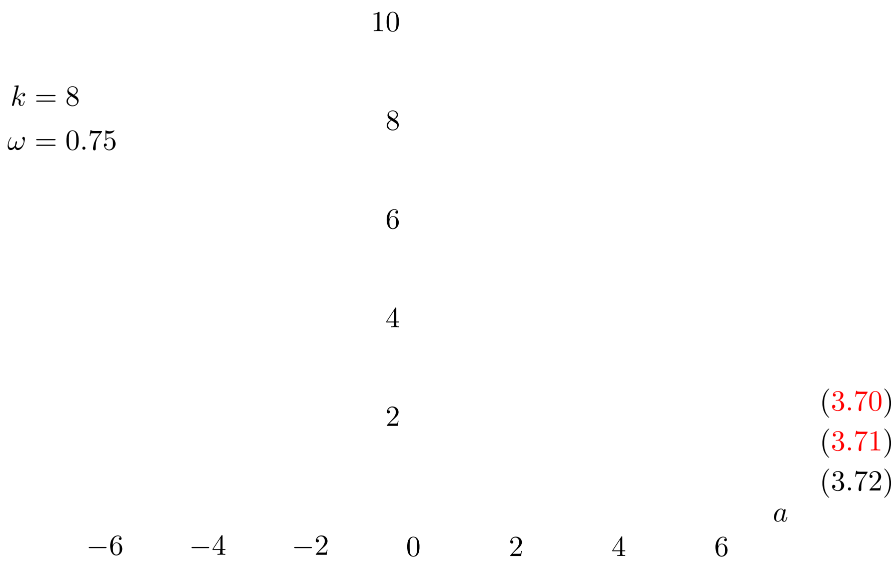

Figure 3.3. The plots show the estimates for $\|B^{-1}\|^{-1}$ given by (3.70), (3.71) and (3.72 a)–(3.72 c) in lemma 3.34 for $\omega = 0.75$ and the wave number $k = 8$. Recall that we have $|\lambda| \geq -|am| + \|B^{-1}\|^{-1}$ for the eigenvalues λ of the angular operator such that a larger bound for $\|B^{-1}\|^{-1}$ provides a stronger result for λ.

3.3.4 Lower bounds for the modulus of the eigenvalues of $\mathcal{A}$

With the help of lemma 3.34 we can improve the lower bound of lemma 3.31 for the absolute value of the eigenvalues λ of the angular operator $\mathcal{A}$.

Theorem 3.35. *For every eigenvalue λ of the angular operator $\mathcal{A}$ we have*

$$|\lambda| \;\geq\; \lambda_G \;:=\; -|am| + \widetilde{\delta}(\omega), \tag{3.75}$$

with

$$\widetilde{\delta}(\omega) \;:=\; \begin{cases} 2C(\omega)^{-1}\,\mathrm{e}^{-\pi|a\omega\nu-(k+\frac{1}{2})|}\,\big|a\omega\nu-(k+\frac{1}{2})\big| & \textit{if}\;\; \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)<0,\\ C(\omega)^{-1}\,|a\omega\nu-(k+\frac{1}{2})| & \textit{if}\;\; \operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)>0,\\ C(\omega)^{-1}\,\frac{\sqrt{2}}{\pi} & \textit{if}\;\; k+\frac{1}{2}-a\omega\nu=0. \end{cases}$$

Proof. The bound (3.75) follows from $|\lambda| \geq -|am| + \|B^{-1}\|^{-1}$ and $\|B^{-1}\| \leq \widetilde{\delta}(k,\omega)^{-1}$ by (3.72 a)–(3.72 c). □

If we compare the result of this theorem with estimate (3.63) in lemma 3.31, we find that the estimate did not improve in the case $\operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)<0$, where the estimate was already of order k. In the case $\operatorname{sign}(k+\frac{1}{2})(k+\frac{1}{2}-a\omega\nu)>0$, however, iteration has improved the estimate insofar as now the bound is also of order k instead of only of order $\sqrt{k}$.

Remark 3.36. Of course, also estimates (3.70) or (3.71) can be used in theorem 3.35. Then we obtain the following lower bounds for the eigenvalues λ:

$$|\lambda| \;\geq\; \lambda_G^{[\mathrm{lin}]} \;:=\; -\,|am| + \tfrac{1}{\pi\,\Gamma(k,\omega)}\,(|k+\tfrac{1}{2}|+1), \tag{3.76}$$

$$|\lambda| \;\geq\; \lambda_G^{[\mathrm{exp}]} \;:=\; -\,|am| + \tfrac{1}{\Gamma(k,\omega)}\,|k+\tfrac{1}{2}|. \tag{3.77}$$

The superscripts [lin] and [exp] refer to the fact that we have used a quotient of linear and exponential functions, respectively, to estimate the quotient of tangent functions involved in the formula for $\|B^{-1}\|$. ◊

Remark 3.37. Even in the case $a=0$, theorem 3.35 does not provide sharp estimates. In fact, inequality (3.75) becomes

$$|\lambda| \;\geq\; \Big|k+\frac{1}{2}\Big|,$$

whereas the exact formula in lemma 3.3 shows that the eigenvalues with smallest modulus are $\lambda_{\pm1}$ with

$$|\lambda_{\pm1}| \;=\; \Big|k+\frac{1}{2}\Big| + \frac{1}{2}.$$ ◊

In lemma 3.16 we showed that λ is an eigenvalue of $\mathcal{A}$ if and only if 0 is an eigenvalue of the operator $T_j(\lambda)$, $j=1,2$, i.e., if there is a function $f\in\mathcal{D}(B)$, such that $f\neq0$ and $T_1(\lambda)f=0$, or equivalently

$$f \;=\; B^{-1}(-D-\lambda)B^{*-1}(D-\lambda)f. \tag{3.78}$$

In the previous section we have used this equation to obtain an estimate for a lower bound of the absolute values of all eigenvalues of $\mathcal{A}$ by simply taking the norm on both sides and solving for $|\lambda|$, see corollary 3.18. The norms of B^{-1} and B^{*-1} have then been estimated by iterating $B^{*-1}B^{-1}$. Instead of applying the iteration process to $B^{*-1}B^{-1}$ only, we can also iterate equation (3.78), thus we obtain

$$f \;=\; \big(-B^{-1}(D+\lambda)B^{*-1}(D-\lambda)\big)^n f, \qquad n\in\mathbb{N}. \tag{3.79}$$

In general we cannot improve the lower bounds for the modulus of the eigenvalues of $\mathcal{A}$ by iterating the complete equation (3.78) for f instead of iterating the operators $B^{*-1}B^{-1}$ only. Under additional assumptions on the physical parameters a, m, ω and k, however, the next lemma shows that (3.78) implies that certain intervals are free of spectrum of $\mathcal{A}$.

Lemma 3.38. *Assume that the parameters k, a, m, ω are such that $|am| < \|B^{-1}\|^{-1}$. Then we have*

$$\begin{aligned}
\big(-\|B^{-1}\|^{-1},\ -|am|\big)\cap\sigma(\mathcal{A}) \;&=\; \emptyset, \qquad \textit{if}\ \ (k+\tfrac12)\,am\ge 0,\\
\big(|am|,\ \|B^{-1}\|^{-1}\big)\cap\sigma(\mathcal{A}) \;&=\; \emptyset, \qquad \textit{if}\ \ (k+\tfrac12)\,am\le 0.
\end{aligned}$$

Proof. We assume $k\ge 0$; the case $k\le -1$ can be treated analogously. If λ is an eigenvalue of the angular operator $\mathcal{A}$, then 0 is an eigenvalue of the operator $T_2(\lambda)$ (cf. lemma 3.16). If $|\lambda|\ge|am|$, then, for every eigenfunction $h\in\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ of $T_2(\lambda)$ with eigenvalue 0, it follows

$$\begin{aligned}
|h(\vartheta)| \;&=\; |B^{*-1}(D-\lambda)B^{-1}(-D-\lambda)h(\vartheta)| \qquad (3.80)\\
&=\; \left|\int_\vartheta^\pi\int_0^t \frac{\varphi(t)\psi(s)}{\varphi(\vartheta)\psi(t)}(am\cos t-\lambda)(-am\cos s-\lambda)h(s)\,\mathrm{d}s\,\mathrm{d}t\right|\\
&\le\; \int_\vartheta^\pi\int_0^t \frac{\varphi(t)\psi(s)}{\varphi(\vartheta)\psi(t)}(-\lambda+am\cos t)(-\lambda-am\cos s)|h(s)|\,\mathrm{d}s\,\mathrm{d}t. \qquad (3.81)
\end{aligned}$$

The monotonicity of the cosine implies $-1\;\le\;\cos t\;\le\;\cos s\;\le\;1$ for $0\le s\le t\le\pi$. Thus, if either $am\ge 0$ and $\lambda\le -am$ or $am\le 0$ and $\lambda\ge am$, it follows that

$$(-\lambda+am\cos t)(-\lambda-am\cos s)\;\le\;(-\lambda+am\cos s)(-\lambda-am\cos s)\;=\;\lambda^2-a^2m^2\cos^2 s\;\le\;\lambda^2.$$

Hence (3.81) can be further estimated by

$$|h(\vartheta)|\;\le\;\lambda^2\int_\vartheta^\pi\int_0^t\frac{\varphi(t)\psi(s)}{\varphi(\vartheta)\psi(t)}\,|h(s)|\,\mathrm{d}s\,\mathrm{d}t\;=\;\lambda^2 B^{*-1}B^{-1}\,|h|(\vartheta).$$

Now, if we take the $\mathscr{L}^2$-norm on both sides, we obtain

$$\|h\|_2\;\le\;\lambda^2\|B^{*-1}B^{-1}\|\,\|h\|_2\;=\;\lambda^2\|B^{-1}\|^2\,\|h\|_2.$$

Solving for λ shows the assertion. □

Estimates for $\|B^{-1}\|^{-1}$ are given in lemma 3.34.

Chapter 4

A variational principle and estimates for the higher eigenvalues of $\mathcal{A}$

In the first section of this chapter an abstract variational principle for a class of selfadjoint block operator matrices

$$\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}$$

on the product Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ is presented. An application of this variational principle to the angular operator $\mathcal{A}$ leads to upper and lower bounds for its eigenvalues with modulus greater than $|am|$.

Note that the classical variational principle applies only to eigenvalues of semibounded operators below or above its essential spectrum, see, e.g., [RS78]. The angular operator $\mathcal{A}$, however, is not semibounded, but the variational principle proved by Eschwé and Langer in [EL04] applies to the Schur complements associated with $\mathcal{A}$, see section 3.3.1 and the subsequent definition 4.15. Here we follow the approach of Langer, Langer and Tretter in [LLT02] where the authors have studied block operator matrices with bounded off-diagonal entries but unbounded diagonal entries. For the angular operator, however, we have to consider the so-called off-diagonal dominant case, i.e., T_{12} is unbounded and dominates the diagonal entries in the sense that $\mathcal{D}(T_{12}) \subseteq \mathcal{D}(T_{22})$ and $\mathcal{D}(T_{12}^*) \subseteq \mathcal{D}(T_{11})$. For bounded diagonal entries this situation has been investigated simultaneously to this work in [KLT04]. Under the assumptions that $T_{12}T_{12}^*$ and $T_{12}^*T_{12}$ are strictly positive, that the spectrum of $\mathcal{T}_0 = \begin{pmatrix} 0 & T_{12} \\ T_{12}^* & 0 \end{pmatrix}$ consist of discrete eigenvalues only and under some additional assumptions on T_{11} and T_{22}, the explicit formula for the eigenvalues of $\mathcal{T}$ provided by the variational principle gives rise to upper and lower bounds for the eigenvalues of $\mathcal{T}$ in terms of the eigenvalues of $\mathcal{T}_0$.

In section 4.2 these results are applied to the angular operator $\mathcal{A}$. Since in this special case the operators on the diagonal are bounded, also standard perturbation theory is applicable and yields upper and lower bounds for the eigenvalues of $\mathcal{A}$. These bounds are compared with the estimate resulting from the variational principle.

If not explicitly stated otherwise, we always assume that all Hilbert spaces are infinite dimensional and separable.

4.1 A variational principle for block operator matrices

In this section we prove a variational principle for the eigenvalues of a certain class of unbounded block operator matrices $\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ on a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. In section 3.3.1 we already gave the formal definition of the Schur complement of a block operator matrix. Recall that for $\lambda \in \rho(T_{22})$ and $\lambda \in \rho(T_{11})$, respectively, they are defined by

$$S_1(\lambda) := T_{11} - \lambda - T_{12}(T_{22}-\lambda)^{-1}T_{21} \quad \text{and} \quad S_2(\lambda) := T_{22} - \lambda - T_{21}(T_{11}-\lambda)^{-1}T_{12}. \tag{4.1}$$

Since in the following we do not assume $\mathcal{D}(T_{11}) \subseteq \mathcal{D}(T_{21})$ or $\mathcal{D}(T_{22}) \subseteq \mathcal{D}(T_{12})$, the domains of the Schur complements $S_1(\lambda)$ and $S_2(\lambda)$ have to be chosen with some care. In fact, in the case of the angular operator $\mathcal{A}$, the operators on the diagonal are bounded, whereas the off-diagonal elements are unbounded and hence are not everywhere defined.
In the following, $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces; by $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ we denote the product Hilbert space equipped with the usual scalar product induced by $\mathcal{H}_1$ and $\mathcal{H}_2$.

Throughout this section we assume that the following conditions on the entries T_{ij} of the block operator matrix $\mathcal{T}$, acting in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, hold:

(B1) T_{12} is a closed densely defined operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ with $T_{12}^* = T_{21}$;

(A1) $\mathcal{D}(T_{12}^*) \subseteq \mathcal{D}(T_{11})$ and T_{11} is symmetric in $\mathcal{H}_1$ and semibounded from below, i.e., there is a constant $c_1 \in \mathbb{R}$ such that

$$(x, T_{11}x) \ \geq \ c_1\|x\|^2, \qquad x \in \mathcal{D}(T_{11});$$

(D1) $\mathcal{D}(T_{12}) \subseteq \mathcal{D}(T_{22})$ and T_{22} is symmetric in $\mathcal{H}_2$ and semibounded from above, i.e., there is a constant $c_2 \in \mathbb{R}$ such that

$$(x, T_{22}x) \ \leq \ c_2\|x\|^2, \qquad x \in \mathcal{D}(T_{22});$$

furthermore, T_{22} is closed and $(c_2, \infty) \subseteq \rho(T_{22})$.

We always assume that the block operator matrix $\mathcal{T}$ is given by

$$(\mathcal{T}1) \qquad \mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}, \qquad \mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}^*) \oplus \mathcal{D}(T_{12}).$$

Remark 4.1. (i) The block operator matrix $\mathcal{T}$ depends only on the restriction $\widetilde{T}_{11} = T_{11}|_{\mathcal{D}(T_{12}^*)}$ of T_{11}. Hence, if T_{11} is not symmetric because its domain is too large, we can replace it by $\widetilde{T}_{11}$. It is easy to see that the restriction $\widetilde{T}_{11}$ is symmetric if $\mathcal{T}$ is symmetric because $(y, \widetilde{T}_{11}x) = \left(\binom{y}{0}, \mathcal{T}\binom{x}{0}\right)$ for all $x, y \in \mathcal{D}(\widetilde{T}_{11}) = \mathcal{D}(T_{12}^*)$; thus the symmetry of $\mathcal{T}$ implies $\mathcal{D}(\widetilde{T}_{11}) \subseteq \mathcal{D}(\widetilde{T}_{11}^*)$.

(ii) Since T_{11} is closable by assumption, the condition concerning its domain implies (see [Kat80, chap. IV, remark 1.5]) that T_{11} is T_{21}-bounded, i.e., that there are positive numbers α and α_{21} such that

$$\|T_{11}x\| \ \leq \ \alpha\,\|x\| + \alpha_{21}\,\|T_{21}x\|, \qquad x \in \mathcal{D}(T_{21}).$$

(iii) Condition (D1) implies that T_{22} is even selfadjoint because the defect index of the closed operator T_{22} is constant on the connected set $\mathbb{C} \setminus \overline{W(T)}$. Now, $\rho(T) \cap \mathbb{C} \setminus \overline{W(T)}$ being nonempty implies that T_{22} has zero defect, hence it is essentially selfadjoint. Since T_{22} is already closed, its selfadjointness is proved.

Observe that the above conditions do not imply that $\mathcal{T}$ is closed.

In this section we study the Schur complement S_1 and the spectrum of $\mathcal{T}$ in some right half plane. The following straightforward definition of the Schur complement can be regarded as a minimal realisation of the Schur complement of the block operator matrix $\mathcal{T}$.

Definition 4.2. Assume that conditions (B1), (A1), (D1) and ($\mathcal{T}$1) hold. Then for all $\lambda > c_2$ the operator

$$\begin{aligned} \mathcal{D}(S_1^{[\min]}(\lambda)) &:= \{x \in \mathcal{D}(T_{12}^*) \,:\, (T_{22}-\lambda)^{-1}T_{12}^*x \in \mathcal{D}(T_{12})\}, \\ S_1^{[\min]}(\lambda) &:= T_{11} - \lambda - T_{12}(T_{22}-\lambda)^{-1}T_{12}^*, \end{aligned} \tag{4.2}$$

on the Hilbert space $\mathcal{H}_1$ is well defined. We call the family $(S_1^{[\min]}(\lambda))_{\lambda > c_2}$ the *minimal Schur complement of $\mathcal{T}$.*

Proposition 4.3. *Assume that conditions (B1), (A1), (D1) and ($\mathcal{T}$1) hold. Then for all $\lambda > c_2$ the operator*

$$\begin{aligned} \mathcal{D}(S_1^{[\min]}(\lambda)) &= \{x \in \mathcal{D}(T_{12}^*) \,:\, (T_{22}-\lambda)^{-1}T_{12}^*x \in \mathcal{D}(T_{12})\}, \\ S_1^{[\min]}(\lambda) &= T_{11} - \lambda - T_{12}(T_{22}-\lambda)^{-1}T_{12}^* \end{aligned}$$

is bounded from below. If in addition one of the conditions

(D2.a) *T_{22} is bounded;*

(D2.b) *the domain of T_{12} is invariant under $(T_{22}-\lambda)^{-1}$, i.e., $(T_{22}-\lambda)^{-1}(\mathcal{D}(T_{12})) \subseteq \mathcal{D}(T_{12})$ for all $\lambda > c_2$;*

holds, then $S_1^{[\min]}(\lambda)$ is also symmetric, and therefore densely defined and closable.

Proof. Because of the inclusion $\mathcal{D}(T_{12}^*) \subseteq \mathcal{D}(T_{11})$, the Schur complement is well defined. To show that $S_1^{[\min]}(\lambda)$ is semibounded, we use that $(\lambda - T_{22})^{-1}$ is a positive operator for $\lambda > c_2$; indeed, for all $x \in \mathcal{D}(S_1^{[\min]}(\lambda))$ we have

$$\begin{aligned} (x,\, S_1^{[\min]}(\lambda)x) &= (x,\, (T_{11}-\lambda)x) - (x,\, T_{12}(T_{22}-\lambda)^{-1}T_{12}^*x) \\ &= (x,\, (T_{11}-\lambda)x) + (T_{12}^*x,\, (\lambda - T_{22})^{-1}T_{12}^*x) \;\geq\; (c_1-\lambda)\|x\|^2. \end{aligned}$$

In particular, the scalar product on the left hand side is real, hence $S_1^{[\min]}(\lambda)$ is formally symmetric. It remains to be shown that $\mathcal{D}(S_1^{[\min]}(\lambda))$ is dense in $\mathcal{H}_1$. First, suppose that (D2.a) holds. By assumption, the operator $(\lambda - T_{22})^{-1}$ is selfadjoint, bounded and positive for fixed $\lambda > c_2$. Hence there exists a positive square root $(\lambda - T_{22})^{-\frac{1}{2}}$ which is also bounded and selfadjoint. Therefore $((\lambda - T_{22})^{-\frac{1}{2}}T_{12}^*)^* = T_{12}(\lambda - T_{22})^{-\frac{1}{2}}$ holds. Condition (D2.a) implies that the operator $(\lambda - T_{22})^{-1}T_{12}^*$ is closed, hence by the theorem of von Neumann (see, for instance, [Kat80, chap. V, theorem 3.24]) the operator $((\lambda - T_{22})^{-\frac{1}{2}}T_{12}^*)^*((\lambda - T_{22})^{-\frac{1}{2}}T_{12}^*) = -T_{12}(T_{22}-\lambda)^{-1}T_{12}^*$ with domain

$$\begin{aligned} &\{x \in \mathcal{D}((\lambda - T_{22})^{-\frac{1}{2}}T_{12}^*) \,:\, (\lambda - T_{22})^{-\frac{1}{2}}T_{12}^*x \in \mathcal{D}(T_{12}(\lambda - T_{22})^{-\frac{1}{2}})\} \\ &\qquad = \{x \in \mathcal{D}(T_{12}^*) \,:\, (T_{22}-\lambda)^{-1}T_{12}^*\,x \in \mathcal{D}(T_{12})\} \;=\; \mathcal{D}(S_1^{[\min]}(\lambda)) \end{aligned}$$

is selfadjoint and its domain is a core of $(T_{22}-\lambda)^{-\frac{1}{2}}T_{12}^*$; in particular, its domain is dense in $\mathcal{H}_1$. Finally, we assume that (D2.b) holds. It follows that $\mathcal{D}(T_{12}(T_{22}-\lambda)^{-1}T_{12}^*) \supseteq \mathcal{D}(T_{12}T_{12}^*)$. Again by von Neumann's theorem, the operator $T_{12}T_{12}^*$ is densely defined, hence $\mathcal{D}(S_1^{[\min]}(\lambda))$ is dense in $\mathcal{H}_1$. □

Remark 4.4. In fact, in condition (D2.b) in the previous lemma, it suffices to assume the inclusion $(T_{22}-\lambda)^{-1}(\mathrm{rg}(T_{12}^*) \cap \mathcal{D}(T_{12})) \subseteq \mathcal{D}(T_{12})$, $\lambda > c_2$ only.

In proposition 4.16, we use sesquilinear forms to realise the Schur complement of $\mathcal{T}$ as a family of selfadjoint operators $(S_1(\lambda))_{\lambda>c_2}$ such that $S_1^{[\min]}(\lambda) \subseteq S_1(\lambda)$, $\lambda > c_2$.
Corollary 4.9 shows that the spectral properties of the block operator matrix $\mathcal{T}$ and its Schur complement S_1 are connected.
The following proposition holds under more general conditions than we actually need.

Proposition 4.5. *For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we consider linear operators $T_{ij}(\mathcal{H}_j \to \mathcal{H}_i)$, $i,j = 1,2$, with $\mathcal{D}(T_{21}) \subseteq \mathcal{D}(T_{11})$ and $\mathcal{D}(T_{12}) \subseteq \mathcal{D}(T_{22})$. Let $\mathcal{T} = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{smallmatrix}\right)$ be the block operator matrix with domain $\mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{21}) \oplus \mathcal{D}(T_{12})$ in the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$. If T_{22} is bijective, then the operator*

$$S \;:=\; T_{11} - T_{12}T_{22}^{-1}T_{21}, \qquad \mathcal{D}(S) := \{x \in \mathcal{D}(T_{21}) \,:\, T_{22}^{-1}T_{21}x \in \mathcal{D}(T_{12})\}$$

is well defined and the following holds:

(i) $\qquad$ *$\mathcal{T}$ is injective $\iff$ S is injective.*

(ii) *If additionally T_{21} is surjective, then $\operatorname{rg}(S) \oplus \{0\} = \operatorname{rg}(\mathcal{T}) \cap (\mathcal{H}_1 \oplus \{0\})$ and*

$$\mathcal{T} \text{ is surjective} \quad\iff\quad S \text{ is surjective.}$$

Proof. (i) First assume that $\mathcal{T}$ is not injective. Then there are $f \in \mathcal{D}(T_{21})$, $g \in \mathcal{D}(T_{12})$ such that

$$T_{11}f + T_{12}g = 0, \qquad T_{21}f + T_{22}g = 0 \qquad \text{and} \qquad \begin{pmatrix} f \\ g \end{pmatrix} \neq 0.$$

From the second equality it follows $T_{22}^{-1}T_{21}f = -g \in \mathcal{D}(T_{12})$. Consequently, f lies in $\mathcal{D}(S)$ and $f \neq 0$. Inserting the expression for g into the first equality gives $Sf = 0$, hence S is not injective.
Now assume that S is not injective and fix an element $f \neq 0$ in its kernel. For $g := -T_{22}^{-1}T_{21}f$ it follows that

$$\begin{aligned} 0 \;&=\; Sf \;=\; T_{11}f - T_{12}T_{22}^{-1}T_{21}f \;=\; T_{11}f + T_{12}g, \\ 0 \;&=\; g + T_{22}^{-1}T_{21}f \;=\; T_{22}^{-1}(T_{22}g + T_{21}f). \end{aligned}$$

Since T_{22}^{-1} is injective, the above equations show $0 \neq \left(\begin{smallmatrix} f \\ g\end{smallmatrix}\right) \in \ker(\mathcal{T})$.

(ii) For every $f \in \mathcal{D}(S)$, it follows that $g := -T_{22}^{-1}T_{21}f$ lies in $\mathcal{D}(T_{12})$. Consequently, $\left(\begin{smallmatrix} f \\ g\end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$ and

$$\mathcal{T}\begin{pmatrix} f \\ g \end{pmatrix} \;=\; \begin{pmatrix} T_{11}f + T_{12}g \\ T_{21}f + T_{22}g \end{pmatrix} \;=\; \begin{pmatrix} T_{11}f - T_{12}T_{22}^{-1}T_{21}f \\ 0 \end{pmatrix} \;=\; \begin{pmatrix} Sf \\ 0 \end{pmatrix}$$

which implies that $\operatorname{rg}(S) \oplus \{0\} \subseteq \operatorname{rg}(\mathcal{T}) \cap (\mathcal{H}_1 \oplus \{0\})$. Conversely, let $\left(\begin{smallmatrix} f \\ g\end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$ such that $\mathcal{T}\left(\begin{smallmatrix} f \\ g\end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ 0\end{smallmatrix}\right)$ for some $x \in \mathcal{H}_1$. From $T_{21}f + T_{22}g = 0$ it follows that $g = -T_{22}^{-1}T_{21}f \in \mathcal{D}(T_{12})$. Thus we have $f \in \mathcal{D}(S)$ and

$$x \;=\; T_{11}f + T_{12}g \;=\; T_{11}f - T_{12}T_{22}^{-1}T_{21}f \;=\; Sf,$$

implying $\operatorname{rg}(\mathcal{T}) \cap (\mathcal{H}_1 \oplus \{0\}) \subseteq \operatorname{rg}(S) \oplus \{0\}$. In particular, the surjectivity of $\mathcal{T}$ implies that of S.
Finally, assume that S is surjective and fix $\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right) \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Since $\operatorname{rg}(T_{21}) = \mathcal{H}_2$ by assumption, there is an $f' \in \mathcal{D}(T_{21}) \subseteq \mathcal{D}(T_{11})$ such that $T_{21}f' = y$. Therefore, $\left(\begin{smallmatrix} f' \\ 0\end{smallmatrix}\right)$ lies in the domain of $\mathcal{T}$ and we have $\mathcal{T}\left(\begin{smallmatrix} f' \\ 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} T_{11}f' \\ y\end{smallmatrix}\right)$. Since we have already shown that $\operatorname{rg}(S) \oplus \{0\} = \operatorname{rg}(\mathcal{T}) \cap (\mathcal{H}_1 \oplus \{0\})$, the surjectivity of S implies $\mathcal{H}_1 \oplus \{0\} = \operatorname{rg}(\mathcal{T}) \cap (\mathcal{H}_1 \oplus \{0\}) \subseteq \operatorname{rg}(\mathcal{T})$, hence we finally have $\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right) = \mathcal{T}\left(\begin{smallmatrix} f' \\ 0\end{smallmatrix}\right) + \left(\begin{smallmatrix} x - T_{11}f' \\ 0\end{smallmatrix}\right) \in \operatorname{rg}\mathcal{T}$ because both terms on the right hand side lie in $\operatorname{rg}(\mathcal{T})$. $\square$

Remark 4.6. In fact, in proposition 4.5 we have shown that

$$\ker(\mathcal{T}) \;=\; \left\{ \begin{pmatrix} f \\ -T_{22}^{-1}T_{21}f \end{pmatrix} \;:\; f \in \ker S \right\} \;\cong\; \ker(S). \qquad \Diamond$$

Remark 4.7. If T_{12} is bounded, then we need not assume that T_{21} is surjective in order to prove the surjectivity of $\mathcal{T}$ in assertion (ii) of proposition 4.5. For any given $\binom{x}{y} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, the element $x - T_{12}T_{22}^{-1}y$ is well defined and the surjectivity of S implies that there is an f such that $Sf = x - T_{12}T_{22}^{-1}y$. Since T_{22} is bijective, we can define $g := T_{22}^{-1}(y - T_{21}f)$. An easy calculation shows $\mathcal{T}\binom{f}{g} = \binom{x}{y}$ and the surjectivity of $\mathcal{T}$ is proved. $\Diamond$

The spectrum and resolvent set of an operator valued function are defined as follows.

Definition 4.8. Let $S = (S(z))_z$ be a family of closed operators, where z varies in some set $U \subseteq \mathbb{C}$. Then the *spectrum*, *point spectrum* and *resolvent set* of S are defined as

$$\begin{aligned}
\sigma(S) &:= \{z \in U \,:\, 0 \in \sigma(S(z))\}, \\
\sigma_p(S) &:= \{z \in U \,:\, 0 \in \sigma_p(S(z))\}, \\
\rho(S) &:= \{z \in U \,:\, 0 \in \rho(S(z))\}.
\end{aligned}$$

Analogous definitions apply to the other parts of the spectrum of S, e.g., the *essential spectrum*.

Recall that for a linear operator S the *essential spectrum* and *discrete spectrum* are defined by

$$\begin{aligned}
\sigma_{ess}(S) &:= \{\lambda \in \mathbb{C} \,:\, \dim(\ker(S-\lambda)) = \infty \text{ or } \operatorname{codim}(\operatorname{rg}(S-\lambda)) = \infty\} \\
\sigma_d(S) &:= \{\lambda \in \mathbb{C} \,:\, \lambda \text{ is an isolated eigenvalue of S with finite multipilcity}\}
\end{aligned}$$

For a selfadjoint operator S we have $\sigma_d(S) = \sigma(S) \setminus \sigma_{ess}(S)$.

Corollary 4.9. *In addition to the assumptions in proposition 4.3, suppose that the operator $\mathcal{T}$ is selfadjoint and that T_{12}^* is surjective. Furthermore, assume that the operator function $S_1^{[\min]}$ defined in (4.2) is holomorphic and that each $S_1^{[\min]}(\lambda)$ is selfadjoint. Then we have*

$$\sigma_p(\mathcal{T}) \cap (c_2, \infty) \;=\; \sigma_p(S_1^{[\min]}), \tag{4.3}$$

$$\sigma_{ess}(\mathcal{T}) \cap (c_2, \infty) \;=\; \sigma_{ess}(S_1^{[\min]}). \tag{4.4}$$

Proof. Proposition 4.5 applied to $\mathcal{T} - \lambda$ shows that $\lambda \in \sigma(\mathcal{T}) \cap (c_2, \infty)$ if and only if $\lambda \in \sigma(S_1^{[\min]})$. Moreover, it follows from remark 4.6 that $\lambda \in \sigma_p(S_1^{[\min]})$ if and only if $\lambda \in \sigma_p(\mathcal{T}) \cap (c_2, \infty)$ with $\dim(\ker(\mathcal{T} - \lambda)) = \infty$ if and only if $\dim(\ker(S_1^{[\min]}(\lambda))) = \infty$. Hence, (4.3) is proved. To show (4.4) it suffices to show $\sigma_d(\mathcal{T}) \cap (c_2, \infty) = \sigma_d(S_1^{[\min]})$. Let $\lambda \in \sigma_d(\mathcal{T}) \cap (c_2, \infty)$. Then we have $\dim \operatorname{rg}(S_1^{[\min]}(\lambda))^\perp = \dim \ker(S_1^{[\min]}(\lambda)) = \dim \ker(\mathcal{T} - \lambda) < \infty$. Further, the range of $\mathcal{T} - \lambda$ is closed because $\lambda \in \sigma_d(\mathcal{T})$. So proposition 4.5 shows that $\operatorname{rg}(S_1^{[\min]}(\lambda)) = \operatorname{rg}(\mathcal{T}) \cap (\mathcal{H}_1 \oplus \{0\})$ is also closed. Hence it follows $0 \in \sigma_d(S_1^{[\min]}(\lambda))$ and consequently $\lambda \in \sigma_d(S_1^{[\min]})$.
Let $\lambda \in \sigma_d(S_1^{[\min]})$. Then $\lambda \in \sigma_p(\mathcal{T})$ with $\dim \ker(\mathcal{T} - \lambda) = \dim \ker(S_1(\lambda)) < \infty$ and we have to show that λ is no accumulation point of $\sigma(\mathcal{T})$. Since $0 \in \sigma_d(S_1^{[\min]}(\lambda))$ and $S_1^{[\min]}$ is holomorphic, there are $\delta > 0$, $\varepsilon > 0$ and holomorphic functions $\mu_j : (\lambda - \delta, \lambda + \delta) \to \mathbb{R}$ with $\mu_j(\lambda) = 0$ for $j = 1, \ldots, \dim \ker(S_1(\lambda))$, such that for all $\widetilde{\lambda} \in (\lambda - \varepsilon,\, \lambda + \varepsilon)$ we have that $\mu \in \sigma(S_1^{[\min]}(\widetilde{\lambda})) \cap (-\varepsilon, \varepsilon)$ if and only if μ is an eigenvalue of $S_1^{[\min]}(\widetilde{\lambda})$ with finite multiplicity and $\mu = \mu_j(\widetilde{\lambda})$ for some j (see [Kat80, chap. IV, §3 and chap. VII]). Furthermore, for $j = 1, \ldots, \dim \ker(S_1^{[\min]}(\lambda))$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mu_j(\lambda) \;=\; \frac{\mathrm{d}}{\mathrm{d}\lambda}(x_j,\, S_1^{[\min]}(\lambda)x_j) \;=\; -\|x_j\|^2 - \|(T_{22} - \lambda)^{-1}T_{12}^*\, x_j\|^2$$

for normalised eigenvectors x_j of $S_1^{[\min]}(\lambda)$ with eigenvalue 0, hence the functions μ_j are not constant in a neighbourhood of λ. Thus there exists a nonempty interval $(\lambda-\widetilde{\delta},\lambda+\widetilde{\delta})$ such that $0\in\rho(S_1^{[\min]}(\widetilde{\lambda}))$ for all $\widetilde{\lambda}\in(\lambda-\widetilde{\delta},\lambda+\widetilde{\delta})\setminus\{\lambda\}$. Consequently, $\sigma(\mathcal{T})\cap(\lambda-\widetilde{\delta},\lambda+\widetilde{\delta})=\{\lambda\}$ which completes the proof. □

In corollary 4.9 we have seen that under certain conditions the spectrum of $\mathcal{T}$ in the interval (c_2,∞) and that of the operator family $S_1^{[\min]}$ coincide. One of the main assumptions was the holomorphy of $S_1^{[\min]}$. In the following we realise the Schur complement as a holomorphic selfadjoint operator function S_1 via sesquilinear forms and establish criteria that guarantee $S_1^{[\min]}(\lambda)=S_1(\lambda)$.

For convenience, we repeat some well known definitions and facts concerning sesquilinear forms on Hilbert spaces, see, e.g., [Kat80, chap. VI]. A mapping

$$\mathfrak{t}:\mathcal{D}(\mathfrak{t})\times\mathcal{D}(\mathfrak{t})\longrightarrow\mathbb{C},\quad (u,v)\mapsto t[u,v]$$

is called a *sesquilinear form* on a complex Hilbert space $\mathcal{H}$ with *domain* $\mathcal{D}(\mathfrak{t})$ if $\mathcal{D}(\mathfrak{t})$ is a linear manifold in $\mathcal{H}$ and if

$$\mathfrak{t}[\alpha u,\,\beta v+w]\;=\;\overline{\alpha}\,(\beta\,\mathfrak{t}[u,\,v]+\mathfrak{t}[u,\,w]),\qquad \alpha,\beta\in\mathbb{C},\;u,v,w\in\mathcal{D}(\mathfrak{t}).$$

The simplest example of a sesquilinear form is the scalar product on $\mathcal{H}$ (observe that we use the convention $(\mathrm{i}x,\,y)=-\mathrm{i}(x,\,y)$). We often use the abbreviation

$$\mathfrak{t}[u]\;:=\;\mathfrak{t}[u,\,u],\qquad u\in\mathcal{D}(\mathfrak{t}).$$

If for forms $\mathfrak{s}$ and $\mathfrak{t}$ on $\mathcal{H}$ the inclusion of domains $\mathcal{D}(\mathfrak{t})\subseteq\mathcal{D}(\mathfrak{s})$ and $\mathfrak{t}[u,v]=\mathfrak{s}[u,v]$ for all $u,v\in\mathcal{D}(\mathfrak{t})$ hold, then $\mathfrak{t}$ is called a *restriction* of $\mathfrak{s}$ and $\mathfrak{s}$ is called an *extension* of $\mathfrak{t}$. We denote this relation by $\mathfrak{t}\subseteq\mathfrak{s}$. A form $\mathfrak{t}$ is called *symmetric* if

$$\mathfrak{t}[u,\,v]\;=\;\overline{\mathfrak{t}[v,\,u]},\qquad u,v\in\mathcal{D}(\mathfrak{t}).$$

The *numerical range of* $\mathfrak{t}$ is the set

$$W(\mathfrak{t})\;:=\;\{\mathfrak{t}[u]\,:\,u\in\mathcal{D}(\mathfrak{t}),\;\|u\|=1\}.$$

Obviously, the numerical range of a symmetric form is a subset of $\mathbb{R}$. A symmetric form is said to be *bounded from below* if there exists a $\gamma\in\mathbb{R}$ such that

$$\mathfrak{t}[u]\;\geq\;\gamma\,\|u\|^2,\qquad u\in\mathcal{D}(\mathfrak{t}).$$

If the numerical range of a sesquilinear form is contained in a sector $\{z\in\mathbb{C}\,:\,|\arg(z-\gamma)|\leq\vartheta\}$ for some $\gamma\in\mathbb{R}$ and $0\leq\vartheta<\frac{\pi}{2}$, then the form is called *sectorial*. Note that we use the convention $\arg(z)\in(-\pi,\pi]$ for $z\in\mathbb{C}$.
A sequence $(u_n)_{n\in\mathbb{N}}\subseteq\mathcal{D}(\mathfrak{t})$ is called $\mathfrak{t}$-*convergent* if it converges to some $u\in\mathcal{H}$ and if $\mathfrak{t}[u_n-u_m]$ tends to zero for $n,m\to\infty$. If $\mathfrak{t}$ is sectorial and $\mathcal{D}(\mathfrak{t})$ is complete with respect to $\mathfrak{t}$-convergence, we call the form $\mathfrak{t}$ *closed*. In other words, $\mathfrak{t}$ is closed if for every $\mathfrak{t}$-convergent sequence $(u_n)_{n\in\mathbb{N}}\subseteq\mathcal{D}(\mathfrak{t})$ also $u:=\lim_{n\to\infty}u_n$ is in the domain of $\mathfrak{t}$ and $\mathfrak{t}[u-u_n]\to0$ for $n\to\infty$.
The form $\mathfrak{t}$ is called *closable* if it admits a closed extension $\mathfrak{s}$. If $\mathfrak{t}[u_n]\to0$ for every $\mathfrak{t}$-convergent sequence $(u_n)_{n\in\mathbb{N}}\subseteq\mathcal{D}(\mathfrak{t})$ with $u_n\to0$, then $\mathfrak{t}$ is closable (see [Kat80, chap. VI, theorem 1.17]). If $\mathfrak{t}$ is closable, then there is a unique smallest closed extension $\widetilde{\mathfrak{t}}$, which is called the *closure* of $\mathfrak{t}$. Its domain consists of all $u\in\mathcal{H}$ such that there is a $\mathfrak{t}$-convergent sequence $(u_n)_{n\in\mathbb{N}}\subseteq\mathcal{D}(\mathfrak{t})$ with $u_n\to u$. It is well known that for a closable form the numerical range is dense in the numerical range of its closure.

By the first representation theorem (see [Kat80, chap. VI, theorem 2.1]), for every densely defined, closed, sectorial form $\mathfrak{t}$ in $\mathcal{H}$ there is a uniquely defined m-sectorial operator T with $\mathcal{D}(T) \subseteq \mathcal{D}(\mathfrak{t})$ and $\mathfrak{t}[u,\, v] = (u,\, Tv)$ for all $u \in \mathcal{D}(\mathfrak{t})$ and $v \in \mathcal{D}(T)$. T is called the *operator associated with* $\mathfrak{t}$. Recall that an operator T is said to be *m-sectorial* if its numerical range is contained in a sector in the right half plane and if the open left half plane is in the resolvent set of T with $\|(T-\lambda)^{-1}\| \le |\operatorname{Re}(\lambda)|^{-1}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) < 0$.

In this section we are also dealing with families of sesquilinear forms and their associated operators.

Definition 4.10. Let U be a domain in $\mathbb{C}$ and H be a Hilbert space. A family $(\mathfrak{t}(\zeta))_{\zeta\in U}$ of sesquilinear forms is called a *holomorphic family of type* (a) if

(i) $\mathcal{D}(\mathfrak{t}(\zeta)) = \mathcal{D}$ is independent of ζ and dense in H and each $\mathfrak{t}(\zeta)$ is sectorial and closed,

(ii) for each fixed $u \in \mathcal{D}$ the function $\zeta \mapsto \mathfrak{t}(\zeta)[u]$ is holomorphic in U.

For each $\zeta \in U$ let $\mathcal{T}(\zeta)$ be the m-sectorial operator associated with $\mathfrak{t}(\zeta)$. Then the family $(\mathcal{T}(\zeta))_{\zeta\in U}$ is called a *holomorphic family of type* (B).

It can be shown that a family of type (B) is holomorphic ([Kat80, chap. VII, theorem 4.2]).

For $\lambda > c_2$, we define

$$\mathcal{D}(\mathfrak{t}_{11}(\lambda)) \;:=\; \mathcal{D}(T_{11}), \qquad \mathfrak{t}_{11}(\lambda)[u,v] \;:=\; \big(u,\, (T_{11}-\lambda)v\big), \tag{4.5}$$

$$\mathcal{D}(\mathfrak{t}_{12}(\lambda)) \;:=\; \mathcal{D}(T_{12}^*), \qquad \mathfrak{t}_{12}(\lambda)[u,v] \;:=\; \big(T_{12}^*u,\, (T_{22}-\lambda)^{-1}T_{12}^*v\big). \tag{4.6}$$

Proposition 4.11. *Assume that the conditions (B1), (A1), (D1) and ($\mathcal{T}$1) hold. Further, let either (D2.a) or (D2.b) be fulfilled. Then the sesquilinear form*

$$\mathcal{D}(\mathfrak{s}_1^{[\min]}(\lambda)) \;:=\; \mathcal{D}(T_{12}^*), \qquad \mathfrak{s}_1^{[\min]}(\lambda)[u,\, v] \;:=\; \big(u,\, (T_{11}-\lambda)v\big) - \big(T_{12}^*u,\, (T_{22}-\lambda)^{-1}T_{12}^*v\big)$$

in $\mathcal{H}_1$ is symmetric, semibounded from below and closable.

Proof. The symmetry and boundedness from below can be shown as in proposition 4.3. Since the operator $T_{11}-\lambda$ is symmetric and bounded from below, it is form-closable, i.e., the symmetric form $\mathfrak{t}_{11}(\lambda)$ defined in (4.5) is closable. Because the sum of closable forms is again closable (see [Kat80, chap.VI, theorem 1.31]), it remains to be shown that the form $\mathfrak{t}_{12}(\lambda)$ is also closable. Consider the restriction $\mathfrak{t}_{12}^{[\min]}(\lambda)$ of $\mathfrak{t}_{12}(\lambda)$ with $\mathcal{D}(\mathfrak{t}_{12}^{[\min]}(\lambda)) = \mathcal{D}(S_1^{[\min]}(\lambda))$. We showed in the proof of proposition 4.3 that the operator $-T_{12}(T_{22}-\lambda)^{-1}T_{12}^*$ with domain $\mathcal{D}(S_1^{[\min]}(\lambda))$ is symmetric and bounded from below. Hence the form $\mathfrak{t}_{12}^{[\min]}(\lambda)$ associated with it is closable; therefore it suffices to show that $\mathfrak{t}_{12}(\lambda) \subseteq \widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda)$ where $\widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda)$ is the closure of $\mathfrak{t}_{12}^{[\min]}(\lambda)$.
To this end we fix $x \in \mathcal{D}(T_{12}^*)$. If we assume that (D2.a) holds, then $\mathcal{D}(S_1^{[\min]}(\lambda))$ is a core of $(\lambda-T_{22})^{-\frac12}T_{12}^*$ (see proof of proposition 4.3). Consequently, there exists a sequence

$$(x_n)_{n\in\mathbb{N}} \;\subseteq\; \mathcal{D}(S_1^{[\min]}(\lambda)) \;\subseteq\; \mathcal{D}((\lambda-T_{22})^{-\frac12}T_{12}^*) \;=\; \mathcal{D}(T_{12}^*)$$

with $x_n \to x$ and $(T_{22}-\lambda)^{-\frac12}T_{12}^*x_n \to (T_{22}-\lambda)^{-\frac12}T_{12}^*x$ for $n\to\infty$. Then it follows that

$$\begin{aligned}\mathfrak{t}_{12}^{[\min]}(\lambda)[x_n - x_m] \;&=\; \big(T_{12}^*(x_n-x_m),\, (\lambda-T_{22})^{-1}T_{12}^*(x_n-x_m)\big)\\ &=\; \big\|(\lambda-T_{22})^{-\frac12}T_{12}^*(x_n-x_m)\big\|^2 \;\longrightarrow\; 0, \qquad n,\, m\to\infty.\end{aligned}$$

Hence $x \in \widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda)$ which implies $\mathfrak{t}_{12}(\lambda) \subseteq \widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda)$.
Finally, we assume that condition (D2.b) holds. Since $\mathcal{D}(T_{12}T_{12}^*)$ is a core of T_{12}^*, there is a sequence

$(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T_{12}T_{12}^*)$ such that $x_n \to x$ and $T_{12}^* x_n \to T_{12}^* x$ for $n \to \infty$. By the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} \mathfrak{t}_{12}^{[\min]}(\lambda)[x_n - x_m] &= (T_{12}^*(x_n - x_m), (\lambda - T_{22})^{-1} T_{12}^*(x_n - x_m)) \\ &\leq \|(\lambda - T_{22})^{-1}\| \, \|T_{12}^*(x_n - x_m)\|^2 \longrightarrow 0, \qquad n, m \to \infty. \end{aligned}$$

This shows $x \in \mathcal{D}(\widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda))$, thus $\mathfrak{t}_{12}(\lambda) \subseteq \widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda)$.
Hence both (D2.a) and (D2.b) imply $\mathfrak{t}_{12}^{[\min]}(\lambda) \subseteq \mathfrak{t}_{12}(\lambda) \subseteq \widetilde{\mathfrak{t}}_{12}^{[\min]}(\lambda)$, and therefore $\mathfrak{t}_{12}(\lambda)$ is closable. □

In the following, we set

$$\mathfrak{s}_1(\lambda) := \widetilde{\mathfrak{s}}_1^{[\min]}(\lambda), \qquad \lambda \in (c_2, \infty),$$

where $\widetilde{\mathfrak{s}}_1^{[\min]}(\lambda)$ is the closure of $\mathfrak{s}_1^{[\min]}(\lambda)$. The next lemma gives conditions which ensure that $\mathfrak{s}_1^{[\min]}(\lambda)$ is already closed. The proof is essentially an application of a perturbation result for closed sesquilinear forms.

Lemma 4.12. *Assume that the conditions (B1), (A1), (D1), (D2.a) and ($\mathcal{T}$1) hold. Then $\mathfrak{s}_1^{[\min]}(\lambda)$ is closed, i.e., $\mathfrak{s}_1^{[\min]}(\lambda) = \mathfrak{s}_1(\lambda)$ for all $\lambda \in (c_2, \infty)$.*

Proof. Since T_{22} is bounded by assumption, it follows that also $(T_{22} - \lambda)^{\frac{1}{2}}$ is bounded and therefore the operator $(\lambda - T_{22})^{-\frac{1}{2}} T_{12}^*$ is closed. For every $\mathfrak{t}_{12}(\lambda)$-convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T_{12}^*)$ we have

$$\|(T_{22} - \lambda)^{-\frac{1}{2}} T_{12}^*(x_n - x_m)\| = \mathfrak{t}_{12}(\lambda)[x_n - x_m] \to 0, \qquad n, m \to \infty.$$

Since $(T_{22} - \lambda)^{-\frac{1}{2}} T_{12}^*$ is closed, it follows that $x := \lim_{n\to\infty} x_n \in \mathcal{D}((T_{22} - \lambda)^{-\frac{1}{2}} T_{12}^*) = \mathcal{D}(T_{12}^*)$. This shows that $\mathfrak{t}_{12}(\lambda)$ is closed. The form $\mathfrak{t}_{11}(\lambda)$ is closable, see proof of proposition 4.11; let $\widetilde{\mathfrak{t}}_{11}(\lambda)$ denote its closure. Then it follows that the form $\mathfrak{s}_1(\lambda) = \widetilde{\mathfrak{t}}_{11}(\lambda) + \mathfrak{t}_{12}(\lambda)$ with domain $\mathcal{D}(\widetilde{\mathfrak{t}}_{11}(\lambda)) \cap \mathcal{D}(\mathfrak{t}_{12}(\lambda))$ is also closed. Since $\mathcal{D}(\mathfrak{t}_{12}(\lambda)) \subseteq \mathcal{D}(\mathfrak{t}_{11}(\lambda)) \subseteq \mathcal{D}(\widetilde{\mathfrak{t}}_{11}(\lambda))$, the form $\mathfrak{s}_1(\lambda)$ with domain $\mathcal{D}(\mathfrak{t}_{12}(\lambda)) = \mathcal{D}(T_{12}^*)$ is closed. □

Remark 4.13. In lemma 4.12, condition (D2.a) cannot be replaced by (D2.b). Consider, for example, the unbounded selfadjoint multiplication operators T_{12} and T_{22} on $\mathcal{H} := \mathscr{L}^2((0,1), \mathrm{d}x)$ with domains $\mathcal{D}(T_{12}) = \mathcal{D}(T_{22}) = \{f \in \mathcal{H} : x \mapsto \frac{1}{x} f(x) \in \mathcal{H}\}$, defined by

$$(T_{12}f)(x) = \frac{1}{x} f(x), \qquad (T_{22}f)(x) = -\frac{1}{x} f(x), \qquad x \in (0,1).$$

Then the block operator matrix $\mathcal{T} := \begin{pmatrix} 0 & T_{12} \\ T_{12} & T_{22} \end{pmatrix}$ with domain $\mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}) \oplus \mathcal{D}(T_{12})$ satisfies all assumptions of lemma 4.12 apart from (D2.a); in particular, $T_{22} \leq c_1 := 0$. Further, condition (D2.b) is fulfilled because for every $\lambda \in (0, \infty)$ and every $f \in \mathcal{D}(T_{12})$ the function $(0,1) \to \mathbb{C}$, $x \mapsto (T_{22} - \lambda)^{-1} f(x) = -\frac{x}{1+\lambda x} f(x)$ lies again in $\mathcal{D}(T_{12})$. Hence, by proposition 4.11, the form $\mathfrak{t}_{12}(\lambda)$ defined in (4.6) is closable for all $\lambda \in (0, \infty)$, but it is not closed. To see this, fix $\varepsilon \in (0, \frac{1}{2})$ and define the sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ by

$$f_n(x) = \begin{cases} x^{\frac{1}{2}-\varepsilon} & \text{if } x \in (\frac{1}{n}, 1) \\ 0 & \text{if } x \in (0, \frac{1}{n}]. \end{cases}$$

Obviously, $\|f_n - f\|_2 \to 0$ for $n \to \infty$ with $f(x) = x^{\frac{1}{2}-\varepsilon}$, $x \in (0,1)$. First we show that the sequence is also $\mathfrak{t}_{12}(\lambda)$-convergent. To this end, observe that for arbitrary $\lambda > 0$ we have

$$\begin{aligned}
|\mathfrak{t}_{12}(\lambda)[f_n - f_m]| &= \left|\left(T_{12}(f_n - f_m),\, (T_{22}-\lambda)^{-1}T_{12}(f_n - f_m)\right)\right| \\
&= \left|\int_{\frac{1}{m}}^{\frac{1}{n}} x^{-2}(-x^{-1}-\lambda)^{-1} f(x)^2 \,\mathrm{d}x\right| = \left|\int_{\frac{1}{m}}^{\frac{1}{n}} x^{-2\varepsilon}(1+\lambda x)^{-1}\,\mathrm{d}x\right| \\
&\le \left|\int_{\frac{1}{m}}^{\frac{1}{n}} x^{-2\varepsilon}\,\mathrm{d}x\right| = \frac{1}{1-2\varepsilon}\left|m^{2\varepsilon-1} - n^{2\varepsilon-1}\right| \longrightarrow 0
\end{aligned}$$

for $n, m \to \infty$. Hence f lies in the domain of the closure $\widetilde{\mathfrak{t}}_{12}(\lambda)$ of $\mathfrak{t}_{12}(\lambda)$. On the other hand, because the integral

$$\int_0^1 \left(\frac{1}{x}f(x)\right)^2 \mathrm{d}x = \int_0^1 x^{-1-2\varepsilon}\,\mathrm{d}x = -\frac{1}{2\varepsilon}\lim_{\delta\to 0}(1-\delta^{-2\varepsilon})$$

is not finite, it follows that $f \notin \mathcal{D}(T_{12})$. Hence we have proved $\mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\lambda)) \neq \mathcal{D}(T_{12})$. ◊

In proposition 4.11 we showed that $\mathfrak{t}_{12}(\lambda)$ is closable if one of the conditions (D2.a) or (D2.b) holds. In particular, if we have (D2.a), then the form $\mathfrak{t}_{12}(\lambda)$ is closed. Lemma 4.12 shows that also $\mathfrak{s}_1^{[\min]}(\lambda)$ is closed and that its domain does not depend on λ. In this case we set

$$\mathcal{D}(\mathfrak{s}_1) := \mathcal{D}(\mathfrak{s}_1^{[\min]}(\lambda)) = \mathcal{D}(T_{12}^*), \qquad \lambda \in (c_2,\infty).$$

Although under condition (D2.b) the domain of the closure does not necessarily coincide with $\mathcal{D}(T_{12}^*)$, the following lemma shows that also in this case the domain of the closure does not depend on λ.

Lemma 4.14. *Assume that the conditions (B1), (A1), (D1) and ($\mathcal{T}$1) are satisfied. Furthermore, suppose that there exists a $\lambda \in (c_2,\infty)$ such that the form $\mathfrak{t}_{12}(\lambda)$ defined in (4.6) is closable with closure $\widetilde{\mathfrak{t}}_{12}(\lambda)$. Then the form $\mathfrak{t}_{12}(\mu)$ is closable for all $\mu \in (c_2, \infty)$ and the domain of its closure $\widetilde{\mathfrak{t}}_{12}(\mu)$ does not depend on μ.*

Proof. Let $\lambda_1,\ \lambda_2 \in (c_2,\infty)$. Using the resolvent equation

$$(T_{22}-\lambda_2)^{-1} - (T_{22}-\lambda_1)^{-1} = (\lambda_2-\lambda_1)(\lambda_1 - T_{22})^{-1}(\lambda_2 - T_{22})^{-1},$$

we find for all $x \in \mathcal{D}(T_{12}^*)$

$$\begin{aligned}
|\mathfrak{t}_{12}(\lambda_1)[x] - \mathfrak{t}_{12}(\lambda_2)[x]| &= \left|\left(T_{12}^*x,\, (-(T_{22}-\lambda_1)^{-1} + (T_{22}-\lambda_2)^{-1})T_{12}^*x\right)\right| \\
&= |\lambda_2-\lambda_1|\left|\left((\lambda_1-T_{22})^{-\frac{1}{2}}T_{12}^*x,\, (\lambda_1-T_{22})^{-\frac{1}{2}}(\lambda_2-T_{22})^{-1}T_{12}^*x\right)\right| \\
&\le |\lambda_2-\lambda_1|\left\|(\lambda_1-T_{22})^{-\frac{1}{2}}T_{12}^*x\right\|\left\|(\lambda_2-T_{22})^{-1}(\lambda_1-T_{22})^{-\frac{1}{2}}T_{12}^*x\right\| \\
&\le |\lambda_2-\lambda_1|\left\|(\lambda_2-T_{22})^{-1}\right\|\left\|(\lambda_1-T_{22})^{-\frac{1}{2}}T_{12}^*x\right\|^2 \\
&= |\lambda_2-\lambda_1|\left\|(\lambda_2-T_{22})^{-1}\right\|\,\mathfrak{t}_{12}(\lambda_1)[x].
\end{aligned}$$

Now let $\lambda \in (c_2,\infty)$ such that $\mathfrak{t}_{12}(\lambda)$ is closable and fix an arbitrary $\mu \in (c_2,\infty)$. We have to show that for every $\mathfrak{t}_{12}(\mu)$-convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathfrak{t}_{12}(\mu)) = \mathcal{D}(T_{12}^*)$ with $x_n \to 0$ for $n \to \infty$ also $\mathfrak{t}_{12}(\mu)[x_n]$ tends to zero. Applying the above inequality with $x = x_n - x_m$ to $\lambda_1 = \lambda$

and $\lambda_2 = \mu$ and to $\lambda_1 = \mu$ and $\lambda_2 = \lambda$ we find that $\mathfrak{t}_{12}(\mu)[x_n - x_m]$ tends to zero if and only if $\mathfrak{t}_{12}(\lambda)[x_n - x_m]$ does. Hence every $\mathfrak{t}_{12}(\mu)$-convergent sequence is also $\mathfrak{t}_{12}(\lambda)$-convergent. Since $\mathfrak{t}_{12}(\lambda)$ is closable, it follows that $\mathfrak{t}_{12}(\lambda)[x_n]$ converges to zero for $n \to \infty$. Application of the above chain of inequalities to x_n shows that also $\mathfrak{t}_{12}(\mu)[x_n] \to 0$ for $n \to \infty$ and the closability of $\mathfrak{t}_{12}(\mu)$ is proved. To show that the domain of the closure of $\mathfrak{t}_{12}(\mu)$ is independent of μ, it suffices to prove the inclusion $\mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\mu)) \subseteq \mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\lambda))$. For $x \in \mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\mu))$ we chose a $\mathfrak{t}_{12}(\mu)$-convergent sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x$. As above, it follows that the sequence is also $\mathfrak{t}_{12}(\lambda)$-convergent, hence the closedness of $\widetilde{\mathfrak{t}}_{12}(\lambda)$ implies $x \in \mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\lambda))$. In the same way we can show the converse inclusion $\mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\lambda)) \subseteq \mathcal{D}(\widetilde{\mathfrak{t}}_{12}(\mu))$, hence the domain of the closure of the form $\mathfrak{t}_{12}(\mu)$ is independent of μ. □

If we assume that T_{11} is $T_{12}(T_{22} - \lambda)^{-1}T_{12}^*$-bounded with relative bound less than 1 (compare proposition 4.16), then the above lemma also holds for the forms $\mathfrak{s}_1(\lambda)$ instead of $\mathfrak{t}_{12}(\lambda)$ because in this case $\mathfrak{s}_1(\lambda)$ is closable (closed) if and only if $\mathfrak{t}_{12}(\lambda)$ is.

Assume now that $\mathfrak{s}_1(\lambda)$ is closed. By the first representation theorem [Kat80, chap. VI, theorem 2.1] there is a uniquely defined selfadjoint operator $S_1(\lambda)$ associated with the closed sesquilinear form $\mathfrak{s}_1(\lambda)$ such that

$$\mathcal{D}(S_1(\lambda)) \subseteq \mathcal{D}(\mathfrak{s}_1) \qquad \text{and} \qquad (v,\, S_1(\lambda)u) \;=\; \mathfrak{s}_1(\lambda)[v, u],\ v \in \mathcal{D}(\mathfrak{s}_1),\ u \in \mathcal{D}(S_1(\lambda)).$$

Moreover, if for fixed $u \in \mathcal{D}(\mathfrak{s}_1)$ the identity $\mathfrak{s}_1(\lambda)[v,\, u] = (v,\, w)$ holds for all v belonging to a core of $\mathfrak{s}_1(\lambda)$, then u is in the domain of $S_1(\lambda)$ and $S_1(\lambda)u = w$.

Definition 4.15. The operator family $(S_1(\lambda))_{\lambda > c_2}$ is called the *Schur complement of $\mathcal{T}$*.

Now we show that the operator $S_1(\lambda)$ is a selfadjoint extension of $S_1^{[\min]}(\lambda)$ defined in proposition 4.3.

Proposition 4.16. *The inclusion $S_1^{[\min]}(\lambda) \subseteq S_1(\lambda)$ holds, where $S_1^{[\min]}(\lambda)$ is the operator of proposition 4.3 and $S_1(\lambda)$ is the operator associated to the form $\mathfrak{s}_1(\lambda)$. If in addition (D2.a) and*

(A2) *T_{11} is symmetric and $T_{12}(T_{22} - \lambda)^{-1}T_{12}^*$-bounded with relative bound less than 1, i.e., there are $\alpha > 0$ and $1 > \widetilde{\alpha} > 0$ (which may depend on λ) such that*

$$\|T_{11}x\| \;\le\; \alpha\,\|x\| + \widetilde{\alpha}\,\|T_{12}(T_{22} - \lambda)^{-1}T_{12}^*x\|, \qquad\qquad x \in \mathcal{D}(T_{12}(T_{22} - \lambda)^{-1}T_{12}^*),$$

then $S_1(\lambda) = S_1^{[\min]}(\lambda)$; in particular, $S_1^{[\min]}(\lambda)$ is selfadjoint.

Proof. Fix some $x \in \mathcal{D}(S_1^{[\min]}(\lambda)) = \{x \in \mathcal{D}(T_{12}^*) \,:\, (T_{22} - \lambda)^{-1}T_{12}^*x \in \mathcal{D}(T_{12})\}$. Then we have for all $v \in \mathcal{D}(\mathfrak{s}_1)$

$$\begin{aligned}\mathfrak{s}_1[v,\, x] \;&=\; (v,\, (T_{11} - \lambda)x) - \big(T_{12}^*v,\, (T_{22} - \lambda)^{-1}T_{12}^*x\big) \;=\; \big(v,\, (T_{11} - \lambda - T_{12}(T_{22} - \lambda)^{-1}T_{12}^*)x\big)\\ &=\; \big(v,\, S_1^{[\min]}(\lambda)x\big),\end{aligned}$$

hence $x \in \mathcal{D}(S_1(\lambda))$ and $S_1(\lambda)x = S_1^{[\min]}(\lambda)x$, therefore $S_1^{[\min]}(\lambda) \subseteq S_1(\lambda)$ is proved.
If we assume that T_{22} is bounded and that T_{11} is bounded with respect to $T_{12}(T_{22}-\lambda)^{-1}T_{12}^*$ with relative bound less than 1, then it follows from the Kato-Rellich theorem [Kat80, chap. V, theorem 4.3] that $S_1^{[\min]}(\lambda)$ is selfadjoint since $T_{12}(T_{22}-\lambda)^{-1}T_{12}^*$ is selfadjoint, see proof of proposition 4.3; hence $S_1^{[\min]}(\lambda) = S_1(\lambda)$ follows. □

Recall that $\mathcal{D}(S_1^{[\min]}(\lambda)) = \mathcal{D}(T_{12}(T_{22} - \lambda)^{-1}T_{12}^*)$. It is not hard to see that under condition (A2) the restriction $\mathfrak{s}_1^{[\min,\mathrm{r}]}(\lambda)$ of $\mathfrak{s}_1^{[\min]}(\lambda)$ to $\mathcal{D}(S_1^{[\min]}(\lambda))$ is also closable and that its closure is given by $\widetilde{\mathfrak{s}}_1^{[\min,\mathrm{r}]}(\lambda) = \mathfrak{s}_1(\lambda)$. This follows because $S_1(\lambda) = S_1^{[\min]}(\lambda)$ is the operator associated with the form $\mathfrak{s}_1(\lambda)$. On the other hand, $\mathfrak{s}_1^{[\min,\mathrm{r}]}(\lambda)$ is the form associated with $S_1^{[\min]}(\lambda)$ and therefore closable.

Since the correspondence between the set of all densely defined, closed sectorial forms and the set of all m-sectorial operators is one-to-one, $\mathfrak{s}_1(\lambda)$ must be the closure of $\mathfrak{s}_1^{[\min,r]}(\lambda)$.

The selfadjoint extension $S_1(\lambda)$ of $S_1^{[\min]}(\lambda)$ which we have constructed via the associated sesquilinear form $\mathfrak{s}_1(\lambda)$ is the so-called *Friedrichs extension* of $S_1^{[\min]}(\lambda)$.

From now on we assume that the conditions ($\mathcal{T}1$), (B1), (A1), (A2), (D1) and (D2.a) hold. Then the family $S_1 = (S_1(\lambda))_\lambda$ with $\lambda \in (c_2, \infty)$ is a family of selfadjoint operators and its spectrum coincides with that of $\mathcal{T}$ in (c_2, ∞), provided that T_{12}^* is surjective. This means that the problem of determining the spectrum of $\mathcal{T}$ in the interval (c_2, ∞) is equivalent to the spectral problem of the operator family S_1.
In the remaining part of this section we prove a minimax principle for the eigenvalues of the operator family S_1 based on a minimax principle in [EL04]. The main result of this paper is stated in the appendix, theorem A.1. The next proposition summarises the assumptions on the block operator matrix $\mathcal{T}$ and provides the main properties of S_1.

Proposition 4.17. *Consider the selfadjoint block operator matrix $\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}$ with domain $\mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}^*) \oplus \mathcal{D}(T_{12})$ on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$. Assume that T_{12}^* is surjective and that the conditions (B1), (A1), (A2), (D1) and (D2.a) hold.*

(i) *For every $\lambda \in (c_2, \infty)$, the form*

$$\mathcal{D}(\mathfrak{s}_1(\lambda)) := \mathcal{D}(T_{12}^*), \quad \mathfrak{s}_1(\lambda)[u,\, v] \;:=\; (u,\, (T_{11} - \lambda)v) - \big((T_{12}^* u,\, (T_{22} - \lambda)^{-1} T_{12}^* v)\big)$$

is closed and its domain is independent of λ. The operator $S_1(\lambda)$ associated with the form is a well defined selfadjoint operator with $S_1^{[\min]}(\lambda) = S_1(\lambda)$, $\lambda \in (c_2, \infty)$.

Define the operator valued function

$$S_1 : (c_2, \infty) \longrightarrow \mathscr{C}(\mathcal{H}_1), \quad \lambda \mapsto S_1(\lambda) \tag{4.7}$$

and, for fixed $x \in \mathcal{D}(\mathfrak{s}_1)$, the function

$$\sigma_1^x : (c_2, \infty) \longrightarrow \mathbb{R}, \qquad \sigma_1^x(\lambda) = \mathfrak{s}_1(\lambda)[x]. \tag{4.8}$$

(ii) *The operator valued function $S_1 : (c_2, \infty) \to \mathscr{C}(\mathcal{H})$ of (4.7) is continuous in the norm resolvent topology, and for every $x \in \mathcal{D}(\mathfrak{s}_1)$ the function $\sigma_1^x : (c_2, \infty) \to \mathbb{R}$ of (4.8) is continuous.*

(iii) *For every $x \in \mathcal{D}(\mathfrak{s}_1) \setminus \{0\}$ the function σ_1^x is decreasing and unbounded from below.*

(iv) *The equalities $\sigma_{ess}(S_1) = \sigma_{ess}(\mathcal{T}) \cap (c_2, \infty)$ and $\sigma_p(S_1) = \sigma_p(\mathcal{T}) \cap (c_2, \infty)$ hold.*

Proof. (i) The assertions concerning $\mathfrak{s}_1(\lambda)$ have been shown in lemma 4.12 while the identity $S_1^{[\min]}(\lambda) = S_1(\lambda)$ was proved in proposition 4.16. In particular, the mapping S_1 is well defined.

(ii) From (i) it follows that the family of sesquilinear forms $(\mathfrak{s}_1(\lambda))_{\lambda \in (c_2, \infty)}$ is of type (a). Hence S_1 is a holomorphic family of type (B), which implies the holomorphy of S_1 in the norm resolvent topology. Obviously, for every $x \in \mathcal{D}(\mathfrak{s}_1)$ the function σ_1^x is even smooth on (c_2, ∞).

(iii) For every $x \in \mathcal{D}(\mathfrak{s}_1)$, $x \neq 0$, the function σ_1^x is monotonously decreasing because

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \sigma_1^x(\lambda) \;=\; \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathfrak{s}_1(\lambda)[x] \;=\; -\|x\|^2 - \|(T_{22} - \lambda)^{-1} T_{12}^* x\|^2 \;\leq\; -\|x\|^2 \;<\; 0. \tag{4.9}$$

(iv) This has been shown in corollary 4.9. $\square$

Proposition 4.18. *Suppose that in addition to the assumptions of proposition 4.17 there is a constant $b > 0$ such that for all $x \in \mathcal{D}(T_{12}^*)$ the estimate*

$$\|T_{12}^* x\| \geq b\|x\| \tag{4.10}$$

holds. For $\lambda \in (c_2, \infty)$ let $d(\lambda)$ be a nonnegative lower bound for $(\lambda - T_{22})^{-1}$, i.e.,

$$\big(x,\, (\lambda - T_{22})^{-1}x\big) \;\geq\; d(\lambda)\|x\|^2 \geq 0, \qquad x \in \mathcal{H}_2,\ \lambda \in (c_2, \infty). \tag{4.11}$$

If there is a $\delta > 0$ with

$$\delta \;<\; d(\lambda)b^2 + c_1 - \lambda \tag{4.12}$$

for all λ in a sufficiently small right neighbourhood $(c_2,\, c_2 + \varepsilon)$ of c_2, then

(v) *the spectral subspace $\mathcal{L}_{(-\infty,0)}S_1(\lambda)$ is trivial for all $\lambda \in (c_2,\, c_2 + \varepsilon)$;*

(vi) *$\sigma_{ess}(S_1) \cap (c_2,\, c_2 + \varepsilon) = \emptyset$.*

If we allow $\delta = 0$ in equation (4.12), then we can show (v) only.

Proof. For $\lambda \in (c_2,\, c_2 + \varepsilon)$, assumptions (4.11) and (4.12) imply for all $x \in \mathcal{D}(S_1(\lambda)) \setminus \{0\}$ that

$$\begin{aligned} (x,\, S_1(\lambda)x) \;&=\; \mathfrak{s}_1(\lambda)[x] \;=\; (x,\, T_{11}x) - \lambda\|x\|^2 + \big(T_{12}^* x,\, (\lambda - T_{22})^{-1}T_{12}^* x\big) \\ &\geq\; (c_1 - \lambda)\|x\|^2 + d(\lambda)\, b^2\, \|x\|^2 \;>\; \delta\, \|x\|^2. \end{aligned} \tag{4.13}$$

(v) If $\delta \geq 0$, then for all $\lambda \in (c_2,\; c_2 + \varepsilon)$ the numerical range of the selfadjoint operator$S_1(\lambda)$, the closure of which equals the closure of the numerical range of $\mathfrak{s}_1(\lambda)$, is contained in the right half plane $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \geq 0\}$, implying $\rho(S_1(\lambda)) \supseteq (-\infty,\; 0)$.

(vi) If we assume the strict inequality $\delta > 0$, then the calculation above shows $(-\infty, \delta) \subseteq \rho(S_1(\lambda))$ for $\lambda \in (c_2, c_2 + \varepsilon)$, hence $(c_2, c_2 + \varepsilon) \cap \sigma(S_1) = \emptyset$. □

Condition (4.10) on T_{12}^* of the previous proposition is fulfilled if, for example, the operator T_{12}^* is boundedly invertible. In this case we can choose $b = \|T_{12}^{*-1}\|^{-1}$. If T_{22} is bounded, then, for $\lambda \in (c_2, \infty)$ and $x \in \mathcal{H}_2$, $x \neq 0$, we find

$$\begin{aligned} \|x\|^2 \;&=\; \|x\|^{-2}\,\Big|\big((\lambda - T_{22})^{\frac{1}{2}}x,\, (\lambda - T_{22})^{-\frac{1}{2}}\,x\big)\Big|^2 \;\leq\; \|x\|^{-2}\,\big\|(\lambda - T_{22})^{\frac{1}{2}}x\big\|^2\,\big\|(\lambda - T_{22})^{-\frac{1}{2}}x\big\|^2 \\ &=\; \|x\|^{-2}\,\big(x,\, (\lambda - T_{22})x\big)\,\big(x,\, (\lambda - T_{22})^{-1}x\big) \;\leq\; \|\lambda - T_{22}\|\,\big(x,\, (\lambda - T_{22})^{-1}x\big) \\ &\leq\; (|\lambda| + \|T_{22}\|)\,\big(x,\, (\lambda - T_{22})^{-1}x\big), \end{aligned}$$

hence we can choose $d(\lambda) = (|\lambda| + \|T_{22}\|)^{-1}$. For λ in a right neighbourhood of c_2, the function d is bounded from below with bound greater than 0. Thus, if b is large enough, then condition (4.12) is satisfied.

Proposition 4.17 (iii) shows that for every $x \in \mathcal{D}(\mathfrak{s}_1) \setminus \{0\}$ the function σ_1^x has at most one zero and that it is not bounded from below. If in addition (4.12) holds with some $\delta > 0$, then σ_1^x is positive for λ in a sufficiently small right neighbourhood of c_2, see (4.13) Thus the continuity of σ_1^x implies that it has exactly one zero. We denote this zero by $p(x)$, i.e.,

$$\sigma_1^x(\lambda) = 0 \quad\Longleftrightarrow\quad \lambda = p(x). \tag{4.14}$$

If relation (4.12) does not hold, then the function σ_1^x need not have a zero. In this case we define $p(x) := -\infty$, so that obviously either $p(x) = -\infty$ or $p(x) > c_2$. Further, $p(x)$ does not depend on the norm of x, i.e., for all $\xi \in \mathbb{C} \setminus \{0\}$ we have $p(x) = p(\xi x)$.

Now fix a linear manifold $\mathcal{D} \subseteq \mathcal{H}_1$, independent of λ, such that

$$\mathcal{D}(S_1(\lambda)) \;\subseteq\; \mathcal{D} \;\subseteq\; \mathcal{D}(\mathfrak{s}_1(\lambda)), \qquad\qquad \lambda \in (c_2, \infty).$$

Such a manifold $\mathcal{D}$ exists; for example, we can choose $\mathcal{D} = \mathcal{D}(T_{12}^*)$.
For $n \in \mathbb{N}$ we define the numbers

$$\mu_n \;:=\; \min_{\substack{L\subseteq\mathcal{D}\\ \dim L=n}} \; \max_{x\in L^\times} \; p(x), \tag{4.15}$$

where $L^\times := L \setminus \{0\}$. Theorem 4.19 shows that these numbers are indeed well defined. Here and in the following, a sequence $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$ with $N = \infty$ has to be understood as the infinite sequence $\lambda_1 \le \lambda_2 \le \dots$.

In the following we need one more notation. For an interval $\Delta \subseteq \mathbb{R}$ and a selfadjoint operator S we denote its spectral subspace corresponding to Δ by $\mathcal{L}_\Delta(S)$. By λ_e we denote the lower bound of the essential spectrum of S_1, i.e.,

$$\lambda_e \;:=\; \begin{cases} \inf \sigma_{ess}(S_1) & \text{if } \sigma_{ess}(S_1) \neq \emptyset, \\ \infty & \text{if } \sigma_{ess}(S_1) = \emptyset. \end{cases}$$

If $(c_2,\, \lambda_e)$ is not empty, then the eigenvalues of S_1 in this interval are characterised by the following minimax principle.

Theorem 4.19. *Let the block operator matrix $\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22}\end{pmatrix}$, $\mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}^*) \oplus \mathcal{D}(T_{12})$ be selfadjoint in the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose that the conditions (B1), (A1), (A2), (D1) and (D2.a) are satisfied and that T_{12}^* is surjective. Further, assume that the set $(c_2,\, \lambda_e)$ is nonempty and that there is a $\lambda_0 \in (c_2,\, \lambda_e)$ such that $\dim \mathcal{L}_{(-\infty,0)} S_1(\lambda_0) < \infty$. Then the index shift*

$$n_0 := \min_{\lambda > c_2} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda) \tag{4.16}$$

is finite and $\sigma(\mathcal{T}) \cap (c_2, \lambda_e)$ consists of a (possibly infinite) sequence of eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$, where $N \in \mathbb{N}_0 \cup \{\infty\}$. If the eigenvalues are counted according to their multiplicity, then

$$\lambda_n = \mu_{n+n_0}, \qquad 1 \le n \le N, \tag{4.17}$$

and $N \in \mathbb{N}_0 \cup \{\infty\}$ is given by

$$N \;=\; n(\lambda_e) - n_0$$

where $n(\lambda_e)$ is the dimension of maximal subspaces of the set

$$\{x \in \mathcal{D} \;:\; \exists\, \lambda > c_2 \text{ with } \mathfrak{s}_1(\lambda)[x] < 0\} \cup \{0\}.$$

If $N = \infty$, then $\lim\limits_{n\to\infty} \lambda_n = \lambda_e$. If $N < \infty$ and $\sigma_{ess}(S_1) = \emptyset$, then $\mu_n = \infty$ for $n > n_0 + N$. If $N < \infty$, $\lambda_e < \infty$, then $\mu_n = \lambda_e$ for $n > n_0 + N$.
If even the stronger assumptions of proposition 4.18 are fulfilled, then $n_0 = 0$.

Proof. Proposition 4.17 shows that all assumptions of theorem [EL04, theorem 2.1] (see theorem A.1) are satisfied. Hence, the numbers μ_n exist and are equal to the eigenvalues of the operator family S_1. By corollary 4.9, we have $\sigma_p(S_1) = \sigma_p(\mathcal{T}) \cap (c_2, \infty)$ and $\sigma_{ess}(S_1) = \sigma_{ess}(\mathcal{T}) \cap (c_2, \infty)$ so that all the assertions follow from theorem [EL04, theorem 2.1].
If even the assumptions of proposition 4.18 are valid, then it follows automatically that $(c_2, \lambda_e) \neq \emptyset$ and that $\dim \mathcal{L}_{(-\infty,0)} S_1(\lambda) = 0$ for λ in a sufficiently small right neighbourhood of c_2, hence the index offset n_0 appearing in formula (4.17) vanishes. □

The numbers $p(x)$ are rather hard to estimate. However, there is a representation of $p(x)$ as the supremum of a functional $\lambda_+ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ where y varies in some subspace of $\mathcal{H}_2$. The functional λ_+ is connected with the so-called quadratic numerical range of block operator matrices, see, for example, [LMMT01]. It was used in [LLT02] to obtain a variational principle for block operator matrices with bounded off-diagonal entries.

Definition 4.20. Let $\mathcal{T} = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$ be a closed block operator matrix on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with domain $\mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{21}) \oplus \mathcal{D}(T_{12})$. Assume that the operators T_{12} and T_{21} are closed and that T_{11} is T_{21}-bounded and that T_{22} is T_{12}-bounded. For $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$, $x \neq 0$, $y \neq 0$, consider the matrices

$$\mathcal{T}_{x,y} \;:=\; \begin{pmatrix} \frac{(x,T_{11}x)}{\|x\|^2} & \frac{(x,T_{12}y)}{\|x\|\,\|y\|} \\ \frac{(y,T_{21}x)}{\|x\|\,\|y\|} & \frac{(y,T_{22}y)}{\|y\|^2} \end{pmatrix} \in M_2(\mathbb{C})$$

with eigenvalues

$$\lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix} \;:=\; \frac{1}{2}\left(\frac{(x,T_{11}x)}{\|x\|^2} + \frac{(y,T_{22}y)}{\|y\|^2} \pm \sqrt{\left(\frac{(x,T_{11}x)}{\|x\|^2} - \frac{(y,T_{22}y)}{\|y\|^2}\right)^2 + \frac{4(x,T_{12}y)(y,T_{21}x)}{\|x\|^2\,\|y\|^2}}\right) \tag{4.18}$$

and define the sets

$$\Lambda_\pm(\mathcal{T}) \;:=\; \left\{ \lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x \in \mathcal{D}(T_{21}),\; y \in \mathcal{D}(T_{12}),\; x, y \neq 0 \right\}.$$

The *quadratic numerical range* $W^2(\mathcal{T})$ of $\mathcal{T}$ is defined as the set of all complex numbers λ that are eigenvalues of some $\mathcal{T}_{x,y}$, that is,

$$W^2(\mathcal{T}) \;:=\; \bigcup_{\substack{x\in\mathcal{D}(T_{21})^\times \\ y\in\mathcal{D}(T_{12})^\times}} \sigma_p(\mathcal{T}_{x,y}) \;=\; \Lambda_+(\mathcal{T}) \cup \Lambda_-(\mathcal{T}).$$

Another way to define the quadratic numerical range would be to consider the operator valued mapping

$$W:\; \mathfrak{D} := \mathbb{C} \times \mathcal{D}(T_{21})^\times \times \mathcal{D}(T_{12})^\times \longrightarrow M_2(\mathbb{C}), \qquad (\lambda, x, y) \mapsto \mathcal{T}_{x,y} - \lambda,$$

where, as usual, $\mathcal{D}(T_{ij})^\times := \mathcal{D}(T_{ij}) \setminus \{0\}$ for $i, j = 1, 2$. In analogy to the point spectrum of an operator valued function, the point spectrum of this mapping can be defined as $\sigma_p(W) := \{(\lambda, x, y) \in \mathfrak{D} : 0 \in \sigma_p(W(\lambda, x, y))\} \subseteq \mathfrak{D}$. The quadratic numerical range of $\mathcal{T}$ is then the projection of $\sigma_p(W)$ onto its first component.

It is easy to see that $\lambda_\pm \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ does not depend on the norm of the vectors x and y. It therefore suffices to restrict the definition of $\lambda_\pm \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to elements $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$ with $\|x\| = \|y\| = 1$.
In the following we characterise $p(x)$, defined in (4.14), in terms of $\lambda_\pm \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Recall that $p(x)$ was the unique zero of the function $\lambda \mapsto \sigma_1^x(\lambda)$ if it exists and $p(x) = -\infty$ otherwise.

Lemma 4.21. *Assume that the conditions of proposition 4.17 hold. Then for all $x \in \mathcal{D}(T_{12}^*) \setminus \{0\}$ with $p(x) \neq -\infty$ we have*

$$p(x) \;=\; \sup\left\{ \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; y \in \mathcal{D}(T_{12}) \setminus \{0\} \right\}. \tag{4.19}$$

If in addition $x \in \mathcal{D}(S_1(p(x)))$, then the supremum is attained, thus we have

$$p(x) \;=\; \max\left\{ \lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; y \in \mathcal{D}(T_{12}) \setminus \{0\} \right\}. \tag{4.20}$$

Proof. Fix $x \in \mathcal{D}(T_{12}^*) \setminus \{0\}$. Since $T_{21} = T_{12}^*$ and the operators T_{11} and T_{22} are symmetric, (4.18) shows that $\lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is real for all $y \in \mathcal{D}(T_{12}) \setminus \{0\}$. For the proof of the assertion, we first show that $p(x) \geq \lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for all $y \in \mathcal{D}(T_{12}) \setminus \{0\}$. So fix $y \in \mathcal{D}(T_{12}) \setminus \{0\}$ and, for simplicity, set $\lambda_+ := \lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. If $\lambda_+ \leq c_2$, then nothing has to be shown since $c_2 \leq p(x)$ by assumption. Now assume $\lambda_+ > c_2$. Since $p(x)$ is the unique zero of the monotonously decreasing function σ_1^x, it suffices to show $\sigma_1^x(\lambda_+) = \mathfrak{s}_1(\lambda_+)[x] \geq 0$. By definition, λ_+ is an eigenvalue of the complex 2×2-matrix $\mathcal{T}_{x,y}$, thus

$$\begin{aligned} 0 &= \|x\|^2 \,\|y\|^2 \det(\mathcal{T}_{x,y} - \lambda_+) \\ &= \big((x,\, T_{11}x) - \lambda_+\|x\|^2\big)\big((y,\, T_{22}y) - \lambda_+\|y\|^2\big) - (x,\, T_{12}y)(y,\, T_{12}^*x) \\ &= \big(y,\, (T_{22} - \lambda_+)y\big)\, \mathfrak{s}_1(\lambda_+)[x] + \big(y,\, (T_{22} - \lambda_+)y\big)\big(T_{12}^*x,\, (T_{22} - \lambda_+)^{-1}T_{12}^*x\big) - |(y,\, T_{12}^*x)|^2. \end{aligned} \tag{4.21}$$

For $\lambda > c_2$ the operator $(\lambda - T_{22})$ is strictly positive and the same holds for the induced sesquilinear form $(u, v) \mapsto (u,\, (\lambda - T_{22})v)$ for $u,\, v \in \mathcal{D}(T_{22})$. For this form, we have the following generalised Cauchy-Schwarz inequality

$$\begin{aligned} |\,(u,\, (\lambda - T_{22})v)\,|^2 &= \left|\,\big((\lambda - T_{22})^{\frac{1}{2}}u,\, (\lambda - T_{22})^{\frac{1}{2}}v\big)\,\right|^2 \;\leq\; \big\|(\lambda - T_{22})^{\frac{1}{2}}u\big\|^2\, \big\|(\lambda - T_{22})^{\frac{1}{2}}v\big\|^2 \\ &= \big(u,\, (\lambda - T_{22})u\big)\big(v,\, (\lambda - T_{22})v\big) \end{aligned}$$

for all $u,\, v \in \mathcal{D}(T_{22})$. Since $y \in \mathcal{D}(T_{12}) \subseteq \mathcal{D}(T_{22})$, we can use this inequality to estimate the last two terms in (4.21):

$$\begin{aligned} &\big(y,\, (T_{22} - \lambda_+)y\big)\big(T_{12}^*x,\, (T_{22} - \lambda_+)^{-1}T_{12}^*x\big) - |(y,\, T_{12}^*x)|^2 \\ &\quad= \big(y,\, (T_{22} - \lambda_+)y\big)\big(T_{12}^*x,\, (T_{22} - \lambda_+)^{-1}T_{12}^*x\big) - \big|\big(y,\, (\lambda_+ - T_{22})(\lambda_+ - T_{22})^{-1}T_{12}^*x\big)\big|^2 \\ &\quad\geq \big(y,\, (\lambda_+ - T_{22})y\big)\big(T_{12}^*x,\, (\lambda_+ - T_{22})^{-1}T_{12}^*x\big) - \big(y,\, (\lambda_+ - T_{22})y\big)\big(T_{12}^*x,\, (\lambda_+ - T_{22})^{-1}T_{12}^*x\big) \\ &\quad= 0. \end{aligned}$$

Because the factor $(y,\, (T_{22} - \lambda_+)y)$ in the first term of (4.21) is negative, it follows that the second factor, $\mathfrak{s}_1(\lambda_+)[x] = \sigma_1^x(\lambda_+)$, must be nonnegative, and thus we have proved the inequality $p(x) \geq \sup\{\lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \,:\, y \in \mathcal{D}(T_{12}) \setminus \{0\}\}$.
If $x \in \mathcal{D}(S_1(p(x)))$, then we can choose an element y such that $p(x) = \lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. To this end, define $y := (T_{22} - p(x))^{-1}T_{12}^*x$. This vector is well defined and it lies in the domain of T_{12} since by assumption $x \in \mathcal{D}(S_1(p(x)))$. If we use

$$\big(y,\, (T_{22} - p(x))y\big)\big(T_{12}^*x,\, (T_{22} - p(x))^{-1}T_{12}^*x\big) - \big|\big(T_{12}^*x,\;\, y\big)\big|^2 \;=\; 0 \tag{4.22}$$

we obtain in analogy to equation (4.21)

$$\|x\|^2 \,\|y\|^2 \det(\mathcal{T}_{x,y} - p(x)) \;=\; \big(y,\, (T_{22} - p(x))y\big)\, \mathfrak{s}_1(p(x))[x] \;=\; 0.$$

This implies that $p(x)$ is an eigenvalue of $\mathcal{T}_{x,y}$. Together with $\lambda_-\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \leq \lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \leq p(x)$ it follows that $p(x) = \lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ which proves (4.19) and (4.20) in the case $x \in \mathcal{D}(S_1(p(x)))$.
It remains to show (4.19) in the case $x \notin \mathcal{D}(S_1(p(x)))$, i.e., for elements $x \in \mathcal{D}(T_{12}^*)$ such that $(T_{22} - p(x))^{-1}T_{12}^*x \notin \mathcal{D}(T_{12})$. For fixed $x \in \mathcal{D}(T_{12}^*) \setminus \mathcal{D}(S_1(p(x)))$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(S_1(p(x)))$ such that

$$x_n \to x \quad \text{and} \quad (T_{22} - \lambda)^{-\frac{1}{2}}T_{12}^*x_n \to (T_{22} - \lambda)^{-\frac{1}{2}}T_{12}^*x, \qquad n \to \infty,$$

since in the proof of proposition 4.3 we saw that $\mathcal{D}(S_1(p(x)))$ is a core of $(T_{22} - p(x))^{-\frac{1}{2}}T_{12}^*$. Set $y_n := (T_{22} - p(x))^{-1}T_{12}^*x_n$, $n \in \mathbb{N}$. Because both $(T_{22} - p(x))^{-1}$ and $T_{22} - p(x)$ are bounded, the limites $y := \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} T_{12}^* x_n$ exist and are not zero; otherwise it would follow that $x \in \mathcal{D}(S_1(p(x)))$ in contradiction to the assumption on x. Moreover, since T_{11} is relatively bounded with respect to T_{12}^*, also the limit $\lim_{n\to\infty} T_{11}x_n$ exists. Therefore, all terms in

$$\begin{aligned}\mathfrak{s}_1(p(x))[x_n] &= \mathfrak{s}_1(p(x))[x] + \mathfrak{s}_1(p(x))[x_n - x, x_n] + \mathfrak{s}_1(p(x))[x_n, x_n - x] + \mathfrak{s}_1(p(x))[x_n - x]\\ &= 2\left(x_n - x,\, (T_{11} - p(x))x_n\right) + \left(x_n - x,\, (T_{11} - p(x))(x_n - x)\right)\\ &\quad - 2\left(T_{12}^*(x_n - x),\, (T_{22} - p(x))^{-1}T_{12}^* x_n\right) - \left(T_{12}^*(x_n - x),\, (T_{22} - p(x))^{-1}T_{12}^*(x_n - x)\right)\end{aligned}$$

converge to zero for $n \to \infty$. As in (4.22), we obtain

$$\left(y_n,\, (T_{22} - p(x))y_n\right)\left(T_{12}^* x_n,\, (T_{22} - p(x))^{-1}T_{12}^* x_n\right) - \left|\left(T_{12}^* x_n,\, y_n\right)\right|^2 = 0, \qquad n \in \mathbb{N},$$

which implies

$$\|x_n\|\,\|y_n\| \det(\mathcal{T}_{x_n,y_n} - p(x)) = \left(y_n,\, (T_{22} - p(x))y_n\right)\mathfrak{s}_1(p(x))[x_n] \longrightarrow 0, \quad n \to \infty.$$

Since neither x_n nor y_n tend to zero, it follows that

$$\Big(p(x) - \lambda_-\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\Big)\Big(p(x) - \lambda_+\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\Big) = \det(\mathcal{T}_{x_n,y_n} - p(x)) \longrightarrow 0, \qquad n \to \infty. \tag{4.23}$$

Each entry of

$$\begin{aligned}\mathcal{T}_{x_n,y_n} - \mathcal{T}_{x,y} &= \begin{pmatrix} (x_n,\, T_{11}(x_n - x)) + (x_n - x,\, T_{11}x) & (x_n,\, T_{12}(y_n - y)) + (x_n - x,\, T_{12}y) \\ (y_n,\, T_{12}^*(x_n - x)) + (y_n - y,\, T_{12}^* x) & (y_n,\, T_{22}(y_n - y)) + (y_n - y,\, T_{22}y) \end{pmatrix}\\ &= \begin{pmatrix} (x_n,\, T_{11}(x_n - x)) + (x_n - x,\, T_{11}x) & (T_{12}^* x_n,\, y_n - y) + (T_{12}^*(x_n - x),\, y) \\ (y_n,\, T_{12}^*(x_n - x)) + (y_n - y,\, T_{12}^* x) & (y_n,\, T_{22}(y_n - y)) + (y_n - y,\, T_{22}y) \end{pmatrix}\end{aligned}$$

converges to zero for $n \to \infty$, hence we have $\mathcal{T}_{x_n,y_n} \to \mathcal{T}_{x,y}$ in norm. Thus the eigenvalues $\lambda_\pm\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)$ converge. Now, if $p(x) > \sup\left\{\lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : y \in \mathcal{D}(T_{12}^*) \setminus \{0\}\right\}$, then there exists an $\varepsilon > 0$ such that $p(x) > \lambda_+\left(\begin{smallmatrix} x \\ \tilde{y} \end{smallmatrix}\right) + \varepsilon$ for all $\tilde{y} \in \mathcal{D}(T_{12}^*) \setminus \{0\}$. Since the functional λ_+ depends continuously on both its independent variables, there exists an N such that $\left|\lambda_+\left(\begin{smallmatrix} x \\ y_n \end{smallmatrix}\right) - \lambda_+\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\right| < \frac{\varepsilon}{2}$ and for all $n \geq N$. Thus we have

$$p(x) - \lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} = p(x) - \lambda_+\begin{pmatrix} x \\ y_n \end{pmatrix} + \lambda_+\begin{pmatrix} x \\ y_n \end{pmatrix} - \lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Because of $p(x) - \lambda_-\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right) \geq p(x) - \lambda_+\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)$ it follows

$$\left(\lambda_-\begin{pmatrix} x_n \\ y_n \end{pmatrix} - p(x)\right)\left(\lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} - p(x)\right) \geq \frac{\varepsilon^2}{4}, \qquad n > N, \tag{4.24}$$

in contradiction to (4.23). □

Remark 4.22. From (4.19) it is clear that if $\lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) > c_2$ for some $y \in \mathcal{D}(T_{12})$, then also $p(x) > c_2$. Vice verse, if there is an $x \in \mathcal{D}(T_{12}^*)$ such that $x \in \mathcal{D}(S_1(p(x)))$ and $p(x) \neq -\infty$, then it follows from (4.20) that there exists a $y \in \mathcal{D}(T_{12})$ such that $\lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = p(x) > c_2$. ◊

Now we can state the main theorem of this section, which is essentially a corollary of the previous lemma and theorem 4.19.

Theorem 4.23. *Suppose that the assumptions of theorem 4.19 hold, that is, suppose that conditions ($\mathcal{T}$1), (B1), (A1), (A2), (D1) and (D2.a) are fulfilled, that T_{12}^* is surjective, that $(c_2, \lambda_e) \neq \emptyset$ and that there is a $\lambda_0 \in (c_2, \lambda_e)$ such that $\dim \mathcal{L}_{(-\infty,0)} S_1(\lambda_0) < \infty$. Additionally assume that the domain of $S_1(\lambda)$ does not depend on λ, i.e.,*

$$\mathcal{D}(S_1(\lambda)) \;=\; \mathcal{D}(S_1), \qquad\qquad \lambda \in (c_2, \infty).$$

Then the eigenvalues of $\mathcal{T}$ in (c_2, λ_e) are given by

$$\begin{aligned}
\lambda_n \;&=\; \min_{\substack{L\subseteq\mathcal{D}(S_1)\\ \dim L = n+n_0}} \; \max_{x\in L^\times} \; \max_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \\
&=\; \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*)\\ \dim L = n+n_0}} \; \max_{x\in L^\times} \; \sup_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}, \qquad\qquad 1 \le n \le N,
\end{aligned} \tag{4.25}$$

where we have adopted the notation of theorem 4.19. □

Proof. From theorem 4.19 it follows that all eigenvalues of $\mathcal{T}$ greater than c_2 are given by

$$\lambda_n \;=\; \min_{\substack{L\subseteq\mathcal{D}\\ \dim L = n+n_0}} \; \max_{x\in L^\times} \, p(x), \qquad\qquad 1 \le n \le N,$$

where $\mathcal{D}$ is any linear manifold with $\mathcal{D}(S_1) \subseteq \mathcal{D} \subseteq \mathcal{D}(\mathfrak{s}_1)$. From proposition 4.17 we know that the forms $\mathfrak{s}_1(\lambda)$, $\lambda \in (c_2, \infty)$ are closed and that $\mathcal{D}(\mathfrak{s}_1(\lambda)) = \mathcal{D}(T_{12}^*)$. Fix $n > 0$ and a subspace $L \subseteq \mathcal{D}(\mathfrak{s}_1)$ with $\dim L = n + n_0$. Then there exists an $x \in L$ with $p(x) \neq -\infty$. Lemma 4.21 and the remark thereafter yield

$$\max_{x\in L^\times} p(x) \;=\; \max_{\substack{x\in L^\times\\ p(x)\neq-\infty}} p(x) \;=\; \max_{\substack{x\in L^\times\\ p(x)\neq-\infty}} \; \sup_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix} \;=\; \max_{x\in L^\times} \; \sup_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}.$$

If we have even $L \subseteq \mathcal{D}(S_1)$, then the supremum can be replaced by the maximum. □

In the next section we use theorem 4.23 to estimate the eigenvalues of the angular operator $\mathcal{A}$ with modulus greater than $|am|$. We know that for the angular operator the spectrum of $T_{12}T_{12}^*$ consists of discrete simple eigenvalues only. So we specialise theorem 4.23 to a class of operators for which the product $T_{12}T_{12}^*$ has only discrete point spectrum $\sigma_p(T_{12}T_{12}^*) \in (0, \infty)$, and all eigenspaces are finite dimensional.

Remark 4.24. Assume that $\mathcal{T}_0 = \begin{pmatrix} 0 & T_{12}\\ T_{12}^* & 0\end{pmatrix}$ with domain $\mathcal{D}(\mathcal{T}_0) = \mathcal{D}(T_{12}^*) \oplus \mathcal{D}(T_{12}) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ is closed and that $T_{12}T_{12}^*$ and $T_{12}^*T_{12}$ are strictly positive. Then $\sigma_p(\mathcal{T}_0) \;=\; \{\lambda \in \mathbb{R} : \lambda^2 \in \sigma_p(T_{12}T_{12}^*)\}$.

Proof. For $\lambda \in \sigma_p(\mathcal{T}_0) \setminus \{0\}$ we have $\lambda^2 \in \sigma_p(T_{12}T_{12}^*) \cap \sigma_p(T_{12}^*T_{12})$ because if $\binom{f}{g}$ is an eigenvector of $\mathcal{T}_0$ with eigenvalue λ, then $f \in \mathcal{D}(T_{12}T_{12}^*)$, $g \in \mathcal{D}(T_{12}^*T_{12})$ and $f, g \neq 0$ and it follows that $0 = (\mathcal{T}_0 + \lambda)(\mathcal{T}_0 - \lambda)\binom{f}{g} = \left(\begin{pmatrix} T_{12}T_{12}^* & 0\\ 0 & T_{12}^*T_{12}\end{pmatrix} - \lambda^2\right)\binom{f}{g} = \begin{pmatrix} (T_{12}T_{12}^*-\lambda^2)f\\ (T_{12}^*T_{12}-\lambda^2)g\end{pmatrix}$.
On the other hand, if $\mu \neq 0$ is an eigenvalue of $T_{12}^*T_{12}$ with eigenfunction g, then it is also an eigenvalue of $T_{12}T_{12}^*$ with eigenfunction $T_{12}g$. For $\sigma = \pm 1$ we define $f = \sigma\mu^{-\frac{1}{2}}\, T_{12}g$. Then we have that $(\mathcal{T}_0 - \sigma\sqrt{\mu})\binom{f}{g} = 0$, hence $\pm\sqrt{\mu}$ are eigenvalues of $\mathcal{T}_0$. □

The next theorem can be regarded as a perturbation result for the eigenvalues of the block operator matrix $\begin{pmatrix} 0 & T_{12}\\ T_{12}^* & 0\end{pmatrix}$ under the unbounded perturbation $\begin{pmatrix} T_{11} & 0\\ 0 & T_{22}\end{pmatrix}$.

Theorem 4.25. *Let $\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}$ with domain $\mathcal{D}(T_{12}) \oplus \mathcal{D}(T_{12}^*) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ be a selfadjoint block operator matrix such that the conditions ($\mathcal{T}$1), (B1), (A1), (A2), (D1) and (D2.a) hold. Then T_{11} is bounded with respect to T_{12}^*; let α and α_{21} such that*

$$\|T_{11}x\| \;\leq\; \alpha\|x\| + \alpha_{21}\|T_{12}^*x\|, \qquad x \in \mathcal{D}(T_{12}^*).$$

Further, let T_{12}^* be bijective and assume that there exists a number $b > 0$ such that $\|T_{12}^*x\| \geq b\|x\|$ for all $x \in \mathcal{D}(T_{12}^*)$. Moreover, assume that for all $\lambda \in (c_2, \infty)$ the Schur complement

$$S_1(\lambda) \;=\; T_{11} - \lambda - T_{12}(T_{22} - \lambda)^{-1}T_{12}^*, \quad \mathcal{D}(S_1(\lambda)) = \{x \in \mathcal{D}(T_{12}^*) \,:\, (T_{22} - \lambda)^{-1}T_{12}^*x \in \mathcal{D}(T_{12})\}$$

is selfadjoint and that $\mathcal{D}(S_1(\lambda)) =: \mathcal{D}(S_1)$ is independent of λ. Additionally suppose that there exists a $\lambda_0 \in (c_2, \infty)$ such that $\dim \mathcal{L}_{(-\infty,0)}S_1(\lambda_0) < \infty$. If the spectrum of the operator $T_{12}T_{12}^*$ satisfies

$$\sigma(T_{12}T_{12}^*) \;=\; \sigma_p(T_{12}T_{12}^*) \;=\; \{\nu_j \,:\, j \in \mathbb{N}\} \quad \text{with} \quad 0 < \nu_1 \leq \nu_2 \leq \dots$$

where the eigenvalues are counted with their multiplicities, then the block operator matrix $\mathcal{T}$ has discrete point spectrum $\lambda_1 \leq \lambda_2 \leq \dots \lambda_N$ in (c_2, λ_e). More precisely, with n_0 as in theorem 4.19, that is, $n_0 = \min\limits_{\lambda > c_2} \dim \mathcal{L}_{(-\infty,0)}S_1(\lambda)$, the eigenvalues λ_n of $\mathcal{T}$ in (c_2, λ_e) satisfy the estimates

$$\lambda_n \;\leq\; \frac{\alpha_{21}}{2}\sqrt{\nu_{n+n_0}} + \sqrt{\nu_{n+n_0} + \tfrac{1}{4}(\alpha_{21}\sqrt{\nu_{n+n_0}} + \|T_{22}\| + \alpha)^2} \; + \tfrac{1}{2}\,(\alpha + c_2)\;, \quad 1 \leq n \leq N, \tag{4.26}$$

$$\lambda_n \;\geq\; \sqrt{\nu_{n+n_0}} \; + \tfrac{1}{2}(c_1 - \|T_{22}\|), \qquad 1 \leq n \leq N. \tag{4.27}$$

Proof. All assumptions of theorem 4.23 are satisfied. In particular, the index shift n_0 is finite. To prove inequalities (4.26) and (4.27), we estimate the right hand side of (4.25). For the proof of (4.26) note that $\mathcal{D}(T_{22}) = \mathcal{H}_2$ and that

(i) $|(x,\, T_{11}x)| \;\leq\; \|x\|\,\|T_{11}x\| \;\leq\; \|x\|\,(\alpha\|x\| + \alpha_{21}\|T_{12}^*x\|), \quad x \in \mathcal{D}(T_{12}^*)$,

(ii) $(y,\, T_{22}y) \;\leq\; c_2\|y\|^2, \quad y \in \mathcal{H}_2$,

(iii) $|(y,\, T_{22}y)| \;\leq\; \|T_{22}\|\,\|y\|^2, \quad y \in \mathcal{H}_2$,

(iv) $|(y,\, T_{12}^*x)|^2 \;\leq \|y\|^2\,\|T_{12}^*x\|^2, \quad x \in \mathcal{D}(T_{12}^*),\; y \in \mathcal{H}_2$.

With the help of these inequalities we find for all $x \in \mathcal{D}(T_{12}^*)$, $y \in \mathcal{D}(T_{12})$ with $x \neq 0$, $y \neq 0$

$$\begin{aligned}
\lambda_+\begin{pmatrix} x \\ y \end{pmatrix} \;&=\; \frac{1}{2}\left(\frac{(x,\,T_{11}x)}{\|x\|^2} + \frac{(y,\,T_{22}y)}{\|y\|^2} + \sqrt{\left(\frac{(x,\,T_{11}x)}{\|x\|^2} - \frac{(y,\,T_{22}y)}{\|y\|^2}\right)^2 + \frac{4|(y,\,T_{12}^*x)|^2}{\|x\|^2\,\|y\|^2}}\;\right) \\
&\leq\; \frac{1}{2}\left(\frac{(x,\,T_{11}x)}{\|x\|^2} + \frac{(y,\,T_{22}y)}{\|y\|^2} + \sqrt{\left(\frac{|(x,\,T_{11}x)|}{\|x\|^2} + \frac{|(y,\,T_{22}y)|}{\|y\|^2}\right)^2 + \frac{4|(y,\,T_{12}^*x)|^2}{\|x\|^2\,\|y\|^2}}\;\right) \\
&\leq\; \frac{1}{2}\left(\alpha + c_2 + \frac{\alpha_{21}\|T_{12}^*x\|}{\|x\|} + \sqrt{\left(\alpha + \frac{\alpha_{21}\|T_{12}^*x\|}{\|x\|} + \|T_{22}\|\right)^2 + \frac{4\,\|T_{12}^*x\|^2}{\|x\|}}\;\right).
\end{aligned}$$

The right hand side is independent of y and monotonously increasing in $\|T_{12}^*x\|$. For given $n \in \mathbb{N}$ let $\mathscr{L}_n$ be an n-dimensional subspace of the spectral space $\mathcal{L}_{[\nu_1,\nu_n]}(T_{12}T_{12}^*)$. Then for every $x \in \mathscr{L}_n$ we have that $\|T_{12}^*x\|^2 = (x,\, T_{12}T_{12}^*x) \leq \nu_n\|x\|^2$.

Observe that $\mathcal{D}(\mathfrak{s}_1) = \mathcal{D}(T_{12})^*$, thus the minimax principle (4.25) shows that

$$\begin{aligned}
\lambda_n &= \min_{\substack{L\subseteq\mathcal{D}(T_{12})^* \\ \dim L = n+n_0}} \max_{x\in L^\times} \sup_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x \\ y\end{pmatrix} \le \max_{x\in\mathscr{L}^\times_{n+n_0}} \sup_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x \\ y\end{pmatrix} \\
&\le \max_{x\in\mathscr{L}^\times_{n+n_0}} \frac{1}{2}\left(\alpha + c_2 + \frac{\alpha_{21}\|T_{12}^* x\|}{\|x\|} + \sqrt{\left(\frac{\alpha_{21}\|T_{12}^* x\|}{\|x\|} + \|T_{22}\| + \alpha\right)^2 + \frac{4\,\|T_{12}^* x\|^2}{\|x\|^2}}\right) \\
&\le \frac{1}{2}\left(\alpha + c_2 + \alpha_{21}\sqrt{\nu_{n+n_0}} + \sqrt{(\alpha_{21}\sqrt{\nu_{n+n_0}} + \|T_{22}\| + \alpha)^2 + 4\,\nu_{n+n_0}}\right)
\end{aligned}$$

which proves (4.26). In order to show (4.27), we choose a particular element $y \in \mathcal{D}(T_{12})$. Since by assumption T_{12}^{*-1} exists and is bounded by b^{-1}, also T_{12}^{-1} exists and is bounded by b^{-1}. For every $x \in \mathcal{D}(T_{12}^*)$ the element $y(x) := T_{12}^{-1}x$ exists and lies in $\mathcal{D}(T_{12})$. Therefore, again by (4.25),

$$\begin{aligned}
\lambda_n &= \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*) \\ \dim L = n+n_0}} \max_{x\in L^\times} \sup_{y\in\mathcal{D}(T_{12})^\times} \lambda_+\begin{pmatrix} x \\ y\end{pmatrix} \\
&\ge \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*) \\ \dim L = n+n_0}} \max_{x\in L^\times} \sup_{y\in\mathcal{D}(T_{12})^\times} \frac{1}{2}\left(\frac{(x,\,T_{11}x)}{\|x\|^2} + \frac{(y,\,T_{22}y)}{\|y\|^2} + \frac{2\,|(x,\,T_{12}y)|}{\|x\|\,\|y\|}\right) && (4.28) \\
&\ge \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*) \\ \dim L = n+n_0}} \max_{x\in L^\times} \frac{1}{2}\left(\frac{(x,\,T_{11}x)}{\|x\|^2} + \frac{(T_{12}^{-1}x,\,T_{22}T_{12}^{-1}x)}{\|T_{12}^{-1}x\|^2} + \frac{2\,(x,\,x)}{\|T_{12}^{-1}x\|\,\|x\|}\right) \\
&\ge \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*) \\ \dim L = n+n_0}} \max_{x\in L^\times} \frac{1}{2}\left(\frac{(x,\,T_{11}x)}{\|x\|} - \|T_{22}\| + 2\,\|T_{12}^{-1}x\|^{-1}\|x\|\right) \\
&\ge \tfrac{1}{2}\left(c_1 - \|T_{22}\|\right) + \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*) \\ \dim L = n+n_0}} \max_{x\in L^\times} \|T_{12}^{-1}x\|^{-1}\,\|x\|. && (4.29)
\end{aligned}$$

For every n-dimensional subspace $L_n \subseteq \mathcal{D}(T_{12}^*)$, also the subspace $T_{12}^{-1}L_n \subseteq \mathcal{D}(T_{12}^*T_{12})$ is n-dimensional. Hence it follows that

$$\begin{aligned}
\min_{L\in\Lambda(n+n_0)} \max_{x\in L^\times} \|T_{12}^{-1}x\|^{-1}\,\|x\| &= \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*) \\ \dim L = n+n_0}} \max_{\xi\in T_{12}^{-1}L^\times} \|\xi\|^{-1}\,\|T_{12}\xi\| \\
&= \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*T_{12}) \\ \dim L = n+n_0}} \max_{\xi\in L^\times} \|\xi\|^{-1}\,\|T_{12}\xi\|.
\end{aligned}$$

By remark 4.24, the squares of the nonzero eigenvalues of $\widetilde{T}_0 = \left(\begin{smallmatrix} 0 & T_{12}^* \\ T_{12} & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & I \\ I & 0\end{smallmatrix}\right) \mathcal{T}_0 \left(\begin{smallmatrix} 0 & I \\ I & 0\end{smallmatrix}\right)$ are the eigenvalues $\nu_1 \le \nu_2 \le \dots$ of $T_{12}T_{12}^*$. On the other hand, the variational principle of theorem 4.23 applied to $\widetilde{T}_0$ shows that

$$\begin{aligned}
\sqrt{\nu_n} = \lambda_n &= \min_{\substack{L\subseteq\mathcal{D}(T_{12}^*T_{12}) \\ \dim L = n}} \max_{\xi\in L^\times} \max_{y\in\mathcal{D}(T_{12}^*)^\times} \frac{|(y,\,T_{12}\xi)|}{\|y\|\,\|\xi\|} \\
&\le \min_{\substack{L\in\mathcal{D}(T_{12}^*T_{12}) \\ \dim L = n}} \max_{\xi\in L^\times} \frac{|(T_{12}\xi,\,T_{12}\xi)|}{\|T_{12}\xi\|\,\|\xi\|} = \min_{\substack{L\in\mathcal{D}(T_{12}^*T_{12}) \\ \dim L = n}} \max_{\xi\in L^\times} \frac{\|T_{12}\xi\|}{\|\xi\|}.
\end{aligned}$$

Inserting into (4.29) yields

$$\lambda_n \;\geq \tfrac{1}{2}(c_1 - \|T_{22}\|) + \sqrt{\nu_{n+n_0}}.$$

□

In theorem 4.25 we saw that we can estimate the eigenvalues of the block operator matrix $\mathcal{T}$ by the eigenvalues of $T_{12}T_{12}^*$, that is, by the eigenvalues of $\left(\begin{smallmatrix} 0 & T_{12} \\ T_{12}^* & 0 \end{smallmatrix}\right)$. In the calculations leading to formulae (4.26) and (4.27) we have used that the operator T_{11} is semibounded. For the angular operator, however, T_{11} is even bounded so we can improve the estimate.

Corollary 4.26. *In addition to the assumptions in theorem 4.25, let T_{11} and T_{22} be bounded. Then there are real numbers c_1, c_1^+, c_2^- and c_2 such that*

$$c_1\,\|x\|^2 \;\leq\; (x,\,T_{11}x) \;\leq\; c_1^+\,\|x\|^2, \quad x \in \mathcal{H}_1 \quad \text{and} \quad c_2^-\,\|y\|^2 \leq\; (y,\,T_{22}y) \;\leq\; c_2\,\|y\|^2, \quad y \in \mathcal{H}_2.$$

Let $n_0 = \min\limits_{\lambda > c_2} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda)$. Then the eigenvalues of the block operator matrix $\mathcal{T}$ in $(c_2,\,\lambda_e)$, enumerated such that $c_2 < \lambda_1 \leq \lambda_2 \leq \dots$, can be estimated by

$$\lambda_n \;\leq\; \sqrt{\nu_{n+n_0} + \tfrac{1}{4}(\|T_{11}\| + \|T_{22}\|)^2} \;+ \tfrac{1}{2}(c_1^+ + c_2), \qquad 1 \leq n \leq N, \tag{4.30}$$

$$\lambda_n \;\geq\; \sqrt{\nu_{n+n_0}} \;+ \tfrac{1}{2}\big(c_1 + c_2^-\big), \qquad 1 \leq n \leq N, \tag{4.31}$$

where $0 < \nu_1 \leq \nu_2 \leq \dots$ are the eigenvalues of $T_{12}T_{12}^*$, see theorem 4.25.

Proof. First, we improve the estimates of λ_+ from above. If we use $(x,\,T_{11}x) \leq c_1^+\,\|x\|^2$ and $|(x,\,T_{11}x)| \leq \|T_{11}\|\,\|x\|^2$, $x \in \mathcal{H}_1$, instead of (i) of theorem 4.25, we obtain

$$\begin{aligned} \lambda_+\begin{pmatrix} x \\ y \end{pmatrix} \;&=\; \frac{1}{2}\left((x,\,T_{11}x) + (y,\,T_{22}y) + \sqrt{((x,\,T_{11}x) - (y,\,T_{22}y))^2 + 4|(y,\,T_{12}^*x)|^2}\right) \\ &\leq\; \frac{1}{2}(c_1^+ + c_2) + \sqrt{\tfrac{1}{4}(\|T_{11}\| + \|T_{22}\|)^2 + \|T_{12}^*x\|^2} \end{aligned}$$

for all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T})$ with $\|x\| = \|y\| = 1$. Also the upper bound for $\lambda_+\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ can be improved if we use $(y,\,T_{22}y) \geq c_2^-$ in (4.28). Now a reasoning analogous to that of theorem 4.25 completes the proof. □

The estimate (4.30) for the eigenvalues can be further improved if we use the fact that all the terms of the formula for λ_+ involving T_{11} and T_{22} are bounded. To this end, we use the following auxiliary lemma.

Lemma 4.27. *For $a_1,\ a_2,\ b_1,\ b_2,\ \gamma \in \mathbb{R}$ with $a_1 < b_1$ and $a_2 < b_2$ we define the function*

$$f : [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R},\;\; f(s,t) = s + t + \sqrt{(s-t)^2 + \gamma^2}.$$

For fixed t, the function f is monotonously increasing in s and vice versa. In particular,

$$f(a_1, a_2) \;\leq\; f(s,t) \;\leq\; f(b_1, b_2), \qquad (s,t) \in [a_1, b_1] \times [a_2, b_2].$$

Proof. Partial differentiation of f with respect to yields

$$\frac{\partial}{\partial s} f(s,t) \;=\; 1 + \frac{s-t}{\sqrt{(s-t)^2 + \gamma^2}} \;\geq\; \frac{\sqrt{(s-t)^2 + \gamma^2} - |s-t|}{\sqrt{(s-t)^2 + \gamma^2}} \;\geq\; 0.$$

□

If we apply this lemma to the functional $\lambda_+\binom{x}{y}$ with $\gamma = 2|(x,\,T_{12}^* y)|$, $s = (x,\,T_{11}x) \in (c_1,\,c_1^+)$ and $t = (y,\,T_{22}y) \in (c_2^-,\,c_2)$ it follows that

$$\frac{1}{2}\Big(c_1 + c_2^- + \sqrt{(c_1 - c_2^-)^2 + 4|(T_{12}^* x,\ y)|^2}\,\Big) \;\le\; \lambda_+\begin{pmatrix} x\\ y\end{pmatrix}$$
$$\le\; \frac{1}{2}\Big(c_1^+ + c_2 + \sqrt{(c_1^+ - c_2)^2 + 4|(T_{12}^* x,\ y)|^2}\,\Big).$$

Now we can use these estimates to improve the result in lemma 4.26. Note that the estimate (4.31) remains unchanged. We finally arrive at the following theorem.

Theorem 4.28. *With the above assumptions and notation the eigenvalues of $\mathcal{T}$ in $(c_2,\,\lambda_e)$ are given by*

$$\lambda_n \;\le\; \sqrt{\nu_{n+n_0} + \tfrac{1}{4}(c_1^+ - c_2)^2}\; + \tfrac{1}{2}(c_1^+ + c_2) \qquad 1 \le n \le N, \tag{4.32}$$

$$\lambda_n \;\ge\; \sqrt{\nu_{n+n_0}}\; + \tfrac{1}{2}(c_1 + c_2^-), \qquad 1 \le n \le N. \tag{4.33}$$

The index shift n_0 is given by $n_0 = \min\limits_{\lambda > c_2} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda)$.

Remark 4.29. In the case of bounded T_{11} and T_{22}, also methods from standard perturbation theory yield estimates for the eigenvalues of $\mathcal{T}$, see lemma 3.9. For all eigenvalues λ_n of $\mathcal{T}$ we obtain the estimate

$$\sqrt{\nu_n} - \left\|\begin{pmatrix} T_{11} & 0\\ 0 & T_{22}\end{pmatrix}\right\| \;\le\; \lambda_n \;\le\; \sqrt{\nu_n} + \left\|\begin{pmatrix} T_{11} & 0\\ 0 & T_{22}\end{pmatrix}\right\|.$$

In the formula above, there is no need to determine an index shift n_0. On the other hand, since

$$\left\|\begin{pmatrix} T_{11} & 0\\ 0 & T_{22}\end{pmatrix}\right\| \;=\; \max\{\|T_{11}\|,\,\|T_{22}\|\} \;=\; \max\{|c_1^+|,\,|c_1|,\,|c_2|,\,|c_2^-|\},$$

the sign of the operators T_{11} and T_{22} are not taken into account. $\Diamond$

Eigenvalues in some left half plane

So far, we have used the Schur complement S_1 to characterise eigenvalues of $\mathcal{T}$ to the right of c_2. For a block operator matrix $\mathcal{T}$ with domain $\mathcal{D}(\mathcal{T}) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$, consider

$$\mathcal{T}^{(-)} \;:=\; \begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}(-\mathcal{T})\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix} \;=\; \begin{pmatrix} -T_{22} & -T_{21}\\ -T_{12} & -T_{11}\end{pmatrix}$$

in $\mathcal{H}_2 \oplus \mathcal{H}_1$ with domain $\mathcal{D}(\mathcal{T}^{(-)}) = \left(\begin{smallmatrix} 0 & I\\ I & 0\end{smallmatrix}\right)\mathcal{D}(\mathcal{T})$. Since $\mathcal{T}^{(-)}$ is unitarily equivalent to $-\mathcal{T}$, it follows that $\sigma(\mathcal{T}) = -\sigma(-\mathcal{T}) = -\sigma(\mathcal{T}^{(-)})$ which implies

$$\sigma(\mathcal{T}) \cap (-\infty,\,c_1) \;=\; -\big(\sigma(\mathcal{T}^{(-)}) \cap (-c_1,\,\infty)\big).$$

Assume that $\mathcal{T}^{(-)}$ satisfies conditions (B1), (A1), (D1) and ($\mathcal{T}$1), i.e., we assume

(B1$^{(-)}$) T_{21} is a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with $T_{21}^* = T_{12}$;

(A1$^{(-)}$) $\mathcal{D}(T_{12}) \subseteq \mathcal{D}(T_{22})$ and T_{22} is symmetric in $\mathcal{H}_2$ and semibounded from above, i.e., there is a constant $c_2 \in \mathbb{R}$ such that

$$(x,\,T_{22}x) \;\le\; c_2\|x\|^2, \qquad x \in \mathcal{D}(T_{22});$$

(D1$^{(-)}$) $\mathcal{D}(T_{21}) \subseteq \mathcal{D}(T_{11})$ and T_{11} is symmetric in $\mathcal{H}_1$ and semibounded from below, i.e., there is a constant $c_1 \in \mathbb{R}$ such that

$$(x,\, T_{11}x) \;\geq\; c_1\|x\|^2, \qquad x \in \mathcal{D}(T_{11});$$

furthermore, T_{11} is closed with $(-\infty, c_1) \subseteq \rho(T_{11})$.

($\mathcal{T}1^{(-)}$) $\mathcal{T} = \begin{pmatrix} -T_{22} & -T_{21} \\ -T_{12} & -T_{11} \end{pmatrix}, \quad \mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}) \oplus \mathcal{D}(T_{21}) \subseteq \mathcal{H}_2 \oplus \mathcal{H}_1.$

Then the Schur complement $S_1^{(-)}$ of $\mathcal{T}^{(-)}$ is well defined for $\lambda \in (-c_1, \infty)$; in particular, we have

$$\begin{aligned} S_1^{(-)}(\lambda) &= -T_{22} - \lambda - (-T_{21})(-T_{11} - \lambda)^{-1}(-T_{12}) \\ &= -\big(T_{22} - (-\lambda) - T_{21}(T_{11} - (-\lambda))^{-1}T_{12}\big) \;=\; -S_2(-\lambda), \qquad \lambda > -c_1. \end{aligned}$$

If in addition $\mathcal{T}$ is selfadjoint and the Schur complement S_2 is a holomorphic operator function such that $S_2(\lambda)$, $\lambda < c_1$, is selfadjoint, then $S_1^{(-)}$ has the same properties, and consequently (see corollary 4.9),

$$\sigma(\mathcal{T}) \cap (-\infty,\, c_1) \;=\; -\big(\sigma(\mathcal{T}^{(-)}) \cap (-c_1,\, \infty)\big) \;=\; -\sigma(S_1^{(-)}) \;=\; \sigma(S_2),$$

so the spectrum of $\mathcal{T}$ to the left of c_1 is given by the spectrum of S_2.

4.2 The variational principle for the angular operator $\mathcal{A}$

In the following we return to the angular operator $\mathcal{A}$, formally given by

$$\mathfrak{A} \;=\; \begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} \qquad \text{on} \qquad (0,\pi).$$

Since in this section no other Hilbert space than $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ and products thereof occur, we set

$$\mathcal{H} \;:=\; \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta).$$

In section 2.1.2 we have seen that for $k \in \mathbb{R}\setminus(-1,0)$ the operator $\mathcal{A}^{\min}$ in the space $\mathcal{H}\times\mathcal{H}$ defined by $\mathcal{A}^{\min}f = \mathfrak{A}f$ with domain $\mathcal{D}(\mathcal{A}^{\min}) = \mathcal{C}_0^\infty(0,\pi)^2$ is essentially selfadjoint. In this case, its closure is given by the block operator matrix

$$\mathcal{A} \;=\; \begin{pmatrix} -D & B \\ B^* & D \end{pmatrix}, \qquad \mathcal{D}(\mathcal{A}) = \mathcal{D}(B^*) \oplus \mathcal{D}(B) \tag{4.34}$$

where D is the operator of multiplication by the function $(0,\pi)\to\mathbb{R}$, $\vartheta\mapsto am\cos\vartheta$ and B is the first order differential operator $\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta$. The operator B with domain

$$\mathcal{D}(B) = \big\{ f\in\mathcal{H} \,:\, f \text{ is absolutely continuous, } \mathfrak{B}_+ f \in \mathcal{H} \big\}$$

is closed; further properties of the operator B have been derived in section 3.3.2.
If not stated explicitly otherwise, it is always assumed that $k\in\mathbb{R}\setminus(-1,0)$.

The aim of this section is to apply the variational principle of section 4.1 to the angular operator $\mathcal{A}$ in order to obtain upper and lower bounds for its eigenvalues. For this task the Schur complements S_1 and S_2 of $\mathcal{A}$ play a crucial role, see (4.2). For $\lambda\in\mathbb{R}\setminus\sigma(D)$ they are given by

$$\mathcal{D}(S_1(\lambda)) = \{f\in\mathcal{D}(B^*) \,:\, (D-\lambda)^{-1}B^*f \in \mathcal{D}(B)\}, \qquad S_1(\lambda) \;:=\; -D-\lambda-B(D-\lambda)^{-1}B^*,$$
$$\mathcal{D}(S_2(\lambda)) = \{f\in\mathcal{D}(B) \,:\, (-D-\lambda)^{-1}Bf \in \mathcal{D}(B^*)\}, \qquad S_2(\lambda) \;:=\; D-\lambda-B^*(-D-\lambda)^{-1}B.$$

The Schur complements are investigated in appendix B with methods from spectral theory for linear differential operators.

4.2.1 Application of the variational principle

In the following, we consider the family S_1 for $\lambda\in(|am|,\infty)$ only.

Lemma 4.30. *The angular operator fulfils conditions ($\mathcal{T}$1), (B1), (A1), (A2), (D1) and (D2.a) of the preceding section, in particular, we have*

$$c_1 \;:=\; -|am| \;\le\; (x,\,-Dx) \;\le\; |am| \;=:\; c_1^+, \qquad x\in\mathcal{H} \tag{A1$'$}$$
$$c_2^- \;:=\; -|am| \;\le\; (x,\,Dx) \;\le\; |am| \;=:\; c_2, \qquad x\in\mathcal{H}, \tag{D1$'$}$$
$$\|-D\| \;=\; \|D\| \;=\; |am|, \tag{D2.a$'$}$$

and $\sigma(D) = \sigma(-D) = \sigma_{ess}(D) = [-|am|,|am|\,]$. For all $\lambda\in(|am|,\infty)$, the form

$$\mathcal{D}(\mathfrak{s}_1(\lambda)) = \mathcal{D}(B^*), \qquad \mathfrak{s}_1(\lambda)[f,g] \;:=\; (f,\,(-D-\lambda)g) - (B^*f,\,(D-\lambda)^{-1}B^*g), \tag{4.35}$$

is symmetric, semibounded from below and closed. Further, the operator $S_1(\lambda)$ is the selfadjoint operator associated with $\mathfrak{s}_1(\lambda)$, and its domain is independent of λ, more precisely, we have

$$\mathcal{D}(S_1(\lambda)) \;=\; \mathcal{D}(BB^*), \qquad \lambda\in(|am|,\infty). \tag{4.36}$$

Proof. Since D is the bounded operator given by multiplication with the continuous, nowhere constant function $am\cos\vartheta$, $\vartheta \in (0,\pi)$, the assertions concerning the spectrum of D and relations (A1′), (D1′) and (D2.a′) are clear. Hence conditions (A1) and (D1) are satisfied with $c_2 = |am|$ and $c_1 = -|am|$, and (A2) and (D2.a) hold because D is bounded with $\|D\| = |am|$. Conditions $(\mathcal{T}1)$ and (B1) hold because we have already shown in section 2.1.2 that the angular operator (4.34) is selfadjoint for every $a \in \mathbb{R}$. Since $\sigma(D) = [-|am|, |am|\,]$, the sesquilinear forms $\mathfrak{s}_1(\lambda)$, $\lambda \in (|am|, \infty)$, are well defined, and, by proposition 4.11 and lemma 4.12, they are symmetric, semibounded from below and closed. Proposition 4.16 implies that for $\lambda \in (|am|, \infty)$ the operator $S_1(\lambda)$ is the selfadjoint operator associated with $\mathfrak{s}_1(\lambda)$.
To prove (4.36), fix $f \in \mathcal{D}(BB^*)$ and $\lambda \in (|am|, \infty)$. We have to show that $(D-\lambda)^{-1}B^*f \in \mathcal{D}(B)$. Since both $(am\cos\vartheta - \lambda)^{-1}$ and B^*f are absolutely continuous, we have

$$\begin{aligned}
&\mathfrak{B}_+(D-\lambda)^{-1}B^*f(\vartheta)\\
&\qquad= \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)(am\cos\vartheta-\lambda)^{-1}\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)f(\vartheta)\\
&\qquad= (am\cos\vartheta-\lambda)^{-1}\Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)f(\vartheta)\\
&\qquad\quad+ \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta}(am\cos\vartheta-\lambda)^{-1}\Big)\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)f(\vartheta)\\
&\qquad= (D-\lambda)^{-1}BB^*f(\vartheta) + \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta}(am\cos\vartheta-\lambda)^{-1}\Big)B^*f(\vartheta).
\end{aligned}$$

Observe that the first term on the first line is the formal differential expression associated with B. Since, by assumption, $f \in \mathcal{D}(BB^*)$ and since both $(D-\lambda)^{-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\vartheta}(am\cos\vartheta-\lambda)^{-1}$ are bounded operators on $\mathcal{H}$, it follows that $(D-\lambda)^{-1}B^*f \in \mathcal{D}(B)$, and consequently $f \in \mathcal{D}(S_1(\lambda))$.
Conversely, assume $f \in \mathcal{D}(S_1(\lambda))$ for some $\lambda \in (|am|, \infty)$. Since the function $am\cos\vartheta - \lambda$ is differentiable on $(0,\pi)$, we have

$$\begin{aligned}
\mathfrak{B}_+B^*f &= -\Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)B^*f(\vartheta)\\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)(am\cos\vartheta-\lambda)(am\cos\vartheta-\lambda)^{-1}\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)f(\vartheta)\\
&= (am\cos\vartheta-\lambda)\Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)(am\cos\vartheta-\lambda)^{-1}\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)f(\vartheta)\\
&\quad+ \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta}(am\cos\vartheta-\lambda)\Big)(am\cos\vartheta-\lambda)^{-1}\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)f(\vartheta)\\
&= (D-\lambda)B(D-\lambda)^{-1}B^*f(\vartheta) + \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta}(am\cos\vartheta-\lambda)\Big)(D-\lambda)^{-1}B^*f(\vartheta).
\end{aligned}$$

Since $D-\lambda$ and $\frac{\mathrm{d}}{\mathrm{d}\vartheta}(am\cos\vartheta-\lambda)$ are bounded operators on $\mathcal{H}$, it follows that the function above is also an element of $\mathcal{H}$, hence we have $B^*f \in \mathcal{D}(B)$, implying $f \in \mathcal{D}(BB^*)$. □

Recall that the spectrum of the angular operator consists only of isolated simple eigenvalues without accumulation points in $(-\infty,\infty)$, see theorem 2.14. We also know that the eigenvalues depend continuously on the parameter a. Hence we can enumerate the eigenvalues λ_n, $n \in \mathbb{Z}\setminus\{0\}$, unambiguously by requiring that λ_n is the analytic continuation of $\lambda_n = \operatorname{sign}(n)\big(k+\frac12| - \frac12 + n\big)$ in the case $a = 0$. Since all eigenvalues are simple, it follows that $\lambda_n < \lambda_m$ for $n < m$.
For fixed Kerr parameter a we define $m_\pm \in \mathbb{Z}$ such that

$$\dots \le \lambda_{m_--2} \le \lambda_{m_--1} < -|am| \le \lambda_{m_-} \le \dots \le \lambda_{m_+} \le |am| < \lambda_{m_++1} \le \lambda_{m_++2} \le \dots$$

i.e., $\sigma(\mathcal{A}) \cap [-|am|, |am|\,] = \{\lambda_n \,:\, m_- \le n \le m_+,\ n \neq 0\}$ and the number of eigenvalues of $\mathcal{A}$ in the interval $[-|am|, |am|\,]$ is given by

$$\#\left(\sigma(\mathcal{A}) \cap [-|am|, |am|\,]\right) \;=\; \begin{cases} m_+ - m_- & \text{if } 0 \in [m_-, m_+], \\ m_+ - m_- + 1 & \text{if } 0 \notin [m_-, m_+]. \end{cases}$$

Observe that m_+ and m_- depend on the physical parameters a, m, ω and k.

Remark 4.31. Since the operator $\mathcal{B} = \left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$ coincides with the angular operator in the case $m = 0$, its spectrum also consists of discrete simple eigenvalues only. Further, $0 \in \rho(\mathcal{B})$, see corollary 3.20, and the spectrum of $\mathcal{B}$ is symmetric with respect to 0, see corollary 2.18. In the following, we always order the eigenvalues μ_n, $n \in \mathbb{Z} \setminus \{0\}$, of $\mathcal{B}$ such that

$$\ldots \;\le\; -\mu_2 \;\le\; -\mu_1 \;<\; 0 \;<\; \mu_1 \;\le\; \mu_2 \;\le\; \ldots; \tag{4.37}$$

then the spectrum of BB^* is given by $\sigma(BB^*) = \{\nu_n = \mu_n^2 \,:\, n \in \mathbb{N}\}$, see also remark 4.24. Furthermore, we have $\|B^{-1}\|^{-1} = \mu_1$. With the enumeration (4.37) the eigenvalues of $\mathcal{B}$ and $\mathcal{A}$ in the case $m = 0$ coincide. Estimates for the eigenvalues ν_n are given in section 4.2.2. ◊

Now we are ready to apply the theorems of the preceding section to characterise the eigenvalues $\lambda_{m_+ + n}$, $n \in \mathbb{N}$.

Theorem 4.32. *Let $n_0 = \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda_0)$ for some $\lambda_0 \in (|am|, \lambda_{m_+ + 1})$. Then the eigenvalues of the angular operator $\mathcal{A}$ to the right of $|am|$ are given by*

$$\lambda_{m_+ + n} \;=\; \min_{\substack{L \subseteq \mathcal{D}(BB^*) \\ \dim L = n + n_0}} \;\max_{x \in L^\times} \;\max_{y \in \mathcal{D}(B)^\times} \;\lambda_+ \begin{pmatrix} x \\ y \end{pmatrix}, \qquad n \in \mathbb{N}. \tag{4.38}$$

Furthermore, the eigenvalues can be estimated by

$$\sqrt{\nu_{n_0 + n}} - |am| \;\le\; \lambda_{m_+ + n} \;\le\; \sqrt{\nu_{n_0 + n}} + |am|, \qquad n \in \mathbb{N}, \tag{4.39}$$

where $\nu_{n + n_0} = \mu_{n + n_0}^2$ are the eigenvalues of BB^*.

Proof. By lemma 4.30, the angular operator satisfies conditions ($\mathcal{T}$1), (B1), (A1), (A2), (D1) and (D2.a), and the domain of the operators $S_1(\lambda)$ does not depend on λ for $\lambda \in (|am|, \infty)$. Further, B^* is surjective because $0 \in \rho(\mathcal{B})$, so we have

$$\sigma_{ess}(S_1) = \sigma_{ess}(\mathcal{A}) \cap (|am|, \infty) = \emptyset$$

by corollary 4.9. Formula (4.38) now follows from theorem 4.23 with $c_2 = |am|$ and $\lambda_e = \infty$. Since D is bounded and

$$-|am|\,\|x\|^2 \;\le\; (x,\, Dx) \;\le\; |am|\,\|x\|^2, \qquad x \in \mathcal{H},$$

application of theorem 4.28 with $c_2 = c_1^+ = |am|$ and $c_2^- = c_1 = -|am|$ yields the estimates (4.39). By theorem 4.23 we have $n_0 = \min\limits_{\lambda \in (|am|, \infty)} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda)$. Since $(|am|, \lambda_{m_+ + 1}) \subseteq \rho(S_1)$ and the index shift n_0 is constant on $\rho(S_1)$, also the assertion concerning n_0 is proved. □

Note that the index shift n_0 does not depend on the choice of $\lambda_0 \in (|am|, \lambda_{m_+ + 1})$ but, of course, it depends on the parameters a and m.
The following lemma gives a sufficient condition for the index shift to be nontrivial.

Lemma 4.33. *If there exists an eigenvalue μ of $\mathcal{A}$ such that*

$$2\,|am| - \lambda_{m_+ +1} \;<\; \mu \;<\; \lambda_{m_+ +1}, \tag{4.40}$$

then we have $n_0 \geq 1$. If in addition $\lambda_{m_+ +1} \leq 3\,|am|$, then there is at least one eigenvalue of $\mathcal{A}$ in $[-|am|, |am|\,]$.

Proof. Recall that $\lambda_{m_+ +1}$ is the first eigenvalue of $\mathcal{A}$ which is greater than $|am|$, hence we have $\mu \leq |am|$. If we also know $\lambda_{m_+ +1} \leq 3\,|am|$, then (4.40) shows that $\mu > 2|am| - \lambda_{m_+ +1} \geq -|am|$. Hence we have that the eigenvalue μ of $\mathcal{A}$ lies in $[-|am|, |am|\,]$.
It remains to be shown that $n_0 \geq 1$. Recall that $n_0 = \min\limits_{\lambda > |am|} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda)$ and that the right hand side is constant on the resolvent set of $\mathcal{A}$ and increasing with increasing λ. Hence it follows that $n_0 = n(\lambda) := \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda)$ for all $\lambda \in (|am|, \lambda_{m_+ +1})$. Let $\mathcal{D}$ be an arbitrary linear manifold such that $\mathcal{D}(S_1) \subseteq \mathcal{D} \subseteq \mathcal{D}(\mathfrak{s}_1)$ independent of λ. In [EL04, lemma 2.5] it has been shown that n_0 is equal to the dimension of every maximal subspace of $\mathcal{N}(\lambda) = \{x \in \mathcal{D} : \mathfrak{s}_1(\lambda)[x] < 0\} \cup \{0\}$. Since $\mathcal{D}(\mathfrak{s}_1) = \mathcal{D}(B^*)$ does not depend on λ, it suffices to show that there exists $x \in \mathcal{D}(B^*)$, $x \neq 0$ such that $\mathfrak{s}_1(\lambda_0)[x] < 0$ for some $\lambda_0 \in (|am|, \lambda_{m_+ +1})$ because then x spans a onedimensional subspace in $\mathcal{N}(\lambda_0)$. Since μ is an eigenvalue of $\mathcal{A}$, there exists an element $\binom{x}{y} \in \mathcal{D}(B^*) \oplus \mathcal{D}(B)$ such that $(\mathcal{A} - \mu)\binom{x}{y} = 0$, i.e.,

$$(-D - \mu)x + By = 0, \qquad\qquad B^*x + (D - \mu)y = 0.$$

In particular, we have $(D-\lambda)^{-1}B^*x = -(D-\lambda)^{-1}(D-\mu)y = -y + (\mu-\lambda)(D-\lambda)^{-1}y$ and $(B^*x,\, y) = (x,\, By) = (x,\, (D+\mu)x)$. Thus for every $\lambda > |am|$

$$\begin{aligned}
\mathfrak{s}_1(\lambda)[x] &= \big(x,\, (-D-\lambda)x\big) - \big(B^*x,\, (D-\lambda)^{-1}B^*x\big) \\
&= -\lambda\|x\|_2^2 - (x,\, Dx) + (B^*x,\, y) - (\mu-\lambda)\big((D-\lambda)^{-1}B^*x,\, y\big) \\
&= (\mu-\lambda)\left(\|x\|_2^2 + \|y\|_2^2\right) - (\mu-\lambda)^2\left((D-\lambda)^{-1}y,\, y\right).
\end{aligned}$$

Since $\lambda > |am| = \|D\|$, we have $0 < -\big((D-\lambda)^{-1}y,\, y\big) \leq (\lambda - |am|)^{-1}\|y\|_2$. Furthermore, we know from lemma 2.17(iii) that $x(\vartheta) = y(\pi - \vartheta)$ for all $\vartheta \in (0,\pi)$ which implies $\|x\|_2 = \|y\|_2$. Thus we have

$$\mathfrak{s}_1(\lambda)[x] \;\leq\; \|x\|_2^2\,(\lambda - |am|)^{-1}(\mu-\lambda)(\mu+\lambda-2|am|).$$

Set $\lambda_0 := \lambda_{m_+ +1} - \frac{1}{2}(\mu + \lambda_{m_+ +1} - 2|am|)$. Then it follows from (4.40) that $\lambda_0 \in (|am|, \lambda_{m_+ +1})$. Furthermore, we have $\mu - \lambda_0 < 0$ and $\mu + \lambda_0 - 2|am| = \frac{1}{2}(\mu + \lambda_{m_+ +1}) - |am| > 0$ by (4.40). Thus it follows

$$\mathfrak{s}_1(\lambda_0)[x] \;\leq\; \|x\|_2^2\,(\lambda_0 - |am|)^{-1}(\mu-\lambda_0)(\mu+\lambda_0-2|am|) \;<\; 0. \qquad \square$$

Recall that ν_n, $n \in \mathbb{N}$, denote the eigenvalues of BB^*.

Lemma 4.34. (i) *If there exists $j_0 \geq 2$ such that*

$$\sqrt{\nu_{n_0+j_0}} - \sqrt{\nu_{n_0+j_0-1}} \;>\; 2\,|am| \qquad \text{and} \qquad \sqrt{\nu_{n_0+j_0+1}} - \sqrt{\nu_{n_0+j_0}} \;>\; 2\,|am|, \tag{4.41}$$

then $n_0 = m_+$.

(ii) *If $\|B^{*-1}\|^{-1} > 2|am|$, then the angular operator $\mathcal{A}$ has no eigenvalues in $[-|am|, |am|\,]$ and we have $n_0 = 0$ and $m_+ = 0$.*

Proof. (i) From standard perturbation theory, see remark 3.12, we know that

$$\operatorname{sign}(n)\sqrt{\nu_{|n|}} - |am| \;\le\; \lambda_n \;\le\; \operatorname{sign}(n)\sqrt{\nu_{|n|}} + |am|, \qquad n \in \mathbb{Z}\setminus\{0\}. \tag{4.42}$$

Hence (4.41) implies that the angular operator $\mathcal{A}$ has exactly one eigenvalue in the interval $[\sqrt{\nu_{n_0+j_0}} - |am|,\ \sqrt{\nu_{n_0+j_0}} + |am|\,]$. Since by (4.41) and (4.39) both $\lambda_{n_0+j_0}$ and $\lambda_{m_++j_0}$ lie in this interval, it follows that $n_0 = m_+$.

(ii) Assume that $\|B^{*-1}\|^{-1} > 2|am|$. Then we have $\sqrt{\nu_1} > 2\,|am|$ for the smallest eigenvalue ν_1 of BB^*. From the estimate (4.42) we obtain that

$$\lambda_{-1} \le -\sqrt{\nu_1} + |am| < -|am|. \quad \text{and} \quad \lambda_1 \ge \sqrt{\nu_1} - |am| > |am|$$

Hence the angular operator $\mathcal{A}$ has no eigenvalues in $[-|am|, |am|\,]$ which implies $m_+ = 0$.
For $\lambda > |am|$ define the set $\mathcal{N}(\lambda)$ as in the proof of lemma 4.33. Since n_0 is equal to the maximal dimension of subspaces of $\mathcal{N}(\lambda)$ for $\lambda \in (|am|, \lambda_{m_++1})$, it suffices to show that $\mathcal{N}(\lambda) = \{0\}$ for λ close enough to $|am|$. To this end fix an arbitrary $x \in \mathcal{D}(\mathfrak{s}_1) = \mathcal{D}(B^*)$. Then it is easy to see that for all $\lambda > |am|$

$$\begin{aligned}
\mathfrak{s}_1(\lambda)[x] &= \big(x, (-D-\lambda)x\big) - \big(B^*x, (D-\lambda)^{-1}B^*x\big) \\
&\ge (-|am| - \lambda)\,\|x\|_2^2 + (|am| + \lambda)^{-1}\,\|B^*x\|_2^2 \\
&\ge (|am| + \lambda)^{-1}\,\|x\|_2^2\,\big(\,\|B^{*-1}\|^{-2} - (\lambda + |am|)^2\big).
\end{aligned}$$

Since by assumption $\|B^{*-1}\|^{-1} > 2|am|$, we have $\mathfrak{s}_1(\lambda)[x] > 0$ for all $x \in \mathcal{D}(B^*)\setminus\{0\}$ if λ is sufficiently close to $|am|$. $\square$

Recall that for fixed parameters a, m and ω, we have $\lim\limits_{|k|\to\infty} \|B^{-1}\| = 0$ by lemma 3.34. Hence, if the norm of the wave number k is large enough, then the angular operator has no eigenvalues in $[-|am|, |am|\,]$ and the index shift n_0 vanishes.

4.2.2 Estimates for the eigenvalues of BB^*

In the minimax principle in theorem 4.32, the eigenvalues of BB^* appear in the formula for the eigenvalues of the angular operator $\mathcal{A}$. The operator BB^* is the selfadjoint realisation of a formal second order differential expression, so Sturm's comparison theorem allows us to find upper and lower bounds for the eigenvalues of BB^*.

Definition 4.35. A formal differential expression τ is called a *Sturm-Liouville differential expression* on $(0,\pi)$ if it has the form

$$\tau u \;:=\; \frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Big(p\,\frac{\mathrm{d}}{\mathrm{d}\vartheta}u\Big) + qu\right) \qquad \text{on} \quad (0,\pi) \tag{SL}$$

where p, q and r are measurable functions of ϑ on $(0,\pi)$ such that p, q are real functions, p has no zeros, r is positive almost everywhere on $(0,\pi)$ and $|r|$ and $|p^{-1}|$ are locally integrable on $(0,\pi)$ (cf. section 2.1). The differential equation $\tau u = 0$ is called a *Sturm-Liouville differential equation* on $(0,\pi)$.

It is well known that for every real λ there is a unique solution of the initial value problem

$$(\tau - \lambda)u = 0, \qquad u(\vartheta_0) = u_0, \quad p(\vartheta_0)u'(\vartheta_0) = u_1$$

for given $\vartheta_0 \in (0,\pi)$ and arbitrary initial values $u_0,\ u_1 \in \mathbb{R}$, see, e.g., [Wei87, chap. 2].

Fix $\lambda \in \mathbb{R}$ and let u be a nontrivial real solution of $(\tau - \lambda)u = 0$. Then there are real valued differentiable functions ρ and δ such that

$$p(\vartheta)u'(\vartheta) + \mathrm{i}u(\vartheta) \;=\; \rho(\vartheta)\,\mathrm{e}^{\mathrm{i}\delta(\vartheta)}, \qquad \vartheta \in (0,\pi).$$

The transformation from u and pu' to the new variables δ and ρ is known as *Prüfer substitution*, see [Wei87, chap. 13]. Although the phase function δ is determined only modulo 2π, we can always choose it such that it is differentiable. Obviously, u vanishes at a point $\vartheta_0 \in (0,\pi)$ if and only if $\delta(\vartheta_0) \in \{n\pi : n \in \mathbb{Z}\}$. Furthermore, δ and ρ satisfy the differential equations

$$\begin{aligned} \delta'(\vartheta) &= \frac{1}{p(\vartheta)}\cos^2\delta(\vartheta) - (q(\vartheta) - \lambda)\sin^2\delta(\vartheta), && \vartheta \in (0,\pi), \\ \rho'(\vartheta) &= \Big(\frac{1}{p(\vartheta)} + q(\vartheta) - \lambda\Big)\sin\delta(\vartheta)\cos\delta(\vartheta), && \vartheta \in (0,\pi). \end{aligned}$$

Consider two Sturm-Liouville differential equations

$$-\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Big(p^{\langle 1\rangle}\frac{\mathrm{d}}{\mathrm{d}\vartheta}u^{\langle 1\rangle}\Big) + q^{\langle 1\rangle}u^{\langle 1\rangle} \;=\; 0, \qquad -\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Big(p^{\langle 2\rangle}\frac{\mathrm{d}}{\mathrm{d}\vartheta}u^{\langle 2\rangle}\Big) + q^{\langle 2\rangle}u^{\langle 2\rangle} \;=\; 0 \tag{4.43}$$

on the interval $(0,\pi)$. For solutions $u^{\langle 1\rangle}$ and $u^{\langle 2\rangle}$ we denote the corresponding phase functions by $\delta^{\langle 1\rangle}$ and $\delta^{\langle 2\rangle}$, respectively. If $p^{\langle 1\rangle} - p^{\langle 2\rangle}$ and $q^{\langle 2\rangle} - q^{\langle 1\rangle}$ are either positive or negative and if the sign of $(\delta^{\langle 1\rangle} - \delta^{\langle 2\rangle})$ is known at one point $\vartheta_0 \in (0,\pi)$, then Sturm's comparison theorem allows us to compare the phase functions on the whole interval $(0,\pi)$.

Theorem 4.36 (Sturm's comparison theorem). *In (4.43), let $p^{\langle 1\rangle} \ge p^{\langle 2\rangle} > 0$ and $q^{\langle 1\rangle} \ge q^{\langle 2\rangle}$.*

(i) *If there exists a $\vartheta_0 \in (0,\pi)$ with $\delta^{\langle 2\rangle}(\vartheta_0) \ge \delta^{\langle 1\rangle}(\vartheta_0)$, then $\delta^{\langle 2\rangle}(\vartheta) \ge \delta^{\langle 1\rangle}(\vartheta)$ for all $\vartheta \in (\vartheta_0,\pi)$. If there exists a $\vartheta_0 \in (0,\pi)$ with $\delta^{\langle 2\rangle}(\vartheta_0) > \delta^{\langle 1\rangle}(\vartheta_0)$, then $\delta^{\langle 2\rangle}(\vartheta) > \delta^{\langle 1\rangle}(\vartheta)$ for all $\vartheta \in (\vartheta_0,\pi)$.*

(ii) *If there exists a $\vartheta_0 \in (0,\pi)$ with $\delta^{\langle 2\rangle}(\vartheta_0) \le \delta^{\langle 1\rangle}(\vartheta_0)$, then $\delta^{\langle 2\rangle}(\vartheta) \le \delta^{\langle 1\rangle}(\vartheta)$ for all $\vartheta \in (0,\vartheta_0)$. If there exists a $\vartheta_0 \in (0,\pi)$ with $\delta^{\langle 2\rangle}(\vartheta_0) < \delta^{\langle 1\rangle}(\vartheta_0)$, then $\delta^{\langle 2\rangle}(\vartheta) < \delta^{\langle 1\rangle}(\vartheta)$ for all $\vartheta \in (0,\vartheta_0)$.*

(iii) *If there exist $\vartheta_0,\ \vartheta_1 \in (0,\pi)$ with $\delta^{\langle 2\rangle}(\vartheta_0) \ge \delta^{\langle 1\rangle}(\vartheta_0)$ and $q^{\langle 1\rangle}(\vartheta) > q^{\langle 2\rangle}(\vartheta)$ for all $\vartheta \in (\vartheta_0,\vartheta_1)$, then $\delta^{\langle 2\rangle}(\vartheta) > \delta^{\langle 1\rangle}(\vartheta)$ for all $\vartheta \in (\vartheta_0,\pi)$.*

(iv) *If there exist $\vartheta_0,\ \vartheta_1 \in (0,\pi)$ with $\delta^{\langle 2\rangle}(\vartheta_0) \le \delta^{\langle 1\rangle}(\vartheta_0)$ and $q^{\langle 1\rangle}(\vartheta) > q^{\langle 2\rangle}(\vartheta)$ for all $\vartheta \in (\vartheta_1,\vartheta_0)$, then $\delta^{\langle 2\rangle}(\vartheta) > \delta^{\langle 1\rangle}(\vartheta)$ for all $\vartheta \in (0,\vartheta_0)$.*

For a proof, we refer the reader to [Wei87, theorem 13.2]. If the solutions $u^{\langle 1\rangle}$ and $u^{\langle 2\rangle}$ have only finitely many zeros in a right neighbourhood of 0, then the phase functions $\delta^{\langle 1\rangle}$ and $\delta^{\langle 2\rangle}$ can be continuously extended to 0 and (iii) holds also for $\vartheta_0 = 0$. Analogously, if the solutions have only finitely many zeros in a left neighbourhood of π, then the phase functions can be continuously extended to π and and (iv) holds also for $\vartheta_0 = \pi$.

If every solution u of $\tau u = 0$ has infinitely many zeros, then the equation is called *oscillatory*. Now we use the comparison theorem to compare the eigenvalues of selfadjoint differential operators associated with Sturm-Liouville differential expressions (SL). The next theorem is an application of theorem [Wei87, theorem 14.10].

Theorem 4.37. *On $(0,\pi)$ we consider the Sturm-Liouville differential expressions*

$$\mathfrak{r}^{\langle 1\rangle} = -\frac{\mathrm{d}}{\mathrm{d}\vartheta}\,p^{\langle 1\rangle}\,\frac{\mathrm{d}}{\mathrm{d}\vartheta} + q^{\langle 1\rangle} \qquad \textit{and} \qquad \mathfrak{r}^{\langle 2\rangle} = -\frac{\mathrm{d}}{\mathrm{d}\vartheta}\,p^{\langle 2\rangle}\,\frac{\mathrm{d}}{\mathrm{d}\vartheta} + q^{\langle 2\rangle}$$

with $p^{\langle 1\rangle} \ge p^{\langle 2\rangle}$ and $q^{\langle 1\rangle} \ge q^{\langle 2\rangle}$.

For $j = 1, 2$, let $R^{\langle j\rangle}$ be a selfadjoint realisation of $\mathfrak{r}^{\langle j\rangle}$ in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ with separated boundary conditions with discrete point spectrum only which is bounded from below. Further, we assume that all eigenfunctions $\psi^{\langle 1\rangle}$ of $\mathfrak{r}^{\langle 1\rangle}$ and $\psi^{\langle 2\rangle}$ of $\mathfrak{r}^{\langle 2\rangle}$ can be continuously extended to 0 and π by $\psi^{\langle 1\rangle}(0) = \psi^{\langle 1\rangle}(\pi) = 0$ and $\psi^{\langle 2\rangle}(0) = \psi^{\langle 2\rangle}(\pi) = 0$. For $j = 1, 2$ let

$$\lambda_1^{\langle j\rangle} < \lambda_2^{\langle j\rangle} < \dots < \lambda_{N^{\langle j\rangle}}^{\langle j\rangle}$$

be the sequence of eigenvalues of $R^{\langle j\rangle}$ where $N^{\langle j\rangle} \in \mathbb{N}\cup\{\infty\}$. If $N^{\langle j\rangle} = \infty$, then the sequence above has to be understood as the infinite series $\lambda_1^{\langle j\rangle} < \lambda_2^{\langle j\rangle} < \dots$
Then for all integers $n \le N^{\langle 1\rangle}$, $N^{\langle 2\rangle}$ we have

$$\lambda_n^{\langle 2\rangle} \le \lambda_n^{\langle 1\rangle}.$$

Proof. Let $\psi_m^{\langle j\rangle}$ be the (up to a constant factor unique) eigenfunction of $R^{\langle j\rangle}$ with eigenvalue $\lambda_m^{\langle j\rangle}$. By [Wei87, theorem 14.10] we know that the mth eigenfunction $\psi_m^{\langle j\rangle}$ has exactly $m-1$ zeros in the interval $(0,\pi)$. If we denote the corresponding phase functions by $\delta(\vartheta, \psi_m^{\langle j\rangle}, \lambda_m^{\langle j\rangle})$ and choose them such that they are zero at $\vartheta = 0$, we obtain

$$\begin{aligned} \delta(0, \psi_m^{\langle 1\rangle}, \lambda_m^{\langle 1\rangle}) &= \delta(0, \psi_m^{\langle 2\rangle}, \lambda_m^{\langle 2\rangle}) = 0, \\ \delta(\pi, \psi_m^{\langle 1\rangle}, \lambda_m^{\langle 1\rangle}) &= \delta(\pi, \psi_m^{\langle 2\rangle}, \lambda_m^{\langle 2\rangle}) = m\pi \end{aligned} \tag{4.44}$$

for all integers $m \le N^{\langle 1\rangle}$, $N^{\langle 2\rangle}$. Now fix $n \in \mathbb{N}$ with $n \le N^{\langle 1\rangle}$, $N^{\langle 2\rangle}$. Then for all $\lambda_m^{\langle 2\rangle}$ with $\lambda_m^{\langle 2\rangle} > \lambda_n^{\langle 1\rangle}$ it follows $q^{\langle 1\rangle} - \lambda_n^{\langle 1\rangle} > q^{\langle 2\rangle} - \lambda_m^{\langle 2\rangle}$. Now (4.44) together with statement (iii) in the comparison theorem 4.36 and the note thereafter yields

$$n\pi = \delta(\pi, \psi_n^{\langle 1\rangle}, \lambda_n^{\langle 1\rangle}) < \delta(\pi, \psi_m^{\langle 2\rangle}, \lambda_m^{\langle 2\rangle}) = m\pi.$$

By the above inequality, we have that $\lambda_m^{\langle 2\rangle} > \lambda_n^{\langle 1\rangle}$ implies $n < m$. Hence $n \ge m$ implies $\lambda_m^{\langle 2\rangle} \le \lambda_n^{\langle 1\rangle}$ and the theorem is proved if we choose $n = m$. □

Next, we use the preceding theorem to estimate the eigenvalues of BB^*. The selfadjoint operator BB^* is associated with the Sturm-Liouville differential expression

$$\mathfrak{b} := \Big(\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big) = -\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} + q(\vartheta), \qquad \vartheta \in (0,\pi).$$

with

$$q(\vartheta) := \frac{(k+\frac12)^2 - (k+\frac12)\cos\vartheta}{\sin^2\vartheta} + a^2\omega^2\sin^2\vartheta + a\omega\cos\vartheta + 2(k+\tfrac12)a\omega, \qquad \vartheta \in (0,\pi). \tag{4.45}$$

Remark 4.38. The operator BB^* fulfils the assumptions on $R^{\langle 1\rangle}$ and $R^{\langle 2\rangle}$ of theorem 4.37 for $k \in \mathbb{R}\setminus(-1,0)$.

Proof. It follows from lemma B.2 in the appendix with $m = 0$ that for $k \in \mathbb{R}\setminus(-2,1)$ the differential expression $\mathfrak{b}$ is in the limit point case both at 0 and at π, hence for a selfadjoint realisation of $\mathfrak{b}$ no boundary conditions are needed. If $k \in (-2,-1]$, then $\mathfrak{b}$ is in the limit point case at 0 and in the limit circle case at π; if $k \in [0,1)$, then $\mathfrak{b}$ is in the limit circle case at π and in the limit point case at 0. In both cases all selfadjoint realisations of $\mathfrak{b}$ are given by separated boundary conditions.
Since the operator BB^* is positive, its spectrum is bounded from below. Furthermore, B^{-1} is compact by lemma 3.22, hence BB^* has discrete point spectrum only. From the subsequent lemma 5.1 it follows that for $k \in \mathbb{R}\setminus(-1,0)$ all functions in the domain of B^* converge to 0 for $\vartheta \to 0$ and $\vartheta \to \pi$, in particular, all eigenfunctions of BB^* can be extended continuously to the points 0 and π with value 0. □

Our goal is to find suitable potentials $q_+ > q$ and $q_- < q$ such that there are corresponding selfadjoint realisations $R_\pm$ of $\mathfrak{b}_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} + q_\pm$ that satisfy the assumptions on the operators $R^{\langle 1\rangle}$ and $R^{\langle 2\rangle}$ of theorem 4.37, respectively. Then the eigenvalues of $R_\pm$ yield estimates for the eigenvalues of BB^*.

We define the following test potentials

$$q^{\langle -\rangle}(\vartheta) \;:=\; \frac{(k+\frac{1}{2})^2 - (k+\frac{1}{2})\cos\vartheta}{\sin^2\vartheta} + \Omega_-, \qquad \vartheta\in(0,\pi), \tag{4.46}$$

$$q^{\langle +\rangle}(\vartheta) \;:=\; \frac{(k+\frac{1}{2})^2 - (k+\frac{1}{2})\cos\vartheta}{\sin^2\vartheta} + \Omega_+, \qquad \vartheta\in(0,\pi), \tag{4.47}$$

with

$$\Omega_- \;:=\; 2(k+\tfrac{1}{2})\,a\omega - |a\omega|, \tag{4.48}$$

$$\Omega_+ \;:=\; \begin{cases} a^2\omega^2 + \frac{1}{4} + 2(k+\frac{1}{2})\,a\omega & \text{if } \ 2a\omega\notin[-1,\,1],\\ 2(k+\frac{1}{2})\,a\omega + |a\omega| & \text{if } \ 2a\omega\in[-1,\,1]. \end{cases} \tag{4.49}$$

Theorem 4.39. *Let $\{\nu_n : n\in\mathbb{N}\}$ be the spectrum of BB^* enumerated as described in remark 4.31. Then the following estimates hold.*

$$\max\bigl\{0,\ \bigl(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\bigr)^2 + \Omega_-\bigr\} \;\le\; \nu_n \;\le\; \bigl(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\bigr)^2 + \Omega_+. \tag{4.50}$$

Proof. Define the Sturm-Liouville differential expressions

$$\mathfrak{r}^{\langle\pm\rangle} \;:=\; -\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} + q^{\langle\pm\rangle} \quad \text{on} \quad (0,\pi)$$

and their corresponding realisations $R^{\langle\pm\rangle}$ in $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ with domain $\mathcal{D}(R^{\langle\pm\rangle}) := \mathcal{D}(BB^*)$. Note that $R^{\langle +\rangle} - BB^*$ and $R^{\langle -\rangle} - BB^*$ are bounded and that for $\omega = 0$ we have $R^{\langle +\rangle} = R^{\langle -\rangle} = BB^*$. Thus it follows from remark 4.38 that also the operators $R^{\langle\pm\rangle}$ satisfy the conditions on $R^{\langle 1\rangle}$ and $R^{\langle 2\rangle}$ of theorem 4.37. Furthermore, we have for all $\vartheta\in(0,\pi)$

$$\begin{aligned} a^2\omega^2\sin^2\vartheta + a\omega\cos\vartheta + 2(k+\tfrac{1}{2})a\omega \;&\ge\; a^2\omega^2 + \tfrac{1}{4} + 2(k+\tfrac{1}{2})\,a\omega + \min_{\vartheta\in[0,\pi]}\{-(a\omega\cos\vartheta - \tfrac{1}{2})^2\}\\ &=\; a^2\omega^2 + \tfrac{1}{4} + 2(k+\tfrac{1}{2})\,a\omega - (|a\omega| + \tfrac{1}{2})^2\\ &=\; 2(k+\tfrac{1}{2})\,a\omega - |a\omega|\\ &=\; \Omega_-, \end{aligned}$$

$$\begin{aligned} a^2\omega^2\sin^2\vartheta + a\omega\cos\vartheta + 2(k+\tfrac{1}{2})a\omega \;&\le\; a^2\omega^2 + \tfrac{1}{4} + 2(k+\tfrac{1}{2})\,a\omega + \max_{\vartheta\in[0,\pi]}\{-(a\omega\cos\vartheta - \tfrac{1}{2})^2\}\\ &=\; a^2\omega^2 + \tfrac{1}{4} + 2(k+\tfrac{1}{2})\,a\omega - \begin{cases} 0 & \text{if } \ 2a\omega\notin[-1,\,1],\\ (|a\omega| - \frac{1}{2})^2 & \text{if } \ 2a\omega\in[-1,\,1] \end{cases}\\ &=\; \begin{cases} a^2\omega^2 + \frac{1}{4} + 2(k+\frac{1}{2})\,a\omega & \text{if } \ 2a\omega\notin[-1,\,1],\\ 2(k+\frac{1}{2})\,a\omega + |a\omega| & \text{if } \ 2a\omega\in[-1,\,1] \end{cases}\\ &=\; \Omega_+, \end{aligned}$$

thus it follows that $q^{\langle -\rangle} \le q \le q^{\langle +\rangle}$.

Finally, we have to determine the eigenvalues of $R^{\langle\pm\rangle}$. The computation of the eigenvalues of $R^{\langle\pm\rangle}$ is similar to the computation of the eigenvalues of the angular operator in the case $a = 0$, see section 3.1. We have to solve the differential equations

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2} + \frac{(k+\frac{1}{2})^2 - (k+\frac{1}{2})\cos\vartheta}{\sin^2\vartheta} + \Omega_\pm - \nu\right)\psi(\vartheta) \;=\; 0, \qquad \vartheta \in (0,\pi) \tag{4.51}$$

with functions $\psi \in \mathcal{D}(R^{\langle\pm\rangle}) = \mathcal{D}(BB^*)$, that is, with functions satisfying the integrability condition

$$\int_0^\pi |\psi(\vartheta)|^2 \,\mathrm{d}\vartheta \;<\infty.$$

If we apply the transformation of the independent variable

$$x \;=\; \tfrac{1}{2}(1+\cos\vartheta), \qquad \vartheta \in (0,\pi),$$

then differentiation with respect to ϑ becomes $\frac{\mathrm{d}}{\mathrm{d}\vartheta} = -\sqrt{x(1-x)}\,\frac{\mathrm{d}}{\mathrm{d}x}$; the ansatz

$$v(x) \;=\; x^\alpha(1-x)^\beta\psi(\vartheta(x)) \qquad \text{with} \qquad \alpha := \tfrac{1}{2}|k| + \tfrac{1}{4}, \quad \beta := \tfrac{1}{2}|k+1| + \tfrac{1}{4}$$

yields the equivalent differential equation

$$x(1-x)v''(x) + (2\alpha + \tfrac{1}{2} - (1+2\alpha+2\beta)x)v'(x) + (\nu - \Omega_\pm - (\alpha+\beta)^2)v(x) \;=\; 0, \qquad x \in (0,1);$$

and the integrability condition becomes

$$\int_0^1 |v(x)|^2\, x^{-(2\alpha+1)}(1-x)^{-(2\beta+1)}\,\mathrm{d}x \;<\; \infty.$$

Note that this system is identical to (3.13 a) with λ^2 substituted by $\nu - \Omega_\pm$. We already saw in lemma 3.3 that for $\nu - \Omega_\pm \geq 0$ this equation has a solution satisfying both the differential equation and the integrability condition if and only if

$$\nu - \Omega_\pm \;\in\; \left\{\left(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\right)^2 \;:\; n \in \mathbb{N}\right\}.$$

The corresponding solutions are the Jacobi polynomials

$$v(x) = F(-(n-1),\, n + |k| + |k+1|;\, |k|+1;\, x).$$

In the case $n = 1$ this polynomial reduces to a constant function implying that the corresponding eigenfunction $\psi_1(\vartheta) = x^\alpha(1-x)^\beta F(0,\,1+|k|+|k+1|;\,|k|+1;x)$ has no zero in $(0,\pi)$. Since the mth eigenvalue of $R^{\langle\pm\rangle}$ has exactly $m-1$ zeros in $(0,\pi)$, it follows that he smallest eigenvalue is $\nu_1^\pm := (|k+\frac{1}{2}| + \frac{1}{2})^2 + \Omega_\pm$.

Thus all eigenvalues of $R^{\langle\pm\rangle}$ are given by

$$\nu_n^\pm \;=\; \left(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\right)^2 + \Omega_\pm, \qquad n \in \mathbb{N}.$$

Application of theorem 4.37 yields $\nu_n^- \leq \nu_n \leq \nu_n^+$. Furthermore, since BB^* is a strictly positive operator (see, e.g., lemma 3.30), we have $\nu_n > 0$ for all $n \in \mathbb{N}$. □

4.2.3 Explicit bounds for the eigenvalues of $\mathcal{A}$

As before, λ_n denotes the analytic continuation of the eigenvalue $\lambda_n = \operatorname{sign}(n)\big(|k+\frac{1}{2}| - \frac{1}{2} + n\big)$ in the case $a = 0$. The following theorem combines theorems 4.39 and 4.32.

Theorem 4.40. *Let $\lambda_{m_+ + n}$ be the nth eigenvalue of the angular operator $\mathcal{A}$ greater than $|am|$. Then*

$$\max\Big\{|am|,\ \operatorname{Re}\Big(\sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n_0 + n\big)^2 + \Omega_-} - |am|\Big)\Big\}$$
$$\leq\ \lambda_{m_+ + n}\ \leq\ \sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n_0 + n\big)^2 + \Omega_+} + |am|, \qquad n \in \mathbb{N}.$$

The quantities

$$\Omega_- \;=\; 2(k+\tfrac{1}{2})\, a\omega - |a\omega|,$$
$$\Omega_+ \;=\; \begin{cases} 2(k+\frac{1}{2})\, a\omega + |a\omega| & \textit{if}\ \ 2a\omega \in [-1,\,1], \\ a^2\omega^2 + \frac{1}{4} + 2(k+\frac{1}{2})\, a\omega & \textit{if}\ \ 2a\omega \notin [-1,\,1] \end{cases}$$

have been introduced in (4.48) and (4.49); the index shift $n_0 = \min\limits_{\lambda > |am|} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda)$ has been defined in theorem 4.19.

That the upper bound is always real is proved in remark 4.43 at the end of this section. A result similar to theorem 4.40 follows directly from standard perturbation theory as explained in section 3.2 even without the need to determine n_0. For convenience, we state this result in the next theorem.

Theorem 4.41. *Let λ_n be the nth eigenvalue of the angular operator $\mathcal{A}$ with the ordering described above. Then for all $n \in \mathbb{N}$ we have*

$$\operatorname{Re}\Big(\sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\big)^2 + \Omega_-}\ \Big) - |am|\ \leq\ \lambda_n\ \leq\ \sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\big)^2 + \Omega_+} + |am|.$$

The functions Ω_- and Ω_+ are the same as in theorem 4.40.

Proof. In theorem 4.39 we have already provided estimates for the eigenvalues of BB^*. Since B and B^* are invertible, the spectrum of $\mathcal{B} = \left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$ is given by $\sigma_p(\mathcal{B}) = \{\pm\sqrt{\nu_n}\ :\ \nu_n \in \sigma(BB^*)\}$. Now, application of analytic perturbation theory to the operators $\mathcal{B}$ and $\mathcal{A}$ with m as perturbation parameter yields $\sqrt{\nu_n} - |am| \leq \lambda_n \leq \sqrt{\nu_n} + |am|$. □

In the following, we denote the lower bounds in theorems 4.40 and 4.41 by $\lambda^{[l]}$ and $\lambda^{[l,\mathrm{SPT}]}$, and the upper bound by $\lambda^{[u]}$ and $\lambda^{[u,\mathrm{SPT}]}$, that is,

$$\lambda_n^{[l]} \;:=\; \max\Big\{|am|,\ \operatorname{Re}\Big(\sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\big)^2 + \Omega_-} - |am|\Big)\Big\}, \tag{4.52}$$

$$\lambda_n^{[l,\mathrm{SPT}]} \;:=\; \operatorname{Re}\Big(\sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\big)^2 + \Omega_-}\ \Big) - |am|, \tag{4.53}$$

$$\lambda_n^{[u]} \;:=\; \lambda_n^{[u,\mathrm{SPT}]} \;:=\; \sqrt{\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n\big)^2 + \Omega_+} + |am|. \tag{4.54}$$

The next lemma follows immediately from 4.34.

Lemma 4.42. (i) *If there exists $j_0 \geq 2$ such that*

$$\lambda^{[l]}_{n_0+j_0} - \lambda^{[u]}_{n_0+j_0-1} \ > \ 0 \qquad \textit{and} \qquad \lambda^{[l]}_{n_0+j_0+1} - \lambda^{[u]}_{n_0+j_0} \ > \ 0, \tag{4.55}$$

then $n_0 = m_+$.

(ii) *If $\lambda^{[l]}_1 > |am|$, then the angular operator $\mathcal{A}$ has no eigenvalues in $[-|am|, |am|\,]$ and we have $n_0 = 0$ and $m_+ = 0$.*

Note that we have

$$\begin{aligned}\lim_{N\to\infty} \lambda^{[l]}_{N+1} - \lambda^{[u]}_N \ &= \ -2|am| + \lim_{N\to\infty} \sqrt{(|k+\tfrac{1}{2}| + N + \tfrac{1}{2})^2 + \Omega_-} - \sqrt{(|k+\tfrac{1}{2}| + N - \tfrac{1}{2})^2 + \Omega_+} \\ &= \ -2|am| + 1;\end{aligned}$$

therefore, (i) of lemma 4.42 holds whenever $|am| < \frac{1}{2}$.

It should be noted that for large k the bounds $\lambda^{[l]}$ and $\lambda^{[u]}$ are approximately linear in the wave number k. The offset $\pm|am|$ is due to the fact that the additive term $am\cos\vartheta \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ arising in the angular operator is treated as a perturbation.

Remark 4.43. The upper bound $\lambda^{[u]}_n$ is always real.

Proof. Assume that $2a\omega \in [-1, 1]$. Since $n \geq 1$ and $n_0 \geq 0$, the radicand in $\lambda^{[u]}_n$ can be estimated by

$$\begin{aligned}\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n_0 + n\big)^2 + \Omega_+ \ &\geq \ (|k+\tfrac{1}{2}| + \tfrac{1}{2})^2 + 2(k+\tfrac{1}{2})\, a\omega + |a\omega| \\ &\geq \ (|k+\tfrac{1}{2}| + \tfrac{1}{2})^2 - |k+\tfrac{1}{2}| \ = \ |k+\tfrac{1}{2}|^2 + \tfrac{1}{4} > 0.\end{aligned}$$

If $2a\omega \notin [-1, 1]$, then the radicand is also positive which can be seen from

$$\begin{aligned}\big(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n_0 + n\big)^2 + \Omega_+ \ &\geq \ \big(|k+\tfrac{1}{2}| + \tfrac{1}{2}\big)^2 + a^2\omega^2 + \tfrac{1}{4} + 2(k+\tfrac{1}{2})\, a\omega \\ &= \ (|k+\tfrac{1}{2}| + \tfrac{1}{2})^2 + (a\omega + k + \tfrac{1}{2})^2 + \tfrac{1}{4} - |k+\tfrac{1}{2}|^2 \\ &= \ |k+\tfrac{1}{2}| + \tfrac{1}{2} + (a\omega + k + \tfrac{1}{2})^2 \ > 0.\end{aligned}$$ □

Observe that for $a = 0$ we obtain $n_0 = 0$ from lemma 4.34 and hence

$$\lambda^{[u]}_n \ = \ \lambda^{[l]}_n \ = \ \tfrac{1}{2}|2k+1| - \tfrac{1}{2} + n, \qquad n \geq 1,$$

which coincides with the exact formula for the eigenvalues λ_n obtained in section 3.1.2 in the case $a = 0$. Hence it can be expected that at least for small a the estimate presented in theorem 4.40 are better than those obtained in sections 3.3.3 and 5.2.

Chapter 5

An alternative lower bound for the modulus of the eigenvalues of $\mathcal{A}$

The bounds for the eigenvalues of the angular operator derived in the previous chapters all contain an additive term $\pm|am|$. In this chapter we establish another lower bound for the modulus of the eigenvalues of $\mathcal{A}$ that does not depend on am.
In the first section, we investigate the behaviour of functions $f \in \mathcal{D}(B^*)$ and $g \in \mathcal{D}(B)$ in a neighbourhood of 0 and π. Then we introduce certain semibounded sesquilinear forms which give rise to a new lower bound for the modulus of the eigenvalues of the angular operator $\mathcal{A}$. To this end, $\mathcal{A}$ is subjected to a unitary transformation such that, under certain assumptions on the parameters k, a and ω, the intersection of the spectra of the operators on the diagonal of the transformed operator is empty. Operator matrices of this type have been investigated in [LT01]. All entries of the transformed operator matrix are unbounded. However, the gap between the spectra of the diagonal entries provides a lower bound for the modulus of the eigenvalues of $\mathcal{A}$.

5.1 Unitary transformation of $\mathcal{A}$

5.1.1 Characterisation of the domains of B and B^*

Recall that the angular operator

$$\mathcal{A} = \begin{pmatrix} -D & B \\ B^* & D \end{pmatrix}, \qquad \mathcal{D}(\mathcal{A}) \;=\; \mathcal{D}(B^*) \oplus \mathcal{D}(B)$$

is selfadjoint on the Hilbert space $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)^2$; the operator D is a bounded multiplication operator and B is a closed differential operator of first order (see (3.1)), formally they are given by

$$D \;=\; am\cos\vartheta, \qquad\qquad B = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta, \qquad \vartheta \in (0,\pi).$$

In the following lemma we describe the behaviour of functions $f \in \mathcal{D}(B^*)$ and $g \in \mathcal{D}(B)$ in a neighbourhood of the endpoints of the interval $(0,\pi)$. We show that for $k \in \mathbb{R}\backslash[-1,0]$ these functions tend to zero at the endpoints at least of order $\sqrt{\vartheta}$ and $\sqrt{\pi-\vartheta}$, respectively. For $k \in \{-1,0\}$ the same asymptotic behaviour is proved in the subsequent remark 5.2 for functions f, g such that $\binom{f}{g}$ is an eigenfunction of $\mathcal{A}$. In the proof of lemma 5.1 we use (3.45 a) to estimate the quotient of tangent functions; recall that this estimate, and alternatively estimate (3.45 c), has been used in lemma 3.30 and lemma 3.34 to estimate the norm of $\|B^{-1}\|$, thus entering the lower bound for the eigenvalues of $\mathcal{A}$ obtained in theorem 3.35. Although, in general, estimate (3.45 c) gives a sharper lower bound for the modulus of the eigenvalues of $\mathcal{A}$, it cannot be used to obtain a result similar to that of lemma 5.1.

In the following lemma we again attach a subscript k to the operators B and B^* to indicate their dependence on the wave number k.

Lemma 5.1. *For $k \in \mathbb{R} \setminus [-1, 0]$ let $f \in \mathcal{D}(B_k^*)$, $g \in \mathcal{D}(B_k)$. Then we have*

$$|f(\vartheta)| \;\le\; c(k)\,\Gamma(k,\omega)\,\|B_k^* f\|_2\,\sqrt{\vartheta(\pi-\vartheta)}, \qquad \vartheta \in (0,\pi), \tag{5.1 a}$$

$$|g(\vartheta)| \;\le\; c(k)\,\Gamma(k,\omega)\,\|B_k g\|_2\,\sqrt{\vartheta(\pi-\vartheta)}, \qquad \vartheta \in (0,\pi) \tag{5.1 b}$$

with $\Gamma(k,\omega) \in \mathbb{R}$ from lemma 3.30 and

$$c(k) \;=\; \frac{2^{\frac{1}{2}(|2k+1|-2)}}{\sqrt{\pi(|2k+1|-1)}}.$$

For $k = 0$ and arbitrary $\varepsilon > 0$ there exist constants $C_{\varepsilon,*}$ and C_ε such that for all $f \in \mathcal{D}(B_0^*)$ and $g \in \mathcal{D}(B_0)$

$$|f(\vartheta)| \;\le C_{\varepsilon,*}\,\|B_0^* f\|_2\,\sqrt{\pi-\vartheta}\,\vartheta^{\frac{1}{2}-\varepsilon}, \quad |g(\vartheta)| \;\le C_\varepsilon\,\|B_0 g\|_2\,\sqrt{\vartheta}\,(\pi-\vartheta)^{\frac{1}{2}-\varepsilon}, \quad \vartheta \in (0,\pi). \tag{5.2}$$

Also for $k = -1$ there exist constants $C_{\varepsilon,*}$ and C_ε such that for all $f \in \mathcal{D}(B_{-1}^*)$ and $g \in \mathcal{D}(B_{-1})$

$$|f(\vartheta)| \;\le C_{\varepsilon,*}\,\|B_{-1}^* f\|_2\,\sqrt{\vartheta}\,(\pi-\vartheta)^{\frac{1}{2}-\varepsilon}, \quad |g(\vartheta)| \;\le C_\varepsilon\,\|B_{-1} g\|_2\,\sqrt{\pi-\vartheta}\,\vartheta^{\frac{1}{2}-\varepsilon}, \quad \vartheta \in (0,\pi). \tag{5.3}$$

Proof. The case $k \in \{-1, 0\}$ is treated separately at the end of the proof.

First we prove (5.1 a). For $k > 0$ and $f \in \mathcal{D}(B_k^*)$ the Cauchy-Schwarz inequality shows for arbitrary $\vartheta \in (0,\pi)$

$$\begin{aligned}
|f(\vartheta)| \;&=\; |B_k^{*-1} B_k^* f(\vartheta)| \;=\; \left|\frac{1}{\varphi(\vartheta)} \int_\vartheta^\pi \varphi(t) B_k^* f(t)\,\mathrm{d}t\right| \\
&\le\; \int_\vartheta^\pi \mathrm{e}^{a\omega(\cos t - \cos\vartheta)} \left(\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\right)^{k+\frac{1}{2}} |B_k^* f(t)|\,\mathrm{d}t \\
&\le\; \sup\bigl\{\mathrm{e}^{a\omega(\cos t - \cos\vartheta)} : 0 < \vartheta \le t < \pi\bigr\}\,\|B_k^* f\|_2 \left(\int_\vartheta^\pi \left(\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\right)^{2k+1} \mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned}$$

Application of the estimate $\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}} \le \frac{\vartheta}{t}$ (see (3.45 a)) yields

$$|f(\vartheta)| \;\le\; \Gamma(k,\omega)\,\|B_k^* f\|_2\,\vartheta^{k+\frac{1}{2}} \left(\int_\vartheta^\pi t^{-2k-1}\,\mathrm{d}t\right)^{\frac{1}{2}} \;=\; \frac{\Gamma(k,\omega)\,\|B_k^* f\|_2}{\sqrt{2k}} \left(1 - \left(\frac{\vartheta}{\pi}\right)^{2k}\right)^{\frac{1}{2}} \sqrt{\vartheta}.$$

The well known equality $z^\nu - 1 \;=\; \prod_{j=1}^{\nu} (z - \mathrm{e}^{\frac{2i\pi j}{\nu}})$ for natural numbers ν yields

$$1 - \left(\frac{\vartheta}{\pi}\right)^{2k} \;=\; -\prod_{j=1}^{2k}\left(\frac{\vartheta}{\pi} - \mathrm{e}^{\frac{2i\pi j}{2k}}\right) \;=\; \frac{1}{\pi}(\pi-\vartheta) \prod_{j=1}^{2k-1}\left(\frac{\vartheta}{\pi} - \mathrm{e}^{\frac{2i\pi j}{2k}}\right).$$

Hence we can continue the estimate of $|f(\vartheta)|$, using $|\frac{\vartheta}{\pi} - \mathrm{e}^{\frac{2\mathrm{i}\pi j}{2k}}| \le 2$ for $j = 1, \dots, 2k-1$:

$$|f(\vartheta)| \;\le\; \frac{\Gamma(k,\omega)\,\|B_k^* f\|_2}{\sqrt{2k\pi}}\,\sqrt{\vartheta(\pi-\vartheta)}\,\prod_{j=1}^{2k-1}\left|\frac{\vartheta}{\pi} - \mathrm{e}^{\frac{2\mathrm{i}\pi j}{2k}}\right|^{\frac{1}{2}} \;\le\; \frac{2^{k-\frac{1}{2}}\Gamma(k,\omega)\,\|B_k^* f\|_2}{\sqrt{2k\pi}}\,\sqrt{\vartheta(\pi-\vartheta)}.$$

For $k < -1$ the assertions are obtained analogously if we use equation (3.45 b) instead of equation (3.45 a) to estimate the term containing the tangent function:

$$\begin{aligned}
|f(\vartheta)| \;&=\; |B_k^{*-1}B_k^* f(\vartheta)| \;=\; \left|\frac{1}{\varphi(\vartheta)}\int_{\vartheta}^{0}\varphi(t)B_k^* f(t)\;\mathrm{d}t\right| \\
&\le \Gamma(k,\omega)\left|\int_{\vartheta}^{0}\left(\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\right)^{-|k+\frac{1}{2}|}|B_k^* f(t)|\;\mathrm{d}t\right| \\
&\le\; \Gamma(k,\omega)\,\|B_k^* f\|_2\left(\int_0^{\vartheta}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{|2k+1|}\mathrm{d}t\right)^{\frac{1}{2}} \;\le\; \Gamma(k,\omega)\,\|B_k^* f\|_2\left(\int_0^{\vartheta}\left(\frac{\pi-\vartheta}{\pi-t}\right)^{|2k+1|}\mathrm{d}t\right)^{\frac{1}{2}} \\
&=\; \frac{\Gamma(k,\omega)\,\|B_k^* f\|_2\,\sqrt{\pi-\vartheta}}{\sqrt{|2k+1|-1}}\left(1-\left(\frac{\pi-\vartheta}{\pi}\right)^{-1+|2k+1|}\right)^{\frac{1}{2}} \\
&=\; \frac{\Gamma(k,\omega)\,\|B_k^* f\|_2\,\sqrt{\pi-\vartheta}}{\sqrt{|2k+1|-1}}\left(\prod_{j=1}^{|2k+1|-1}\frac{\pi-\vartheta}{\pi} - \mathrm{e}^{\frac{2\mathrm{i}\pi j}{|2k+1|-1}}\right)^{\frac{1}{2}} \\
&\le\; \frac{2^{\frac{1}{2}(|2k+1|-2)}\,\Gamma(k,\omega)\,\|B_k^* f\|_2}{\sqrt{\pi(|2k+1|-1)}}\;\sqrt{\vartheta(\pi-\vartheta)}.
\end{aligned}$$

Next we prove (5.1 b). For $k > 0$ and $g \in \mathcal{D}(B_k)$

$$\begin{aligned}
|g(\vartheta)| \;&=\; |B_k^{-1}B_k g| \;=\; \left|\frac{1}{\psi(\vartheta)}\int_0^{\vartheta}\psi(t)B_k g(t)\;\mathrm{d}t\right| \\
&\le\; \int_0^{\vartheta}\mathrm{e}^{a\omega(\cos\vartheta-\cos t)}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{k+\frac{1}{2}}|B_k g(t)|\;\mathrm{d}t \\
&\le\; \sup\{\mathrm{e}^{a\omega(\cos\vartheta-\cos t)} : 0 < t \le \vartheta < \pi\}\;\|B_k g\|_2\left(\int_0^{\vartheta}\left(\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\right)^{2k+1}\right)^{\frac{1}{2}}.
\end{aligned}$$

The estimate $\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}} \le \frac{\pi-\vartheta}{\pi-t}$ (see equation (3.45 b)) yields

$$\begin{aligned}
|g(\vartheta)| &\le \Gamma(k,\omega)\,\|B_k g\|_2\,(\pi-\vartheta)^{k+\frac{1}{2}}\left(\int_0^{\vartheta}(\pi-t)^{-2k-1}\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&= \frac{\Gamma(k,\omega)\|B_k g\|_2}{\sqrt{2k}}\,\sqrt{\pi-\vartheta}\,\left(1-\frac{(\pi-\vartheta)^{2k}}{\pi^{2k}}\right)^{\frac{1}{2}}\\
&= \frac{\Gamma(k,\omega)\|B_k g\|_2}{\sqrt{2\pi k}}\,\sqrt{\vartheta(\pi-\vartheta)}\,\prod_{j=1}^{2k-1}\left|\frac{\pi-\vartheta}{\pi}-\mathrm{e}^{\frac{2\mathrm{i}\pi j}{2k}}\right|^{\frac{1}{2}}\\
&\le \frac{2^{k-\frac{1}{2}}\,\Gamma(k,\omega)\,\|B_k g\|_2}{\sqrt{2\pi k}}\,\sqrt{\vartheta(\pi-\vartheta)}.
\end{aligned}$$

The calculation for $k<-1$ is similar.

It remains to prove the assertions for $k\in\{-1,0\}$. We consider the case $k=0$ only. For $k=-1$, the result can either be obtained by similar calculations or they can be derived by symmetry arguments, cf. remark 3.29. So we assume $k=0$ and fix $\varepsilon>0$.
Since $0<\tan\frac{\vartheta}{2}\le\tan\frac{t}{2}<1$ for all $0<\vartheta\le t<\pi$, it follows immediately that

$$\begin{aligned}
|f(\vartheta)| &\le \Gamma(0,\omega)\,\|B_0^* f\|_2\left(\int_{\vartheta}^{\pi}\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\,\mathrm{d}t\right)^{\frac{1}{2}} \le \sqrt{2}\,\Gamma(0,\omega)\,\|B_0^* f\|_2\,\sqrt{\pi-\vartheta},\\
|g(\vartheta)| &\le \Gamma(0,\omega)\,\|B_0 g\|_2\left(\int_0^{\vartheta}\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\,\mathrm{d}t\right)^{\frac{1}{2}} \le \sqrt{2}\,\Gamma(0,\omega)\,\|B_0 g\|_2\,\sqrt{\vartheta},
\end{aligned}\tag{5.4}$$

hence there are constants $\widetilde{C}_*$ and $\widetilde{C}$ such that $|f(\vartheta)|\le\widetilde{C}_*\|B_0^*f\|_2\sqrt{\pi-\vartheta}$ and $|g(\vartheta)|\le\widetilde{C}\|B_0g\|_2\sqrt{\vartheta}$ for $\vartheta\in(0,\pi)$. Since for every $a\in(0,\pi)$ the functions

$$(0,a\,]:\vartheta\ \mapsto\ (\pi-\vartheta)^{\frac{1}{2}-\varepsilon}\qquad\text{and}\qquad[\,a,\pi):\vartheta\ \mapsto\ \vartheta^{\frac{1}{2}-\varepsilon}$$

are strictly positive, it suffices for the proof of (5.2) to show that the limites $\lim\limits_{\vartheta\to 0}(\vartheta^{-1+2\varepsilon}|f(\vartheta)|^2)$ and $\lim\limits_{\vartheta\to\pi}((\pi-\vartheta)^{-1+2\varepsilon}|g(\vartheta)|^2)$ exist and are finite.
A straightforward evaluation of the integrals in (5.4) leads to

$$\begin{aligned}
|f(\vartheta)| &\le \Gamma(0,\omega)\,\|B_0^* f\|_2\left(\int_{\vartheta}^{\pi}\frac{\tan\frac{\vartheta}{2}}{\tan\frac{t}{2}}\,\mathrm{d}t\right)^{\frac{1}{2}} = \sqrt{2}\,\Gamma(0,\omega)\,\|B_0^* f\|_2\,(-\tan\tfrac{\vartheta}{2}\,\ln(\sin\tfrac{\vartheta}{2}))^{\frac{1}{2}},\\
|g(\vartheta)| &\le \Gamma(0,\omega)\,\|B_0 g\|_2\left(\int_0^{\vartheta}\frac{\tan\frac{t}{2}}{\tan\frac{\vartheta}{2}}\,\mathrm{d}t\right)^{\frac{1}{2}} = \sqrt{2}\,\Gamma(0,\omega)\,\|B_0 g\|_2\,(-\cot\tfrac{\vartheta}{2}\,\ln(\cos\tfrac{\vartheta}{2}))^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{\vartheta\to 0}\vartheta^{-1+2\varepsilon}\tan\tfrac{\vartheta}{2}\ln(\sin\vartheta) &= \lim_{\vartheta\to 0}\frac{\tan\frac{\vartheta}{2}}{\vartheta}\,(\vartheta^{2\varepsilon}\ln(\sin\vartheta)) = 0,\\
\lim_{\vartheta\to\pi}(\pi-\vartheta)^{-1+2\varepsilon}\cot\tfrac{\vartheta}{2}\ln(\cos\vartheta) &= \lim_{\vartheta\to\pi}\frac{\cot\frac{\vartheta}{2}}{\pi-\vartheta}\,((\pi-\vartheta)^{2\varepsilon}\ln(\cos\vartheta)) = 0,
\end{aligned}\tag{5.5}$$

the assertion is proved. □

For $\varepsilon = 0$ the limites in (5.5) do not exist, hence a decay, e.g., of arbitrary functions $f \in \mathcal{D}(B^*)$ of order $\sqrt{\vartheta}$ for $\vartheta \to 0$ and $k = 0$ cannot be obtained with the methods applied in the proof of the lemma. However, with the theory of ordinary differential equations, we can prove the following remark.

Remark 5.2. Let $k \in \{-1, 0\}$ and $\Psi := \binom{f}{g}$ be an eigenfunction of $\mathcal{A}$. Then both f and g decay of order $\sqrt{\vartheta(\pi-\vartheta)}$ at $\vartheta = 0$ and $\vartheta = \pi$.

Proof. We show that for every eigenfunction Ψ of $\mathcal{A}$ there exists a positive number c such that $\|\Psi(\vartheta)\| \leq c\sqrt{\vartheta(\pi-\vartheta)}$; here $\|\cdot\|$ denotes the usual norm on $\mathbb{C}^2$. To this end, we first rewrite the differential equation $(\mathcal{A}-\lambda)u = 0$ in the equivalent form $\sigma_1\sigma_3(\mathcal{A}-\lambda)u = 0$ such that the differential operators are placed on the diagonal of the new block operator matrix; explicitly

$$\begin{aligned} 0 &= \sigma_1\sigma_3(\mathcal{A}-\lambda)u \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} & 0 \end{pmatrix} + \frac{k+\frac{1}{2}}{\sin\vartheta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -am\cos\vartheta & a\omega\sin\vartheta \\ a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} - \lambda\right)u \\ &= \left(\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\vartheta} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} \end{pmatrix} + \frac{k+\frac{1}{2}}{\sin\vartheta}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -a\omega\sin\vartheta & -am\cos\vartheta \\ -am\cos\vartheta & a\omega\sin\vartheta \end{pmatrix} - \lambda\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)u. \end{aligned}$$

Now we show that for $k \in \{-1, 0\}$ there is at least one solution that decays like $\sqrt{\vartheta}$ near 0. The decay like $\sqrt{\pi-\vartheta}$ in a neighbourhood of π can be proved analogously. Note that

$$\frac{k+\frac{1}{2}}{\sin\vartheta}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{k+\frac{1}{2}}{\vartheta}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{k+\frac{1}{2}}{\sin\vartheta} - \frac{k+\frac{1}{2}}{\vartheta}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the second term on the right and side is analytic in a complex neighbourhood of 0. To ease notation, we define the function

$$T(\vartheta) := \left(\frac{k+\frac{1}{2}}{\sin\vartheta} - \frac{k+\frac{1}{2}}{\vartheta}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -a\omega\sin\vartheta & -am\cos\vartheta \\ -am\cos\vartheta & a\omega\sin\vartheta \end{pmatrix} - \lambda\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in a sufficiently small neighbourhood of 0 such that T is analytic. Now we apply a transformation such that the eigenvalues of the matrix in front of the singular term $\frac{k+\frac{1}{2}}{\vartheta}$ do not differ by an integer. We assume $k = 0$ and set $U_0(\vartheta) := \left(\begin{smallmatrix} \vartheta & 0 \\ 0 & 1 \end{smallmatrix}\right)$. (For $k = -1$ an appropriate transformation matrix is $U_{-1}(\vartheta) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \vartheta \end{smallmatrix}\right)$.) We obtain

$$\begin{aligned} 0 &= U_0^{-1}\sigma_1\sigma_3(\mathcal{A}-\lambda)U_0U_0^{-1}u \\ &= \left(\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\vartheta} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} \end{pmatrix} + \frac{1}{\vartheta}\begin{pmatrix} -(k+\frac{1}{2})+1 & 0 \\ 0 & k+\frac{1}{2} \end{pmatrix} + U_0^{-1}TU_0\right)U_0^{-1}u \\ &= \left(\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\vartheta} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta} \end{pmatrix} + \frac{1}{\vartheta}\begin{pmatrix} -k+\frac{1}{2} & 0 \\ 0 & k+\frac{1}{2} \end{pmatrix} + U_0^{-1}TU_0\right)U_0^{-1}u. \end{aligned}$$

Since $k = 0$, both diagonal entries in the coefficient matrix of $\frac{1}{\vartheta}$ are equal to $\frac{1}{2}$. Evaluation of the third term gives

$$\begin{aligned} U_0^{-1}T(\vartheta)U_0 &= \frac{1}{\vartheta}\begin{pmatrix} 0 & -(am\cos\vartheta-\lambda) \\ 0 & 0 \end{pmatrix} \\ &\quad + \left(\tfrac{k+\frac12}{\sin\vartheta} - \tfrac{k+\frac12}{\vartheta}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} -a\omega\sin\vartheta & 0 \\ -am\,\vartheta\,\cos\vartheta & a\omega\sin\vartheta\end{pmatrix} - \lambda\begin{pmatrix} 0 & 0 \\ \vartheta & 0\end{pmatrix} \\ &= \frac{1}{\vartheta}\begin{pmatrix} 0 & -(am-\lambda) \\ 0 & 0 \end{pmatrix} + \tfrac{am(1-\cos\vartheta)}{\vartheta}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} \\ &\quad + \left(\tfrac{k+\frac12}{\sin\vartheta} - \tfrac{k+\frac12}{\vartheta}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} -a\omega\sin\vartheta & 0 \\ -am\,\vartheta\,\cos\vartheta & a\omega\sin\vartheta\end{pmatrix} - \lambda\begin{pmatrix} 0 & 0 \\ \vartheta & 0\end{pmatrix} \\ &=: \frac{1}{\vartheta}\begin{pmatrix} 0 & -(am-\lambda) \\ 0 & 0 \end{pmatrix} + T_0(\vartheta). \end{aligned} \tag{5.6}$$

Note that T_0 is analytic in a sufficiently small complex neighbourhood of 0. Summarising the transformations above, we find that the equation $(\mathcal{A}-\lambda)u = 0$ is equivalent to

$$\mathfrak{T}\,U_0^{-1}u \;:=\; \left(\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\vartheta} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\vartheta}\end{pmatrix} + \tfrac{1}{\vartheta}\,T_1 + T_0(\vartheta)\right)U_0^{-1}u \;=\; 0 \tag{5.7}$$

with the constant matrix $T_1 = \begin{pmatrix} -k+\frac12 & -(am+\lambda) \\ 0 & k+\frac12\end{pmatrix}$ and the matrix function T_0 defined in (5.6). According to [CL55, chap. 4, theorem 4.1], a fundamental system W of the differential equation $\mathfrak{T}w = 0$ is given by

$$W(\vartheta) \;=\; P(\vartheta)\vartheta^{-T_1}$$

where $P(\vartheta) = 1 + \sum\limits_{j=1}^{\infty} P_j\vartheta^j$ is an analytic function and

$$\vartheta^{-T_1} := \mathrm{e}^{-\log\vartheta\;T_1} \;=\; \vartheta^{-\frac12}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} 0 & (am-\lambda)\log\vartheta \\ 0 & 0\end{pmatrix}\right).$$

Thus a particular solution Ψ_0 of the differential equation $(\mathcal{A}-\lambda)u = 0$ in a neighbourhood of 0 is given by

$$\begin{aligned} \Psi_0(\vartheta) &= U_0W(\vartheta)\begin{pmatrix}1\\0\end{pmatrix} \;=\; \vartheta^{-\frac12}U_0\,P(\vartheta)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \begin{pmatrix} 0 & (am-\lambda)\log\vartheta \\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix} \\ &= \vartheta^{-\frac12}U_0\,P(\vartheta)\begin{pmatrix}1\\0\end{pmatrix} \;=\; \vartheta^{-\frac12}\,U_0\begin{pmatrix}1\\0\end{pmatrix} + \vartheta^{\frac12}\,U_0\sum_{j=0}^{\infty}P_{j+1}\vartheta^j\begin{pmatrix}1\\0\end{pmatrix} \\ &= \vartheta^{\frac12}\begin{pmatrix}1\\p_{21}\end{pmatrix} + \vartheta^{\frac32}\left(\begin{pmatrix}p_{11}\\0\end{pmatrix} + U_0\sum_{j=0}^{\infty}P_{j+2}\vartheta^j\begin{pmatrix}1\\0\end{pmatrix}\right), \end{aligned}$$

where $P_1 =: (p_{ij})_{i,j=1,2}$. Since the second term is analytic in a neighbourhood of 0 and tends to 0 for $\vartheta \to 0$, there is a constant c_0 such that $\|\Psi_0(\vartheta)\| \le c_0\,|\vartheta^{\frac12}|$ in a neighbourhood of 0. Analogously, we find a solution Ψ_π of $(\mathcal{A}-\lambda)u = 0$ and a constant c_π such that $\|\Psi_\pi(\vartheta)\| \le c_\pi|(\pi-\vartheta)^{\frac12}|$ in a neighbourhood of $\pi-\vartheta$. If λ is an eigenvalue of $\mathcal{A}$ with eigenfunction Ψ, then we must have $\Psi = d_0\,\Psi_0 = d_\pi\Psi_\pi$ for some constants $d_0, d_\pi \in \mathbb{C}$ because $\mathcal{A}$ is in the limit point case both at 0 and π. Combining the estimates for the decay of Ψ_0 and Ψ_π, we conclude that there must be a constant c, such that $\|\Psi(\vartheta)\| \le c\sqrt{\vartheta(\pi-\vartheta)}$ for all $\vartheta \in (0,\pi)$. $\square$

Remark 5.3. If we estimate the integrands in the proof of lemma 5.1 by means of exponential functions (that is, if we use (3.45 c) instead of (3.45 a) and (3.45 b)), we can show that elements of $\mathcal{D}(B^*)$ and $\mathcal{D}(B)$ decay like the square root only for one endpoint of the interval $(0, \pi)$. For instance, for $k \geq 0$ we obtain

$$\text{(i)} \quad |f(\vartheta)| \leq \frac{\Gamma(\omega)\,\|B^* f\|}{\sqrt{2k+1}}\, \mathrm{e}^{-(2k+1)\pi}\,(\mathrm{e}^{(2k+1)(\pi-\vartheta)} - 1)^{\frac{1}{2}}, \quad f \in \mathcal{D}(B^*),$$

$$\text{(ii)} \quad |g(\vartheta)| \leq \frac{\Gamma(\omega)\,\|Bg\|}{\sqrt{2k+1}}(1 - \mathrm{e}^{-(2k+1)\vartheta})^{\frac{1}{2}}, \quad g \in \mathcal{D}(B). \qquad \Diamond$$

5.1.2 The transformed operator $\mathcal{A}_U$

As already mentioned at the beginning of this chapter, there is a unitary transformation that transforms the angular operator into a block operator matrix whose diagonal entries have nonintersecting spectrum. For the angular operator in the form

$$\mathcal{A} = \begin{pmatrix} -am\cos\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & am\cos\vartheta \end{pmatrix} = \begin{pmatrix} -D & B \\ B^* & D \end{pmatrix},$$

only the off-diagonal entries are unbounded. The operators $-D$ and D on the diagonal are bounded, but their spectra are not separated, they even coincide:

$$\sigma(D) = \sigma(-D) = [\,-|am|,\, |am|\,].$$

Transformation of $\mathcal{A}$ with the unitary matrix $U = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} I & I \\ -I & I \end{smallmatrix}\right)$ leads to

$$\mathcal{A}_U := U\mathcal{A}U^{-1} = \begin{pmatrix} \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta & \frac{\mathrm{d}}{\mathrm{d}\vartheta} + am\cos\vartheta \\ -\frac{\mathrm{d}}{\mathrm{d}\vartheta} + am\cos\vartheta & -(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta) \end{pmatrix} =: \begin{pmatrix} -D_U & B_U \\ B_U^* & D_U \end{pmatrix}, \tag{5.8}$$

where

$$\begin{aligned} B_U &:= \tfrac{1}{2}(B - B^*) + D = \tfrac{\mathrm{d}}{\mathrm{d}\vartheta} + am\cos\vartheta, \\ D_U &:= -\tfrac{1}{2}(B + B^*) = -\Big(\tfrac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta\Big). \end{aligned}$$

Recall that this transformation was applied already earlier in section 2.2. Since $\mathcal{A}$ is selfadjoint and U is unitary, the operator $\mathcal{A}_U$ with domain

$$\mathcal{D}(\mathcal{A}_U) = U\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} g+f \\ g-f \end{pmatrix} : f \in \mathcal{D}(B^*),\ g \in \mathcal{D}(B) \right\} \tag{5.9}$$

is also selfadjoint. Although the angular operator $\mathcal{A}$ is a block operator matrix, it is not clear whether the operator matrix $\mathcal{A}_U$ is also a block operator matrix since its domain is not given explicitly as a direct sum of two submanifolds. In fact, the next remark shows that for $k \in \{-1, 0\}$ the block operator $\mathcal{A}_U$ is not a block operator matrix.

Remark 5.4. For $k \in \{-1, 0\}$ the operator $\mathcal{A}_U$ is not a block operator matrix.

Proof. We have to show that for wave numbers $k \in \{-1, 0\}$ the domain $\mathcal{D}(\mathcal{A}_U) = U\mathcal{D}(\mathcal{A})$ of $\mathcal{A}_U$ cannot be written as a direct sum $\mathcal{D}(\mathcal{A}_U) = \mathcal{D}_{U,1} \oplus \mathcal{D}_{U,2}$ with $\mathcal{D}_{U,1},\ \mathcal{D}_{U,2} \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. We show the assertion for $k = 0$ only; the case $k = -1$ follows similarly. Let χ be a smooth function on $(0, \pi)$ with values in $[0, 1]$, such that

$$\chi \equiv \begin{cases} 1 & \text{in a neighbourhood of } 0, \\ 0 & \text{in a neighbourhood of } \pi. \end{cases}$$

We consider the vector valued function $\chi\Psi_1$ with $\Psi_1(\vartheta) = t(\vartheta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $t(\vartheta) := \left(\tan\frac{\vartheta}{2}\right)^{\frac{1}{2}}$. Recall that Ψ_1, already defined in (2.16), is a solution of the differential equation $\mathfrak{A}\Psi = 0$ and that it lies left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. Hence $\chi\Psi_1$ lies in the domain of $\mathcal{A}$, implying that $U(\chi\Psi_1) = \frac{1}{\sqrt{2}}\chi t\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ lies in the domain of $\mathcal{A}_U$. On the other hand, $D_U(\chi t)$ diverges of order $\vartheta^{-\frac{1}{2}}$ for $\vartheta \to 0$, hence it is not square integrable. Consequently, the vector valued function $\chi t\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ cannot be an element of the domain of $\mathcal{A}_U$ which proves the assertion. □

Although in general $\mathcal{A}_U$ is not a block operator matrix, the minimal operator $\mathcal{A}_U^{\min}$, defined by

$$\mathcal{D}(\mathcal{A}_U^{\min}) := U\mathcal{D}(\mathcal{A}^{\min}) = U\left\{\begin{pmatrix} f \\ g \end{pmatrix} : f, g \in \mathcal{C}_0^\infty(0,\pi)\right\} = \mathcal{D}(\mathcal{A}^{\min}),$$
$$\mathcal{A}_U^{\min}\begin{pmatrix} f \\ g \end{pmatrix} := \mathcal{A}_U\begin{pmatrix} f \\ g \end{pmatrix}$$

always is. We have already shown that $\mathcal{A}^{\min}$ is essentially selfadjoint and that $\mathcal{A}$ is its selfadjoint extension (see section 2.1.2). Therefore also $\mathcal{A}_U^{\min}$ is essentially selfadjoint with closure $\mathcal{A}_U$.

For $k \in \mathbb{R}\setminus[-1, 0]$, we associate sesquilinear forms with the entries of $\mathcal{A}_U$ as follows:

$$\mathfrak{d}_U[u, v] := (u, D_U v), \qquad u, v \in \mathcal{D}(\mathfrak{d}_U) \tag{5.10}$$

$$\mathfrak{b}_U[u, v] := (u, B_U v). \qquad u, v \in \mathcal{D}(\mathfrak{b}_U). \tag{5.11}$$

As domains we choose either $\mathcal{D}(\mathfrak{d}_U) = \mathcal{D}(\mathfrak{b}_U) := \mathcal{D}(B)$ or $\mathcal{D}(\mathfrak{d}_U) = \mathcal{D}(\mathfrak{b}_U) := \mathcal{D}(B^*)$.

Remark 5.5. (i) For $k \in \mathbb{R}\setminus[-1, 0]$, the forms are well defined with either domain; in fact, if we use equation (5.1 b), it is easy to see that for all $k \in \mathbb{R}\setminus[-1, 0]$ and $u, v \in \mathcal{D}(B)$

$$\begin{aligned} |\mathfrak{d}_U[u, v]| &= |(u, D_U v)| = \left|\int_0^\pi \left(a\omega\sin\vartheta + \tfrac{k+\frac{1}{2}}{\sin\vartheta}\right)\overline{u(\vartheta)}v(\vartheta)\mathrm{d}\vartheta\right| \\ &\le c(k)^2\,\Gamma(k,\omega)^2\|Bu\|\,\|Bv\|\int_0^\pi\left(a\omega\sin\vartheta + \tfrac{k+\frac{1}{2}}{\sin\vartheta}\right)\vartheta(\pi-\vartheta)\,\mathrm{d}\vartheta < \infty. \end{aligned}$$

Since the integrand is bounded, we have proved that the form $\mathfrak{d}_U$ with domain $\mathcal{D}(B)$ is well defined. Using this result and $v \in \mathcal{D}(B)$ we also find that

$$\begin{aligned} |\mathfrak{b}_U[u, v]| &= |(u, B_U v)| = |(u, (B + D_U + D)v)| \\ &\le |(u, Bv)| + |(u, D_U v)| + |(u, Dv)| \le \|u\|\,\|Bv\| + |\mathfrak{d}_U[u, v]| + |am|\|u\|\,\|v\| < \infty. \end{aligned}$$

For $u, v \in \mathcal{D}(B^*)$ similar considerations show that $\mathfrak{d}_U$ and $\mathfrak{b}_U$ with domain $\mathcal{D}(B^*)$ are also well defined.

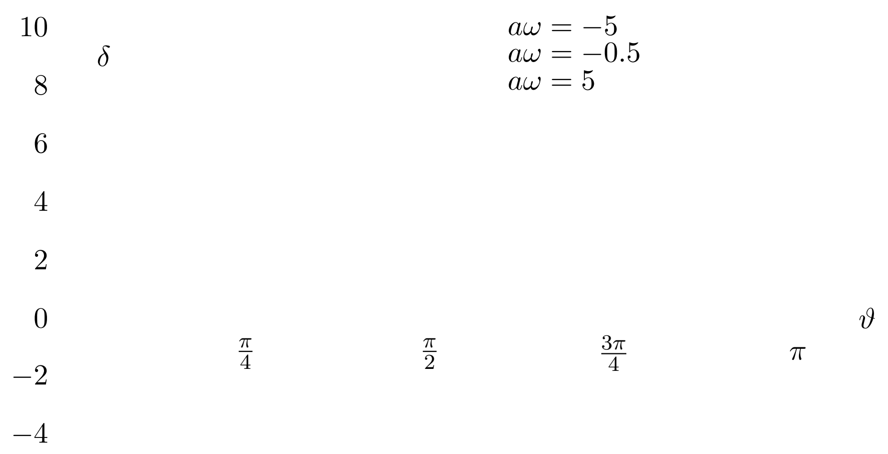

Figure 5.1. The plot shows the function δ of lemma 5.6 for $k = 0$ and $a\omega = -5,\ -0.5,\ 5$.

(ii) Also in the cases $k = -1$ and $k = 0$ the scalar products

$$(u,\, D_U v) \qquad \text{and} \qquad (u,\, B_U v)$$

are well defined if u and v are components of eigenfunctions of $\mathcal{A}_U$ because in that case they also show a decay proportional to $\sqrt{\vartheta(\pi - \vartheta)}$ at $\vartheta = 0$ and $\vartheta = \pi$, see remark 5.2. $\Diamond$

In order to determine the spectrum of the operators $\pm D_U = \mp(a\omega \sin\vartheta + \frac{k+\frac{1}{2}}{\sin\vartheta})$ we have to find the range of the function

$$\delta : (0, \pi) \longrightarrow \mathbb{R}, \qquad \delta(\vartheta) = a\omega \sin\vartheta + \frac{k + \frac{1}{2}}{\sin\vartheta}. \tag{5.12}$$

Sample plots of δ for the values $k = 0$ and $a\omega \in \{-5,\ -0.5,\ 5\}$ are shown in figure 5.1.

Lemma 5.6. *The function δ is not bounded from above if $k \geq 0$ and not bounded from below if $k \leq -1$. In either case it has a global extremum $\delta(\vartheta_0)$ at ϑ_0 given by*

$$\begin{cases} \vartheta_0 = \dfrac{\pi}{2} & \textit{with} \quad \delta_0 := \delta(\vartheta_0) = k + \frac{1}{2} + a\omega \qquad \textit{if} \ \ \varepsilon_k a\omega \leq |k + \frac{1}{2}|, \\[2ex] \sin\vartheta_0 = \sqrt{\dfrac{a\omega}{k + \frac{1}{2}}} & \textit{with} \quad \delta_0 := \delta(\vartheta_0) = 2\varepsilon_k \sqrt{a\omega\left(k + \frac{1}{2}\right)} \qquad \textit{if} \ \ \varepsilon_k a\omega \geq |k + \frac{1}{2}|, \end{cases}$$

where $\varepsilon_k := \operatorname{sign}(k + \frac{1}{2})$.

Proof. Assume $k \geq 0$. Then it is easy to see that

$$\lim_{\vartheta \to 0} \delta(\vartheta) = \lim_{\vartheta \to \pi} \delta(\vartheta) = \infty. \tag{5.13}$$

Since δ is continuously differentiable on $(0, \pi)$, it is necessarily bounded from below. Its derivative

$$\frac{\mathrm{d}\delta}{\mathrm{d}\vartheta}(\vartheta) = \frac{\cos\vartheta(a\omega \sin^2\vartheta - (k + \frac{1}{2}))}{\sin^2\vartheta} \tag{5.14}$$

shows that for $a\omega \le k+\frac{1}{2}$ there is only one local extremal point, namely $\vartheta_0 = \frac{\pi}{2}$, and the behaviour of δ at the boundary points 0 and π shows that the extremum is a global minimum.

For $a\omega > k+\frac{1}{2}$ there are three extremal points of δ, given by $\vartheta_1 = \frac{\pi}{2}$ and $\vartheta_2 \ne \vartheta_3$, such that $\sin\vartheta_2 = \sin\vartheta_3 = \sqrt{\frac{k+\frac{1}{2}}{a\omega}}$. In order to find the global minimum of δ we evaluate the function δ at the points ϑ_1, ϑ_1 and ϑ_3:

$$\begin{aligned}\delta(\vartheta_1) &= a\omega + k + \tfrac{1}{2},\\ \delta(\vartheta_2) = \delta(\vartheta_3) &= 2\sqrt{a\omega\Big(k+\tfrac{1}{2}\Big)}\end{aligned}$$

and compare them:

$$\begin{aligned}\delta(\vartheta_1)^2 &= (a\omega + k + \tfrac{1}{2})^2 = a^2\omega^2 + (k+\tfrac{1}{2})^2 + 2a\omega(k+\tfrac{1}{2}) = (a\omega - (k+\tfrac{1}{2}))^2 + 4a\omega(k+\tfrac{1}{2})\\ &\ge 4a\omega(k+\tfrac{1}{2}) = \delta^2(\vartheta_2).\end{aligned}$$

By the assumption on $a\omega$ and k we have that $\delta(\vartheta_1) > 0$ and $\delta(\vartheta_2) > 0$, therefore it follows that δ is minimal for ϑ_2 and ϑ_3. The case $k \le -1$ can be treated analogously. $\square$

The preceding lemma allows us to locate the spectrum of D_U.

Corollary 5.7. *For $\varepsilon_k a\omega \le -|k+\frac{1}{2}|$ the spectra of D_U and $-D_U$ intersect, more precisely we have*

$$\sigma(D_U)\cap\sigma(-D_U) = [-|k+\tfrac{1}{2}+a\omega|,\ |k+\tfrac{1}{2}+a\omega|] \ne \emptyset.$$

For $a\omega = -\varepsilon_k|k+\frac{1}{2}| = -(k+\frac{1}{2})$ they have exactly the point 0 in common and for $\varepsilon_k a\omega > -|k+\frac{1}{2}|$ the spectra do not intersect.

Proof. Since $-D_U$ is the operator of multiplication by the function δ, its spectrum is the closure of the range of δ which, by the previous lemma, is $[\delta_0, \infty)$ for $k \ge 0$ and $(-\infty, \delta_0]$ for $k \le -1$. If $\delta_0 = 0$, then the spectra of D_U and $-D_U$ have exactly the point 0 in common; this is the case if and only if $a\omega = -(k+\frac{1}{2})$. If $k \ge 0$ and $\delta_0 < 0$ or $k \le -1$ and $\delta_0 > 0$, then we have $\sigma(D_U)\cap\sigma(-D_U) = [\,-|\delta_0|,\,|\delta_0|\,]$. From the previous lemma it follows that the first case occurs if and only if $k \ge 0$ and $a\omega < -(k+\frac{1}{2})$, the second case holds if and only if $k \le -1$ and $a\omega > -(k+\frac{1}{2})$. In all other cases the spectra of D_U and $-D_U$ do not intersect. Using the expressions for δ_0 of the previous lemma yields the assertion. $\square$

Corollary 5.8. *Assume $k \ge 0$ and $a\omega \ge 0$. Then we have*

$$(-D_U x, x) \ \ge\ k+\tfrac{1}{2}, \qquad x \in \mathcal{D}(D_U),\ \|x\| = 1. \tag{5.15}$$

For $a\omega > k+\frac{1}{2}$ we even have

$$(-D_U x, x) \ \ge\ 2\big(k+\tfrac{1}{2}\big), \qquad x \in \mathcal{D}(D_U),\ \|x\| = 1. \tag{5.16}$$

Similar results hold for $k \le -1$ and $a\omega \le 0$.

Proof. For $k \ge 0$ this follows from $(-D_U x,\, x) \ge \delta_0\|x\|^2$ for all $x \in \mathcal{D}(D_U)$. For $k \le -1$ we use $(-D_U x,\, x) \le \delta_0\|x\|^2$ for all $x \in \mathcal{D}(D_U)$. $\square$

5.2 A spectral gap around zero

Having located the spectrum of the operators on the diagonal of $\mathcal{A}_U$, we are now in a position to establish a lower bound for the absolute values of the eigenvalues of $\mathcal{A}_U$ and consequently of $\mathcal{A}$ since the spectra of $\mathcal{A}$ and $\mathcal{A}_U$ coincide.
In the case $\varepsilon_k a\omega \geq -|k+\frac{1}{2}|$ the following general theorem provides a lower bound for the modulus of the eigenvalues of $\mathcal{A}_U$. In the case $\varepsilon_k a\omega < -|k+\frac{1}{2}|$, however, the theorem does not apply to the angular operator because the ranges of the functions δ and $-\delta$, and therefore the spectra of $-D_U$ and D_U, intersect.

Theorem 5.9. *Let $\mathcal{H} = \widetilde{\mathcal{H}} \oplus \widetilde{\mathcal{H}}$ be the direct sum of a Hilbert space with itself and let $\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with domain $\mathcal{D}(\mathcal{T}) \subseteq \mathcal{H}$ be a selfadjoint block operator matrix such that T_{11} and T_{22} are symmetric and that $T_{21} = T_{12}^*$. Further, suppose that for $i,j = 1,2$ there exist sesquilinear forms $\mathfrak{t}_{ij}$ on $\widetilde{\mathcal{H}}$ with domain $\mathcal{D}(\mathfrak{t}_{ij}) \supseteq \mathcal{D}(T_{ij})$ such that $\mathfrak{t}_{ij}[u,v] = (u, T_{ij}v)$ for all $u \in \mathcal{D}(\mathfrak{t}_{ij})$ and $v \in \mathcal{D}(T_{ij})$, and that for all eigenfunctions $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ of $\mathcal{T}$ we have $\Psi_i \in \mathcal{D}(\mathfrak{t}_{ij})$; in addition suppose $\mathfrak{t}_{ij} = \mathfrak{t}_{ji}^*$ Finally, let*

$$E_i \;:=\; \{\Psi_i \in \widetilde{\mathcal{H}} : \text{ there exists a } \Psi_2 \in \widetilde{\mathcal{H}} \text{ such that } \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \text{ is an eigenfunction of } \mathcal{T}\}.$$

(i) *Suppose that there are numbers $s_1 \geq s_2$ such that*

$$\mathfrak{t}_{11}[\Psi_1] \;>\; s_1\,\|\Psi_1\|^2, \qquad \mathfrak{t}_{22}[\Psi_2] \;<\; s_2\,\|\Psi_2\|^2, \qquad \Psi_1 \in E_1 \setminus \{0\},\; \Psi_2 \in E_2 \setminus \{0\}.$$

If λ is an eigenvalue of $\mathcal{T}$ with eigenfunction $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$, then

$$\lambda \,>\, s_1 \quad \text{and} \quad \|\Psi_1\| \,>\, \|\Psi_2\| \qquad \text{or} \qquad \lambda \,<\, s_2 \quad \text{and} \quad \|\Psi_1\| \,<\, \|\Psi_2\|. \tag{5.17}$$

(ii) *Suppose that there are numbers $s_1 \leq s_2$ such that*

$$\mathfrak{t}_{11}[\Psi_1] \;<\; s_1\,\|\Psi_1\|^2, \qquad \mathfrak{t}_{22}[\Psi_2] \;>\; s_2\,\|\Psi_2\|^2 \qquad \Psi_1 \in E_1 \setminus \{0\},\; \Psi_2 \in E_2 \setminus \{0\}.$$

If λ is a eigenvalue of $\mathcal{T}$ with eigenfunction $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$, then

$$\lambda \,<\, s_1 \quad \text{and} \quad \|\Psi_1\| \,>\, \|\Psi_2\| \qquad \text{or} \qquad \lambda \,>\, s_2 \quad \text{and} \quad \|\Psi_1\| \,<\, \|\Psi_2\|. \tag{5.18}$$

Proof. We prove (i) only, assertion (ii) follows analogously. Let $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ be an eigenfunction of $\mathcal{T}$ with eigenvalue λ. We write the eigenvalue equation $(\mathcal{T} - \lambda)\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0$ as a system of coupled linear equations

$$\begin{aligned} T_{11}\Psi_1 - \lambda\Psi_1 + T_{12}\Psi_2 \;&=\; 0, \\ T_{12}^*\Psi_1 + T_{22}\Psi_2 - \lambda\Psi_2 \;&=\; 0. \end{aligned}$$

If we take the scalar product of Ψ_1 with the first row of the above system, and that of Ψ_2 with the second row, we obtain the following linear system

$$\begin{aligned} \mathfrak{t}_{11}[\Psi_1] - \lambda\|\Psi_1\|^2 + \mathfrak{t}_{12}[\Psi_1,\, \Psi_2] \;&=\; 0, \\ \mathfrak{t}_{22}[\Psi_2] - \lambda\|\Psi_2\|^2 + \mathfrak{t}_{12}^*[\Psi_2,\, \Psi_1] \;&=\; 0. \end{aligned} \tag{5.19}$$

By assumptions, all terms in this system exist. The terms $\mathfrak{t}_{11}[\Psi_1]$ and $\mathfrak{t}_{22}[\Psi_2]$ are real, therefore also $\mathfrak{t}_{12}[\Psi_1,\, \Psi_2]$ and $\mathfrak{t}_{12}^*[\Psi_2,\, \Psi_1]$ must be real which implies $\mathfrak{t}_{12}[\Psi_1,\, \Psi_2] = \overline{\mathfrak{t}_{12}[\Psi_1,\, \Psi_2]} = \mathfrak{t}_{12}^*[\Psi_2,\, \Psi_1]$.

Note that system (5.19) is no longer a system of equations for vectors in some Hilbert space, but a system of linear equations for real numbers.
If $\Psi_1 = 0$, it follows that $\lambda\|\Psi_2\|^2 = \mathfrak{t}_{22}[\Psi_2] < s_2$; for $\Psi_2 = 0$ we have $\lambda\|\Psi_1\|^2 = \mathfrak{t}_{22}[\Psi_1] > s_1$. Now we assume $\Psi_1 \neq 0$ and $\Psi_2 \neq 0$. With $\mathfrak{t}_{12}[\Psi_1, \Psi_2] = \mathfrak{t}_{12}^*[\Psi_2, \Psi_1]$ it follows from system (5.19) that

$$\mathfrak{t}_{11}[\Psi_1] - \mathfrak{t}_{22}[\Psi_2] - \lambda\left(\|\Psi_1\|^2 - \|\Psi_2\|^2\right) \;=\; 0. \tag{5.20}$$

Observe that $\|\Psi_1\| \neq \|\Psi_2\|$, otherwise we have the contradiction

$$0 \;=\; \mathfrak{t}_{11}[\Psi_1] - \mathfrak{t}_{22}[\Psi_2] \;>\; (s_1 - s_2)\,\|\Psi_1\|^2 \;\geq 0.$$

Hence it follows from (5.20) that

$$\lambda \;=\; \frac{\mathfrak{t}_{11}[\Psi_1] - \mathfrak{t}_{22}[\Psi_2]}{\|\Psi_1\|^2 - \|\Psi_2\|^2}.$$

Assume that $\|\Psi_1\| > \|\Psi_2\|$. Using $\mathfrak{t}_{11}[\Psi_1] - \mathfrak{t}_{22}[\Psi_2] > s_1\,\|\Psi_1\|^2 - s_2\,\|\Psi_2\|$, we obtain

$$\lambda \;>\; \frac{s_1\|\Psi_1\|^2 - s_2\|\Psi_2\|^2}{\|\Psi_1\|^2 - \|\Psi_2\|} \;=\; s_1 + \frac{(s_1 - s_2)\,\|\Psi_2\|^2}{\|\Psi_1\|^2 - \|\Psi_2\|^2} \;\geq\; s_1.$$

Analogously, we can show $\lambda < s_2$ if $\|\Psi_1\| < \|\Psi_2\|$. □

The previous theorem, applied to the transformed angular operator $\mathcal{A}_U$, yields a lower bound for the modulus of the eigenvalues of the angular operator $\mathcal{A}$.

Theorem 5.10. *Let λ be an eigenvalue of the angular operator $\mathcal{A}$. Then*

$$|\lambda| \;>\; \lambda_Q \;:=\; \begin{cases} \varepsilon_k(a\omega + k + \frac{1}{2}) \;=\; |a\omega + k + \frac{1}{2}| & \text{if} \quad a\omega \in [-|k+\frac{1}{2}|,\; |k+\frac{1}{2}|\,], \\ 2\sqrt{a\omega(k+\frac{1}{2})} & \text{if} \quad \varepsilon_k a\omega \geq |k+\frac{1}{2}|. \end{cases} \tag{5.21}$$

Corollary 5.11. *For $\varepsilon_k a\omega \geq 0$ and all eigenvalues λ of $\mathcal{A}$ we have*

$$|\lambda| \;>\; \bigl|k + \tfrac{1}{2}\bigr|. \tag{5.22}$$

For $\varepsilon_k a\omega \geq |k+\frac{1}{2}|$ we even have

$$|\lambda| \;>\; 2\bigl|k + \tfrac{1}{2}\bigr|. \tag{5.23}$$

Note that for $a = 0$ the estimate provided in (5.21) coincides with the one obtained in (3.75) by the off-diagonalisation of the angular operator.

Proof of theorem 5.10. Recall that $\varepsilon_k = \operatorname{sign}(k+\frac{1}{2})$. In remark 5.5 we have seen that for all eigenfunctions $\binom{u}{v}$ of $\mathcal{A}_U$ the numbers $\mathfrak{d}_U[u]$, $\mathfrak{d}_U[v]$ and $\mathfrak{b}_U[u,\, v]$ exist, thus $\mathcal{A}_U$ satisfies the assumptions of theorem 5.9. If $\varepsilon_k a\omega \geq -|k+\frac{1}{2}|$, then the function δ representing the multiplication operator $-D_U$ is either nonpositive or nonnegative and takes on its extremal value δ_0 exactly once in $(0, \vartheta)$, see lemma 5.6. Consequently, the spectra of $-D_U$ and D_U are separated or have only the point 0 in common. If $k \geq 0$, then $\delta(\vartheta) \geq \delta_0 \geq 0$, $\vartheta \in (0,\pi)$, with $\delta(\vartheta) = \delta_0$ for exactly one $\vartheta \in (0,\pi)$. Hence for every eigenfunction $\binom{u}{v}$ of $\mathcal{A}_U$ with $u \neq 0$ it follows that

$$\mathfrak{d}_U[u] \;=\; (u,\, -D_U u) \;=\; \int_0^\pi \delta(\vartheta)|u(\vartheta)|^2\,\mathrm{d}\vartheta \;>\; \delta_0\,\|u\|_2^2.$$

Application of theorem 5.9 (i) with $s_1 = -s_2 = \delta_0$ yields that $\lambda > s_1 = \delta_0$ for all eigenvalues of $\mathcal{A}_U$. If we insert the explicit expression for δ_0 given in lemma 5.6 we obtain $|\lambda| \geq \lambda_Q$ for all eigenvalues of $\mathcal{A}_U$. Since $\mathcal{A}$ and $\mathcal{A}_U$ are unitarily equivalent, the assertion is proved for $k \geq 0$. If $k \leq -1$, the assertion follows analogously, if we use theorem 5.9 (ii). □

Chapter 6

Comparison of the eigenvalue bounds

6.1 Negative eigenvalues

In chapter 3 and chapter 5 we obtained lower bounds λ_G and λ_Q for the modulus of the eigenvalues of the angular operator. Hence $-\lambda_G$ and $-\lambda_Q$ are upper bounds for the negative eigenvalues of $\mathcal{A}$. In section 4.2 we used a variational principle to obtain an exact formula for the eigenvalues of $\mathcal{A}$ that are greater than $\|D\| = |am|$. From that formula, upper and lower bounds for the eigenvalues have been derived. Since also the operator $-\mathcal{A}$ satisfies all the assumptions of the relevant theorems for the variational principle, they can be applied to $-\mathcal{A}$ thus resulting in a formula for the eigenvalues of $\mathcal{A}$ that are less than $-\|-D\| = -|am|$.
For convenience, we summarise the estimates for the negative eigenvalues in the following theorem.

Theorem 6.1. *All negative eigenvalues λ of the angular operator $\mathcal{A}$ are bounded from above by*

$$\lambda \;\leq\; \min\{-\lambda_G,\; -\lambda_Q\} \tag{6.1}$$

with the estimates λ_G from theorem 3.35 and λ_Q from theorem 5.10. For the nth eigenvalue $\lambda_{m_- - n}$ smaller than $-|am|$ we have

$$-\sqrt{\left(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n_0 + n\right)^2 + \Omega_+} - |am| \;\leq\; \lambda_{m_- - n} \;\leq\;$$

$$\min\Big\{-|am|,\; \mathrm{Re}\Big(-\sqrt{\left(|k+\tfrac{1}{2}| - \tfrac{1}{2} + n_0 + n\right)^2 + \Omega_-} + |am|\Big)\Big\}, \quad n \in \mathbb{N}, \tag{6.2}$$

with $\Omega_\pm$ given in theorem 4.40 and the offset $n_0 = \min\limits_{\lambda < -|am|} \dim \mathcal{L}_{(0,\infty)} S_1(\lambda)$.

Proof. The estimates in (6.1) follow from theorems 3.35 and 5.10 which provide lower bounds for the modulus of the eigenvalues of $\mathcal{A}$. In order to obtain relation (6.2) we consider $-\mathcal{A}$; let $S_1^{(-)}(\lambda)$ denote the Schur complement of $-\mathcal{A}$. Observe that $-\mathcal{A}$ satisfies the assumptions needed for the variational principle with the same constants $c_1,\ c_1^+,\ c_2^-,\ c_2$ as $\mathcal{A}$ so that formula (4.39) gives estimates for the eigenvalues of $-\mathcal{A}$ to the right of $\|-D\| = |am|$. Since λ is an eigenvalue of $-\mathcal{A}$ if and only if $-\lambda$ is an eigenvalue of $\mathcal{A}$, estimate (6.2) with the index shift $n_0 = n_0(-\mathcal{A}) = \min\limits_{\lambda > |am|} \dim \mathcal{L}_{(-\infty,0)} S_1^{(-)}(\lambda)$ according to theorem 4.32 is proved. It remains to verify the formula for the index shift n_0 in the assertion. The Schur complement $S_1^{(-)}(\lambda)$ of $-\mathcal{A}$ is related to the corresponding Schur complement of $\mathcal{A}$ by

$$S_1^{(-)}(\lambda) \;=\; D - \lambda - (-B)(-D-\lambda)^{-1}(-B^*) \;=\; -(-D + \lambda - B(D+\lambda)^{-1}B^*) \;=\; -S_1(-\lambda).$$

Hence

$$n_0 \;=\; \min_{\lambda>|am|} \dim \mathcal{L}_{(-\infty,\,0)} S_1^{(-)}(\lambda) \;=\; \min_{\lambda>|am|} \dim \mathcal{L}_{(0,\,\infty)} S_1(-\lambda) \;=\; \min_{\lambda<-|am|} \dim \mathcal{L}_{(0,\,\infty)} S_1(\lambda).$$

□

Analytic perturbation theory yields the following theorem in analogy to theorem 4.41.

Theorem 6.2. *For $n \in \mathbb{N}$ let λ_{-n} be the $-n$th eigenvalue of $\mathcal{A}$ which, if considered as analytic function of a, is the $-n$th negative eigenvalue of $\mathcal{A}$ in the case $a = 0$. Then for all $n \in \mathbb{N}$ we have that*

$$-\sqrt{\left(|k+\tfrac{1}{2}|-\tfrac{1}{2}+n\right)^2+\Omega_+}\; - |am| \;\le\; \lambda_{-n} \;\le\; -\operatorname{Re}\sqrt{\left(|k+\tfrac{1}{2}|-\tfrac{1}{2}+n\right)^2+\Omega_-}\; + |am|.$$

Remark 6.3. Another way to obtain the formula for the eigenvalues to the left of $-|am|$ is to use the symmetry properties of $\mathcal{A}$ with respect to change of the physical parameters, see chapter 2.3. We use the notation $\mathcal{A}(k,\omega)$, $\lambda_n(k,\omega)$, $S_1(\lambda;\, k,\omega)$ etc. to express the dependence on the physical parameters k and ω. It is easy to check that $\Omega_\pm(-(k+1),-\omega) = \Omega_\pm(k,\omega)$. Using remark 3.29 we obtain

$$S_1(\lambda;\, -(k+1),-\omega) \;=\; -RS_1(-\lambda;\, k,\omega)R.$$

Since R is unitary, it follows that

$$n_0(-(k+1),-\omega) \;=\; \min_{\lambda>|am|} \dim \mathcal{L}_{(-\infty,0)} S_1(\lambda;\, -(k+1),-\omega) \;=\; \min_{\lambda<-|am|} \dim \mathcal{L}_{(0,\infty)} S_1(\lambda;\, k,\omega)$$

which is equal to the index shift n_0 asserted in theorem 6.1. From lemma 2.17(ii) it follows that $\lambda_n(k,\omega)$ is an eigenvalue of $\mathcal{A}(k,\omega)$ if and only if $-\lambda_n(k,\omega)$ is an eigenvalue of $\mathcal{A}(-(k+1),-\omega)$. Hence, for the nth eigenvalue $\lambda_{m_--n}(k,\omega)$ of $\mathcal{A}(k,\omega)$ smaller than $-\|D\| = -|am|$ we obtain

$$\begin{aligned}
\lambda_{m_--n}(k,\omega) &= \; -\lambda_{m_++n}(-(k+1),-\omega) \\
&\ge \; -\Big(\sqrt{(|-(k+1)+\tfrac{1}{2}|-\tfrac{1}{2}+n_0(-(k+1),-\omega)+n)^2+\Omega_+(-(k+1),-\omega)}+|am|\Big) \\
&= \; -\sqrt{(|k+\tfrac{1}{2}|-\tfrac{1}{2}+n_0(-(k+1),-\omega)+n)^2+\Omega_+(k,\omega)}-|am|, \qquad n\in\mathbb{N}.
\end{aligned}$$

Analogously, the lower bound for $\lambda_{m_--n}(k,\omega)$ in (6.2) can be derived. ◊

6.2 Comparison of the estimates

In this section we first compare the various analytical bounds for the eigenvalues of $\mathcal{A}$ with each other, then we compare them with numerically calculated values obtained by Suffern, Fackerell and Cosgrove in [SFC83] and Chakrabarti in [Cha84]. Using an ansatz of [SFC83] we independently obtain numerical approximations of the eigenvalues.

Notation

The eigenvalues λ_n of the angular operator are enumerated such that they are the analytic continuation of the exact eigenvalues $\lambda_n = |k+\frac{1}{2}| - \frac{1}{2} + n$ in the case $a = 0$, see lemma 3.3.
All numerically obtained values are marked with a superscript *num*. For the values provided by Suffern et al. [SFC83], a subscript S, for those of Chakrabarti [Cha84], a subscript C is added.
Evaluating the series ansatz for the eigenvalues suggested in [SFC83] we obtain numerical values for λ, cf. the section 6.2.2; these values have no additional subscript.
The analytical bounds obtained in theorems 4.40 and 4.41 are denoted by $\lambda^{[l]}$ and $\lambda^{[u]}$, and $\lambda^{[l,\mathrm{SPT}]}$ and $\lambda^{[u,\mathrm{SPT}]}$, respectively. So we have

$$
\begin{array}{lllllllll}
\lambda^{[l]}_{n_0+n} & \leq & \lambda_{m_++n} & \leq & \lambda^{[u]}_{n_0+n}, & -\lambda^{[u]}_{n_0^-+n} & \leq & \lambda_{m_--n} & \leq -\lambda^{[l]}_{n_0^-+n}, & n\in\mathbb{N},\\
\lambda^{[l,\mathrm{SPT}]}_{n} & \leq & \lambda_{n} & \leq & \lambda^{[u,\mathrm{SPT}]}_{n}, & -\lambda^{[u,\mathrm{SPT}]}_{n} & \leq & \lambda_{-n} & \leq -\lambda^{[l,\mathrm{SPT}]}_{n}, & n\in\mathbb{N},
\end{array}
$$

with the index shifts $n_0 = \min_{\lambda>|am|}\dim\mathcal{L}_{(-\infty,0)}S_1(\lambda)$ and $n_0^- = \min_{\lambda<-|am|}\dim\mathcal{L}_{(0,\infty)}S_1(\lambda)$, see theorems 4.40 and 6.1.
Further we recall that we have the lower bounds λ_G and λ_Q of theorems 3.35 and 5.10: For all eigenvalues λ_n, $n\in\mathbb{Z}\setminus\{0\}$, of the angular operator we have

$$|\lambda_n| \;\geq\; \max\{\lambda_G,\ \lambda_Q\};$$

note, however, that λ_Q is not defined for all $k, a\omega$ and that λ_G might be negative for large values of $|am|$.

6.2.1 Analytic lower bounds for the modulus of the eigenvalues

Each of the bounds λ_G, $\lambda_G^{[\mathrm{lin}]}$ and $\lambda_G^{[\mathrm{exp}]}$ from theorem 3.35 and the subsequent remark, and λ_Q from theorem 5.10 is a lower bound for the modulus of the eigenvalues of the angular operator $\mathcal{A}$. Hence for every eigenvalue λ of $\mathcal{A}$ we have

$$|\lambda| \;\geq \max\{\lambda_G,\ \lambda_G^{[\mathrm{lin}]},\ \lambda_G^{[\mathrm{exp}]},\ \lambda_Q\}. \tag{6.3}$$

Since the explicit formulae for λ_G and λ_Q depend on the interval in which $a\omega$ lies, we set

$$
\begin{aligned}
\lambda_G^{(1)} &:= -|am| + 2C(\omega)^{-1}\mathrm{e}^{-2\pi|a\omega\nu-(k+\frac{1}{2})|}\,|a\omega\nu-(k+\tfrac{1}{2})|,\\
\lambda_G^{(2)} &:= -|am| + C^{-1}(\omega)\,|a\omega\nu-(k+\tfrac{1}{2})|\,,\\
\lambda_Q^{(1)} &:= |a\omega+k+\tfrac{1}{2}|\,,\\
\lambda_Q^{(2)} &:= 2\sqrt{a\omega(k+\tfrac{1}{2})}\ ;
\end{aligned}
$$

further recall that

$$\lambda_G^{[\mathrm{lin}]} \;=\; -|am| + \tfrac{1}{\pi\,\Gamma(k,\omega)}\,(|k+\tfrac{1}{2}|+1),\qquad \lambda_G^{[\mathrm{exp}]} \;=\; -|am| + \tfrac{1}{\Gamma(k,\omega)}\,|k+\tfrac{1}{2}|,$$

$k \leq 0$			$k \geq 0$		
$a\omega \in$	λ_G	λ_Q	$a\omega \in$	λ_G	λ_Q
$\left(-\infty,\ k+\frac{1}{2}\right)$	$\lambda_G^{(2)}$	$\lambda_Q^{(2)}$	$\left(-\infty,\ \frac{k+\frac{1}{2}}{\nu}\right)$	$\lambda_G^{(1)}$	not defined
$\left(k+\frac{1}{2},\ -(k+\frac{1}{2})\right)$	$\lambda_G^{(2)}$	$\lambda_Q^{(1)}$	$\left(\frac{k+\frac{1}{2}}{\nu},\ -(k+\frac{1}{2})\right)$	$\lambda_G^{(2)}$	not defined
$\left(-(k+\frac{1}{2}),\ \frac{k+\frac{1}{2}}{\nu}\right)$	$\lambda_G^{(2)}$	not defined	$\left(-(k+\frac{1}{2}),\ k+\frac{1}{2}\right)$	$\lambda_G^{(2)}$	$\lambda_Q^{(1)}$
$\left(\frac{(k+\frac{1}{2})}{\nu},\ \infty\right)$	$\lambda_G^{(1)}$	not defined	$\left(k+\frac{1}{2},\ \infty\right)$	$\lambda_G^{(2)}$	$\lambda_Q^{(2)}$

Table 6.1. Bounds for the modulus of the eigenvalues λ of the angular operator $\mathcal{A}$.

with $C(\omega) = \mathrm{e}^{|a\omega(c_+ - c_-)|}$, see theorems 3.35 and 5.10. The formulae for λ_G and λ_Q are summarised in table 6.1.

Unfortunately, the estimates above only give a lower bound for the modulus of the eigenvalues of $\mathcal{A}$; they do not distinguish between positive and negative eigenvalues λ. For λ_G, this is due to the fact that we had to solve an equation for $|\lambda|$. For λ_Q it follows from the symmetry of the gap between the spectrum of D_U and $-D_U$ with respect to 0. The only exception where the sign of λ plays a role is lemma 3.38. Numerical results, however, show that for $a \neq 0$ the eigenvalues λ are not symmetric with respect to 0.

Which of the given estimates is the better one depends on $a\omega$ and am. As we have already observed, in the case $a = 0$ the bounds λ_G and λ_Q coincide. For $a \neq 0$ the lower bound λ_Q will in general yield better results provided that it is defined since for fixed k the lower bound λ_G is decaying exponentially with increasing $|a|$. On the other hand, if we fix a and let k grow, then λ_Q is given by $\lambda_Q = k + \frac{1}{2} + a\omega$, whereas λ_G grows like $C(\omega)^{-1}(k + \frac{1}{2})$ with $C(\omega)^{-1} < 1$ only.
For fixed k and ω the lower bound λ_G may even become negative for large m or large a due to the term $-|am|$ that arises because of the perturbation nature of the estimate. Nevertheless, for $a\omega$ such that λ_Q fails to exist, λ_G still provides a lower bound for the modulus of the eigenvalues if $|am|$ is small enough.

Another lower bound for the modulus of the eigenvalues of $\mathcal{A}$ is given by $\lambda_1^{[l]}$ where it is nonnegative, see theorem 4.40.

In figures 6.1 and 6.2 the lower bounds are plotted as functions of the Kerr parameter a with the wave number k, mass m and frequency ω fixed. The bound $\lambda_1^{[l]}$ is plotted only for those values of a where the index shift n_0 is zero according to lemma 4.34 (ii). Interestingly, if the physical parameters are chosen $m = 0.025$, $\omega = 0.75$ and $k = 0$, see figure 6.1, then for each of the lower bounds λ_G, $\lambda_G^{[\exp]}$, λ_Q and $\lambda_1^{[l]}$ there is an interval where it provides a better lower bound than the other three bounds.

However, it seems that for large $|am|$ the bound λ_Q generally provides the best lower bound for the modulus of the eigenvalues of $\mathcal{A}$ provided that λ_Q is defined.

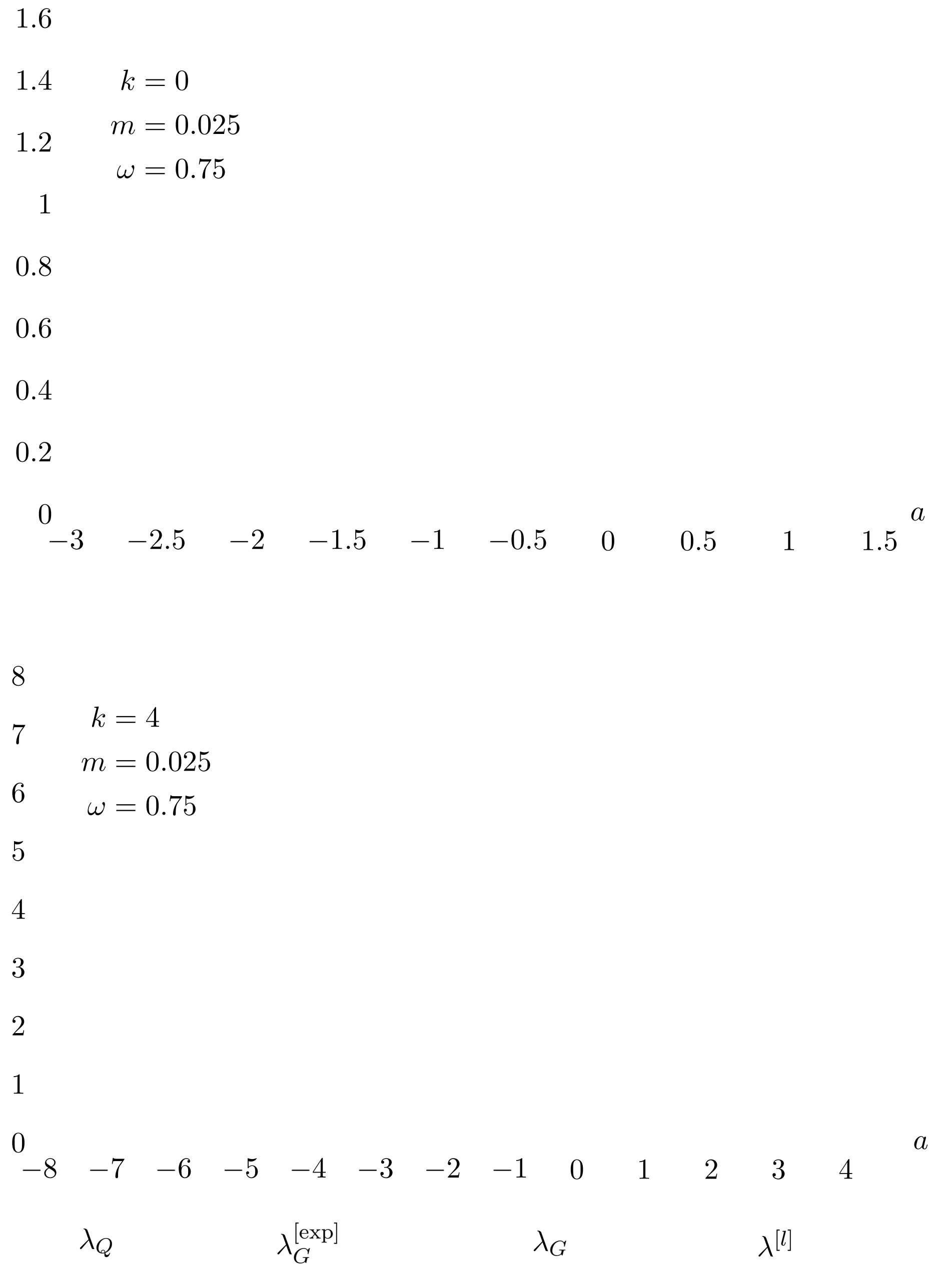

Figure 6.1. Lower bounds for the eigenvalue of $\mathcal{A}$ with smallest modulus for k, m and ω fixed.Note that for $k = 0$, for each of the plotted estimates there is an interval for a where it provides a larger lower bound for the modulus of the eigenvalues of $\mathcal{A}$ than the other three bounds: for $a \lesssim -1.06900$ the bound λ_G yields the best result; for $-1.06900 \lesssim a \lesssim -0.642408$, $\lambda_G^{[\exp]}$ gives the sharpest lower bound; then, for $-0.642408 \lesssim a \lesssim 0.66667$, $\lambda_1^{[l]}$ is the best lower bound; finally, for $0.66667 \lesssim a$, the best lower bound is provided by λ_Q.

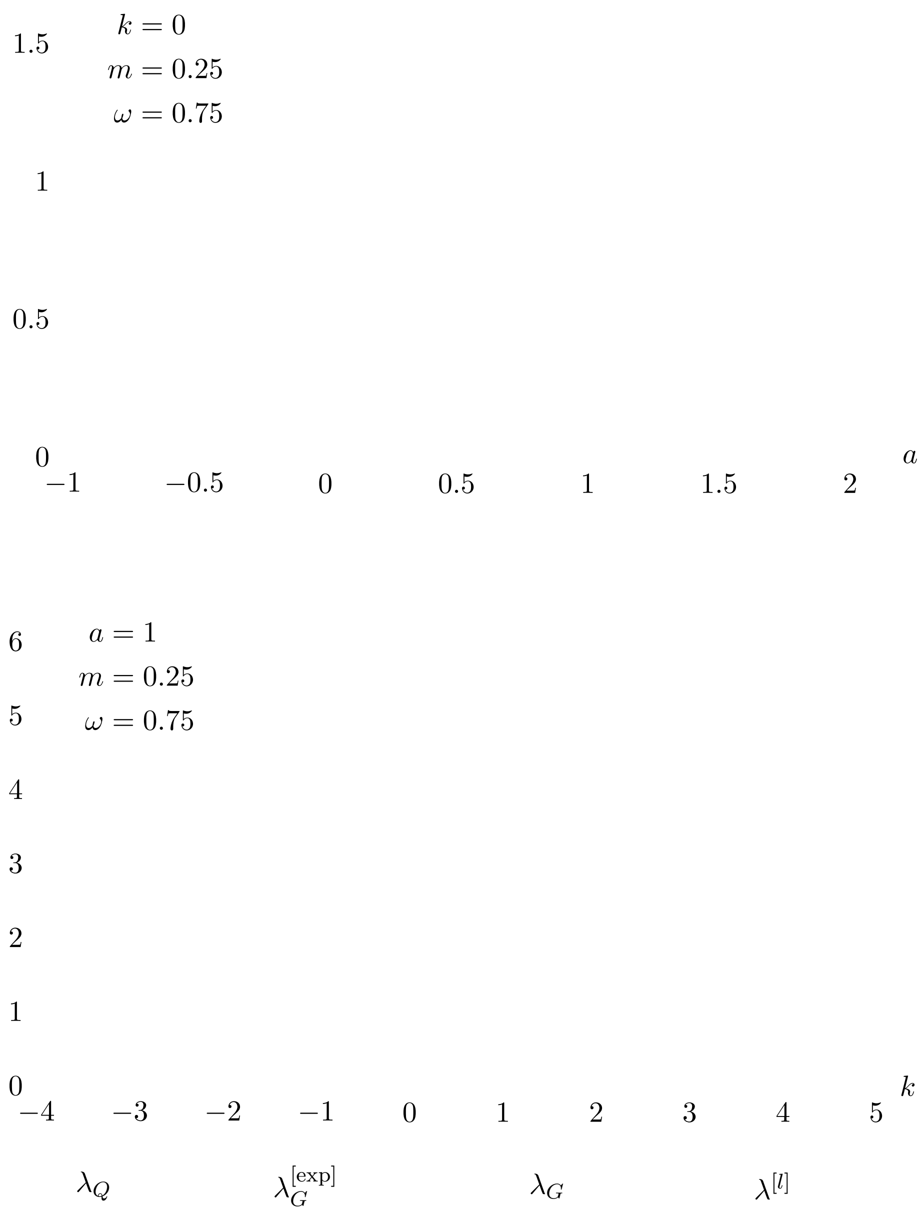

Figure 6.2. Lower bounds for the modulus of the eigenvalues λ of $\mathcal{A}$ for $m = 0.25$ and $\omega = 0.75$ fixed.In the first graph, the bounds are plotted as functions of a for $k = 0$ fixed. The second graph shows the bounds as functions of k with $a = 1$ fixed. Note that physically only the values for integral values of k make sense.

6.2.2 Comparison with numerical values

In the following we compare the various analytic bounds for the eigenvalues of the angular operator $\mathcal{A}$ proved in this work with numerical values from the papers [SFC83] and [Cha84]. When using the estimates $\lambda^{[l,\mathrm{SPT}]}$ and $\lambda^{[u,\mathrm{SPT}]}$ from theorem 4.41, the main problem is to prove that for given physical parameters the first positive eigenvalue is the analytic continuation of the first positive eigenvalue in the case $a = 0$. For the bounds $\lambda^{[l]}$ and $\lambda^{[u]}$ from theorem 4.40 we additionally have to determine the index shifts m_+ and n_0.

Comparison with numerical values in [SFC83]

Suffern, Fackerell and Cosgrove [SFC83] obtained numerical approximations of the eigenvalues of the angular operator by expanding the solution of the angular equation in terms of hypergeometric functions, resulting in a three term recurrence relation for the coefficients in the series ansatz. Then λ is expanded with respect to $a(m-\omega)$ and $a(m+\omega)$ as

$$\lambda = \sum_{r,s} C_{r,s} a^{r+s} (m-\omega)^r (m+\omega)^s$$

with the coefficients $C_{r,s}$ obtained from the recurrence relation. Observe that in [SFC83] the authors denote the wave number by m (in our terminology it is denoted by k), and that, due to the form of the differential equations in the cited article, their eigenvalues (which we denote here by $\lambda^{[num]}_{S,n}$) differ from the eigenvalues given in this work by a factor -1.
Instead of using the coefficients $C_{r,s}$, it is also possible to find a numerical approximation for λ from the continued fraction equation for λ given in [SFC83] directly. We find that the eigenvalues $\lambda^{[num]}$ computed in this way with a short Maple programme differ at most slightly from the tabulated values in [SFC83] for small n; for higher values of am, $a\omega$ and n, however, there are significant differences. For instance, for $am = 0.25$, $a\omega = 0.75$ and $k = 0$ Suffern et al. list the fourth positive eigenvalue as $\lambda^{[num]}_{S,4} = 4.13127$, whereas an evaluation of the recurrence relation for λ gives $\lambda^{[num]}_4 = 4.13969$. The results in [BSW05] seem to favour the latter value for λ, cf. in particular the appendix in the cited article. In the following, the numerical values $\lambda^{[num]}_{S,n}$, $\lambda^{[num]}_{S,-n}$ are all taken from tables in the article [SFC83].

For fixed values of am and $a\omega$, tables 6.2 and 6.3 contain the numerical values for the first positive and first negative eigenvalues tabulated in [SFC83] and the analytical bounds λ_G and λ_Q obtained in theorems 3.35 and 5.10, together with the lower and upper bounds $\lambda^{[l]}_1$ and $\lambda^{[u]}_1$ from theorem 4.40 for wave numbers $k = -5, \ldots, 4$. For all physical parameters under consideration, apart from the case $am = 0.25$, $a\omega = 0.75$, $k = -1$, we always have $\varepsilon_k a\omega \geq -|k + \frac{1}{2}|$ so that λ_Q is defined; furthermore,

$$\|B^{-1}\|^{-1} \;=\; \sqrt{\nu_1} \;\geq\; \mathrm{Re}\Big(\sqrt{(|k+\tfrac{1}{2}| + \tfrac{1}{2})^2 + \Omega_-}\;\Big) \;>\; 2|am|$$

where ν_1 is the first eigenvalue of BB^*, see theorem 4.39, so that we have $n_0 = 0$ and $m_+ = 0$ by lemma 4.34 (ii). Therefore, the first positive eigenvalue is indeed the analytic continuation of the first positive eigenvalue in the case $a = 0$. The case $am = 0.25$, $a\omega = 0.75$, $k = -1$ is discussed in the subsequent remark.

Remark 6.4 ($am = 0.25$**,** $a\omega = 0.75$**,** $k = -1$**).**

(i) In this case, the bound λ_Q is not defined because of $\varepsilon_k a\omega = -0.75 < -\frac{1}{2} = -|k + \frac{1}{2}|$.

(ii) Theorem 4.39 yields no positive upper bound for $\|B^{-1}\|$ so that we cannot use lemma 4.34 (ii) to conclude $n_0 = m_+ = 0$. However, since $|am| < \frac{1}{2}$, it follows from lemma 4.42 (i) that $n_0 = m_+$. By theorem 4.41 we still have

$$-0.25 \;\leq\; \sqrt{\nu_1} - |am| \leq\; \lambda_1 \leq \sqrt{\nu_1} + |am| \;\leq 1.28078 \tag{6.4}$$

where λ_1 is the analytic continuation of the first positive eigenvalue in the case $a = 0$ and ν_1 is the first eigenvalue of BB^* which we have estimated according to (4.50) of theorem 4.39. The lower bound can be further improved if we use $\|B^{-1}\|^{-1} = \sqrt{\nu_1}$ and observe that for the given physical parameters estimate (3.72 a) from lemma 3.34 yields a positive lower bound for $\|B^{-1}\|^{-1}$. Thus we obtain the sharper estimate

$$-0.13843 \leq\; -\sqrt{\nu_1} - |am| \leq\; \lambda_1 \leq \sqrt{\nu_1} + |am| \;\leq 1.28078.$$

(iii) Even a positive lower bound for λ_1 can be obtained by means of analytic perturbation theory if a is treated as the perturbation parameter. For $a = 0$ we have $\lambda_n = \operatorname{sign}(n)\big(|k+\frac{1}{2}|-\frac{1}{2}+n\big) = n$; hence for the given physical parameters we obtain from lemma 3.9 that

$$n - 0.75 \;\leq \lambda_n \;\leq\; n + 0.75, \qquad n \in \mathbb{Z}\setminus\{0\}. \tag{6.5}$$

In particular it follows that $0.25 \leq \lambda_1$. For all other values of n, however, the bounds $\lambda_n^{[l]}$ and $\lambda_n^{[u]}$ obtained from the more elaborate estimates in theorem 4.40 (where m plays the role of the perturbation parameter) yield tighter bounds than the formula above as can be seen in table 6.4.

Combining (6.4) and (6.5) we obtain $0.25 \;\leq\; \lambda_1 \;\leq\; 1.28078$. ◊

Remark 6.5. In some cases, the bounds can be further improved. For $am = 0.005$ and $a\omega = 0.015$ and $k \in \{-5, \dots, 4\}$ we have $\sigma(\mathcal{A}) \cap [-|am|, |am|\,] = \emptyset$ because of $|\lambda_{\pm 1}| \geq \lambda_1^{[l]} = \sqrt{\nu_1} - |am| > |am|$. Furthermore, $\|B^{-1}\|^{-1} = \sqrt{\nu_1} > |am|$ so that the assumption of lemma 3.38 is satisfied. Hence it follows:

(i) For $k = 0, \dots, 4$ we have $\big(-\|B^{-1}\|^{-1}, -|am|\big) \cap \sigma(\mathcal{A}) = \emptyset$ by lemma 3.38, hence

$$\lambda_{-1} \;\leq\; \|B^{-1}\|^{-1} \;=\; -\lambda_1^{[u]} - |am|.$$

(ii) For $k = -5, \dots, -1$ we have $\big(-\|B^{-1}\|^{-1}, -|am|\big) \cap \sigma(\mathcal{A}) = \emptyset$ by lemma 3.38, hence

$$\lambda_1 \;\geq\; \|B^{-1}\|^{-1} \;=\; \lambda_1^{[l]} + |am|.$$

Analogously, for $am = 0.25$, $a\omega = 0.75$ the upper bound for λ_{-1} can be improved if $k = 0, \dots, 4$ and the lower bound for λ_1 can be improved if $k = -5, \dots, -2$.
Note, however, that for $k = -1$ the assumptions of lemma 3.38 are not fulfilled. ◊

The discussion in remarks 6.4 and 6.5 shows that it is very hard to decide a priori which analytic bound gives the sharpest bound for the eigenvalues of $\mathcal{A}$. It seems that often a combination of the various estimates yields the best result.
It can be seen from the tables that in most cases the estimate $\lambda_1^{[l]}$ yields the sharpest lower bound. On the other hand, figures 6.1 and 6.2 suggest that for increasing am and $a\omega$ the estimate λ_Q provides a better lower bound for the smallest positive eigenvalue than $\lambda^{[l]}$ does.
In tables 6.2 and 6.3 we have also listed the numerical values, denoted by $\lambda_1^{[num]}$ and $\lambda_{-1}^{[num]}$, that

we have obtained by solving the continued fractions directly since they seem to be more reliable than the values originally given by the authors in [SFC83].
In figures 6.3 and 6.4 the numerical values $\lambda_{S,1}^{[num]}$ and $\lambda_{S,-1}^{[num]}$ together with the analytical bounds $\pm\lambda_G$ and $\pm\lambda_Q$ as functions of k are plotted.
Figures 6.5 and 6.6 show the upper and lower bounds $\lambda_1^{[l]}$ and $\lambda_1^{[u]}$ for the lowest eigenvalues as functions of the wave number k together with the numerical values $\lambda_{-1}^{[num]}$ and $\lambda_1^{[num]}$ from tables 6.2 and 6.3.

$am = 0.25,\quad a\omega = 0.75$								
	λ_G	λ_Q	$\lambda_1^{[l]}$	$\lambda_{S,1}^{[num]}$	$\lambda_1^{[num]}$	$\lambda_{S,-1}^{[num]}$	$\lambda_{-1}^{[num]}$	$\lambda_1^{[u]}$
$k = -5$	2.96570	3.75000	3.93330	4.29756	4.29756	-4.34936	-4.34936	4.61606
-4	2.15294	2.75000	2.91228	3.30870	3.30870	-3.37371	-3.37370	3.65037
-3	1.34019	1.75000	1.87132	2.32657	2.32658	-2.41349	-2.41348	2.71221
-2	0.52743	0.75000	0.75000	1.35984	1.35980	-1.48903	-1.48898	1.85078
-1	< 0	undefined	(0.25000)	0.44058	0.44025	-0.67315	-0.67284	(1.28078)
0	0.59808	1.22474	0.75000	1.59764	1.59745	-1.47645	-1.47627	1.85078
1	1.41083	2.25000	2.09521	2.65654	2.65651	-2.57663	-2.57661	2.90754
2	2.22359	3.25000	3.21410	3.68229	3.68229	-3.62219	-3.62218	3.93273
3	3.03634	4.25000	4.27769	4.69685	4.69684	-4.64856	-4.64856	4.94707
4	3.84910	5.25000	5.31776	5.70622	5.70622	-5.66583	-5.66583	5.95636

Table 6.2. Analytic bounds and numerical approximations for the first positive and first negative eigenvalue of $\mathcal{A}$. The estimates λ_G and λ_Q from theorems 3.35 and 5.10 are lower bounds for $|\lambda_{\pm 1}|$. $\lambda_1^{[l]}$ and $\lambda_1^{[u]}$ from theorem 4.40 are upper and lower bounds for $\lambda_{\pm 1}$. The values $\lambda_{S,1}^{[num]}$ and $\lambda_{S,-1}^{[num]}$ for the first positive and the first negative eigenvalue of $\mathcal{A}$ are taken from [SFC83]. We have obtained the numerical values $\lambda_1^{[num]}$ and $\lambda_{-1}^{[num]}$ by approximating a solution of the continued fraction equation for λ. Note that for $k = 0, \dots, 4$ the upper bound for λ_{-1} can be further improved, while for $k = -2, \dots, -5$ the lower bound for λ_1 can be improved, see remark 6.5. For $k = -1$ see the discussion in remark 6.4.

$am = 0.005,\quad a\omega = 0.015$								
	λ_G	λ_Q	$\lambda_1^{[l]}$	$\lambda_{S,1}^{[num]}$	$\lambda_1^{[num]}$	$\lambda_{S,-1}^{[num]}$	$\lambda_{-1}^{[num]}$	$\lambda_1^{[u]}$
$k = -5$	4.46556	4.48500	4.97998	4.98591	4.98591	-4.98682	-4.98682	4.99299
-4	3.46969	3.48500	3.97997	3.98611	3.98611	-3.98723	-3.98723	3.99373
-3	2.47383	2.48500	2.97996	2.98643	2.98643	-2.98786	-2.98786	2.99498
-2	1.47797	1.48500	1.97994	1.98700	1.98700	-1.98901	-1.98901	1.99749
-1	0.48211	0.48500	0.97989	0.98834	0.98834	-0.99170	-0.99170	1.00500
0	0.50376	0.51500	0.99500	1.01167	1.01167	-1.00836	-1.00836	1.01989
1	1.49962	1.51500	2.00249	2.01300	2.01300	-2.01101	-2.01101	2.01994
2	2.49548	2.51500	3.00498	3.01357	3.01357	-3.01215	-3.01215	3.01996
3	3.49134	3.51500	4.00623	4.01389	4.01389	-4.01278	-4.01278	4.01997
4	4.48720	4.51500	5.00699	5.01409	5.01409	-5.01318	-5.01318	5.01998

Table 6.3. Analytic bounds and numerical approximations for the first positive and first negative eigenvalue of $\mathcal{A}$. The estimates λ_G and λ_Q from theorems 3.35 and 5.10 are lower bounds for $|\lambda_{\pm 1}|$. $\lambda_1^{[l]}$ and $\lambda_1^{[u]}$ obtained in theorem 4.40 are upper and lower bounds for $\lambda_{\pm 1}$. The values $\lambda_{S,1}^{[num]}$ and $\lambda_{S,-1}^{[num]}$ are the first positive and the first negative eigenvalue of $\mathcal{A}$ calculated numerically by Suffern et al. [SFC83], while we have obtained the values $\lambda_1^{[num]}$ and $\lambda_{-1}^{[num]}$ by approximating a solution of the continued fraction equation for λ. Note that for $k = 0, \dots, 4$ the upper bound for λ_{-1} can be further improved, while for $k = -1, \dots, -5$ the lower bound for λ_1 can be improved, see remark 6.5.

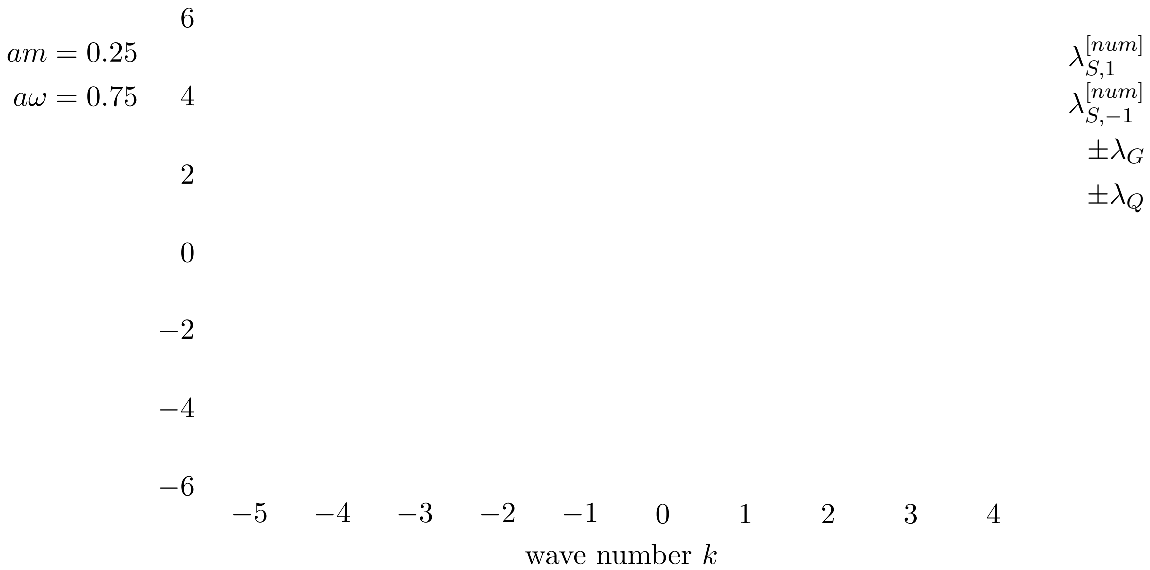

Figure 6.3. The plot shows the lower bounds λ_G and λ_Q for the absolute value of the eigenvalues of $\mathcal{A}$ and the numerical values for the first positive and the first negative eigenvalue from [SFC83] in the case $am = 0.25$ and $a\omega = 0.75$. The bound λ_Q is not defined for $k \in (-1.25, 0)$. For $-1.35 \lesssim k \leq -0.5$ the bound λ_G is negative, so it is replaced by zero in this interval.

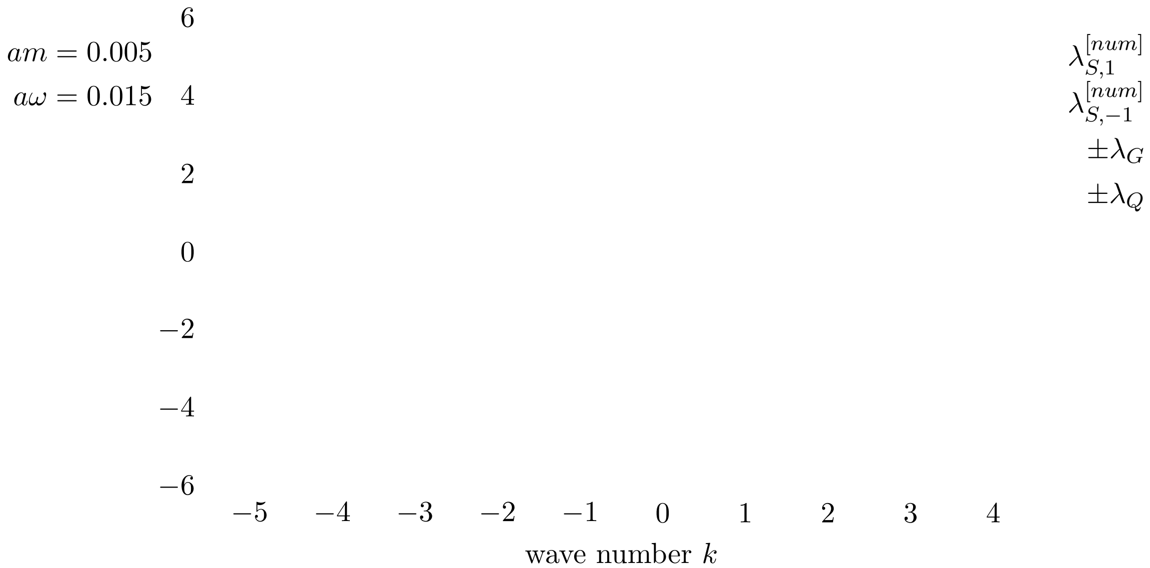

Figure 6.4. The plot shows the lower bounds λ_G and λ_Q for the absolute value of the eigenvalues of $\mathcal{A}$ and the numerical values for the first positive and first negative eigenvalue of $\mathcal{A}$ from [SFC83] in the case $am = 0.005$ and $a\omega = 0.015$. The bounds λ_G and λ_Q are very close to each other so that they seem to coincide in the plot above. The bounds have not been plotted in the interval $(-1, 0)$ because for wave numbers k in that interval the angular operator is not uniquely defined as a selfadjoint operator.

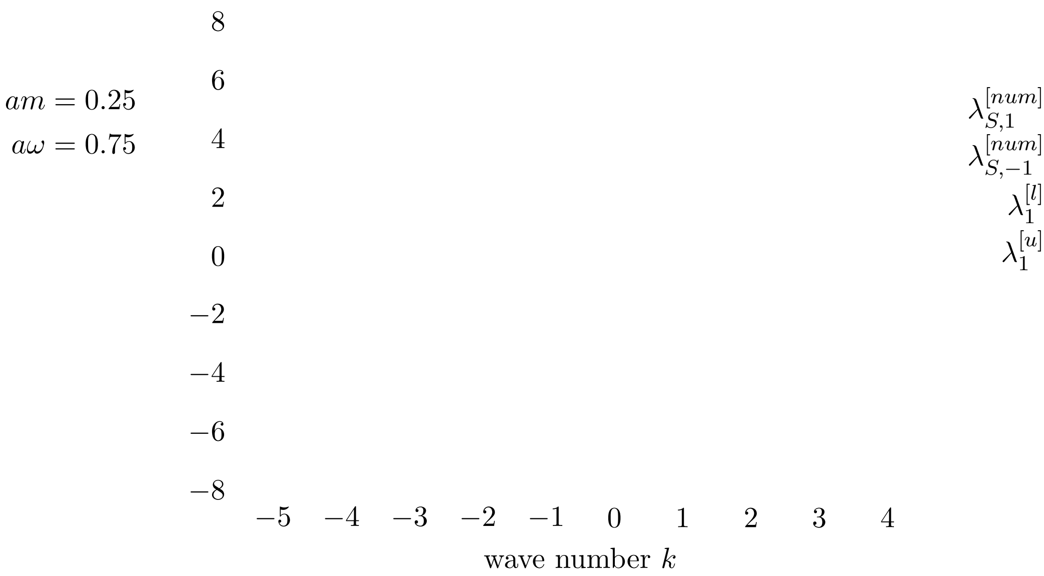

Figure 6.5. The plot shows the numerical values $\lambda_{S,\pm 1}^{[num]}$ for the first positive and the first negative eigenvalue of [SFC83] for $am = 0.25$ and $a\omega = 0.75$. They are enclosed by the analytic upper and lower bounds $\pm\lambda_1^{[u]}$ and $\pm\lambda_1^{[l]}$ which are plotted as functions of k. Note that for $-1.65 \lesssim k \lesssim -0.23$ the estimate for $\|B^{-1}\|^{-1}$ obtained from theorem 4.39 is not large enough to guarantee $n_0 = 0$ by lemma 4.34 (ii), so the analytic bounds are plotted only for $-1.65 \lesssim k$ and $k \geq 0$.

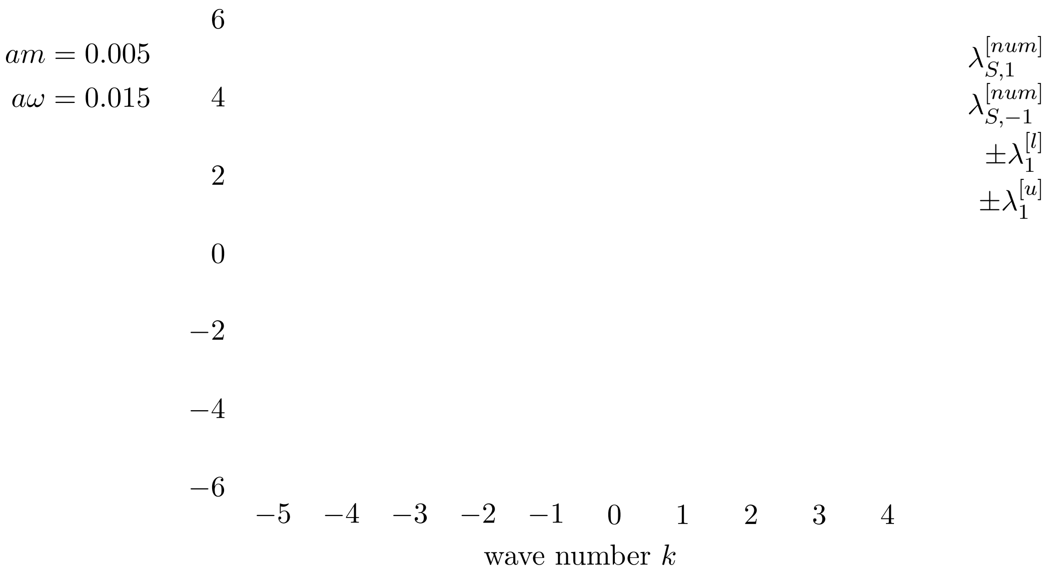

Figure 6.6. The plot shows the numerical values $\lambda_{S,\pm 1}^{[num]}$ of [SFC83] for the first positive and the first negative eigenvalue of $\mathcal{A}$ for $am = 0.005$ and $a\omega = 0.015$. Here the analytic upper and lower bounds $\pm\lambda_1^{[u]}$ and $\pm\lambda_1^{[l]}$ enclose the numerical values so tightly that in this resolution they seem to coincide.

Finally, we compare higher eigenvalues $\lambda_{S,n}^{[num]}$ given in [SFC83] with the analytic bounds $\lambda_n^{[u]}$ and $\lambda_n^{[l]}$. In tables 6.5 and 6.4 we have listed the analytical bounds $\lambda_n^{[l]}$ and $\lambda_n^{[u]}$ for $n = 1, \ldots, 5$ with the numerically obtained eigenvalues in the cases $am = 0.005$, $a\omega = 0.015$ and $am = 0.25$, $a\omega = 0.75$ where the wave number $k \in \{-1, 0\}$ is fixed. The numerical results $\lambda_{S,n}^{[num]}$ and $\lambda_{S,-n}^{[num]}$ are taken from [SFC83], the numerical values $\lambda_n^{[num]}$ and $\lambda_{-n}^{[num]}$ have been obtained by solving the continued fractions equation for λ. The data of these tables are visualised in figures 6.8 and 6.7.
For the case $am = 0.25$, $a\omega = 0.75$, $k = -1$ we refer to remarks 6.4 and 6.5.

$am = 0.25, \quad a\omega = 0.75$						
$k = 0$	$\lambda_n^{[l]}$	$\lambda_{S,n}^{[num]}$	$\lambda_n^{[num]}$	$\lambda_{S,-n}^{[num]}$	$\lambda_{-n}^{[num]}$	$\lambda_n^{[u]}$
$n = 1$	0.75000	1.59764	1.59745	-1.47645	-1.47627	1.85078
2	1.75000	2.22587	2.28748	-2.23549	-2.24221	2.60850
3	2.75000	3.17408	3.18894	-3.16265	-3.16640	3.50000
4	3.75000	4.13127	4.13969	-4.12446	-4.12658	4.44076
5	4.75000	5.10533	5.11069	-5.10083	-5.10217	5.40388
$k = -1$	$\lambda_n^{[l]}$	$\lambda_{S,n}^{[num]}$	$\lambda_n^{[num]}$	$\lambda_{S,-n}^{[num]}$	$\lambda_{-n}^{[num]}$	$\lambda_n^{[u]}$
$n = 1$	(0.25000)	0.44058	0.44025	-0.67315	-0.67284	(1.28078)
2	1.33114	1.84225	1.88562	-1.87948	-1.89090	2.26556
3	2.48861	2.90717	2.92395	-2.92301	-2.92728	3.26040
4	3.55789	3.93475	3.94370	-3.94336	-3.94562	4.25780
5	4.59768	4.94973	4.95529	-4.95513	-4.95653	5.25625

Table 6.4. For $am = 0.25$, $a\omega = 0.75$ and $k = 0, -1$ the numerical values $\lambda_{S,-n}^{[num]}$ and $\lambda_{S,n}^{[num]}$ and the lower and upper bounds $\lambda_n^{[l]}$ and $\lambda_n^{[u]}$ from theorem 4.40 are shown. For $k = -1$, $n = 1$, we refer to remark 6.4.

$am = 0.005, \quad a\omega = 0.015$						
$k = 0$	$\lambda_n^{[l]}$	$\lambda_{S,n}^{[num]}$	$\lambda_n^{[num]}$	$\lambda_{S,-n}^{[num]}$	$\lambda_{-n}^{[num]}$	$\lambda_n^{[u]}$
$n = 1$	0.99500	1.01167	1.01167	-1.00836	-1.00836	1.01989
2	1.99500	2.00435	2.00437	-2.00369	-2.00369	2.01249
3	2.99500	3.00273	3.00274	-3.00245	-3.00245	3.01000
4	3.99500	4.00180	4.00200	-4.00184	-4.00184	4.00875
5	4.99500	5.00158	5.00158	-5.00148	-5.00148	5.00800
$k = -1$	$\lambda_n^{[l]}$	$\lambda_{S,n}^{[num]}$	$\lambda_n^{[num]}$	$\lambda_{S,-n}^{[num]}$	$\lambda_{-n}^{[num]}$	$\lambda_n^{[u]}$
$n = 1$	0.97989	0.98834	0.98834	-0.99170	-0.99170	1.00500
2	1.98749	1.99567	1.99570	-1.99636	-1.99636	2.00500
3	2.99000	2.99730	2.99731	-2.99759	-2.99759	3.00500
4	3.99125	3.99803	3.99803	-3.99819	-3.99819	4.00500
5	4.99200	4.99845	4.99845	-4.99855	-4.99855	5.00500

Table 6.5. For $am = 0.015$, $a\omega = 0.025$ and $k = 0, -1$ the numerical values $\lambda_{S,-n}^{[num]}$ and $\lambda_{S,n}^{[num]}$ and the lower and upper bounds $\lambda_n^{[l]}$ and $\lambda_n^{[u]}$ from theorem 4.40 are shown.

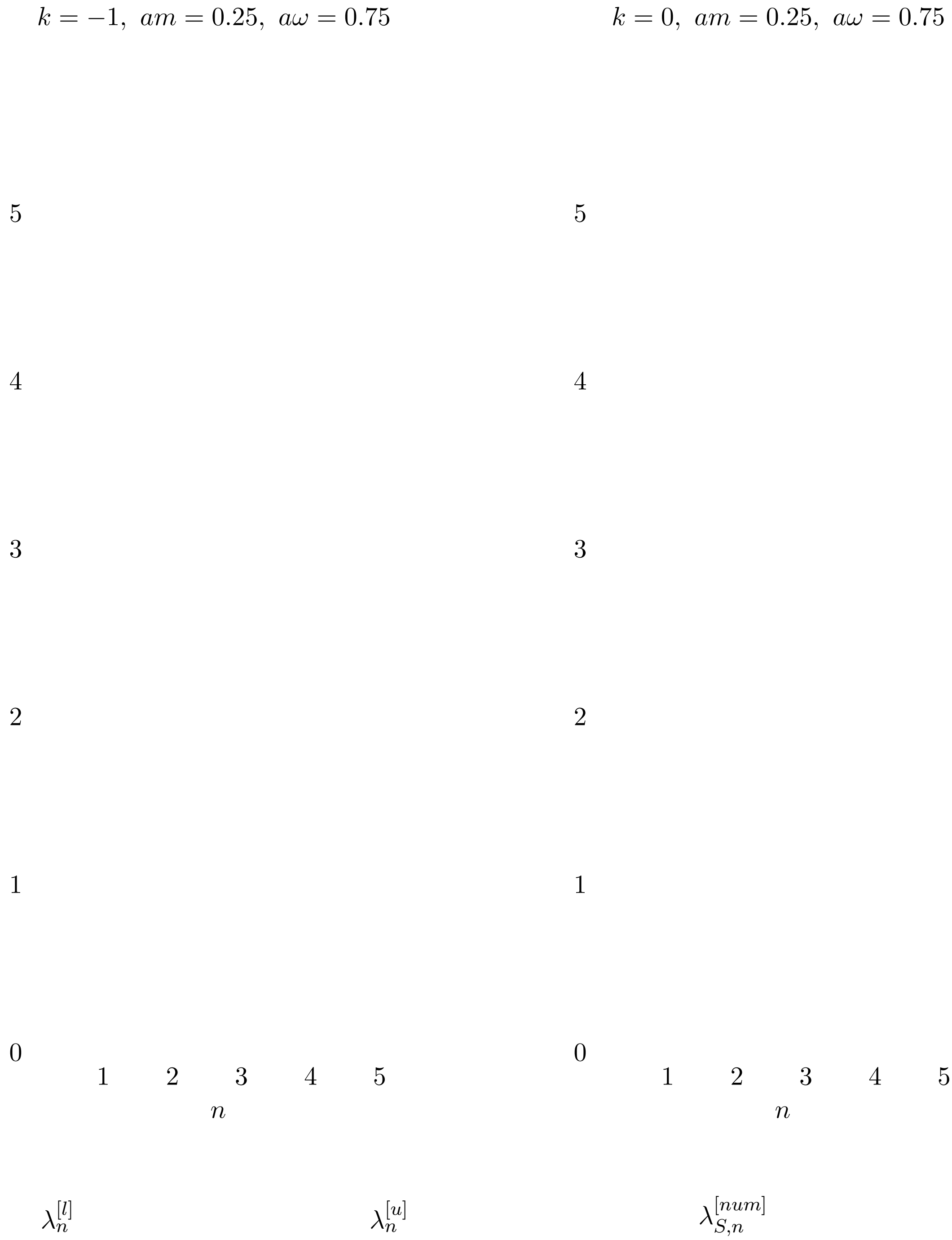

Figure 6.7. Higher eigenvalues. For $am = 0.25$ and $a\omega = 0.75$, the plots show the numerical values $\lambda_{S,-n}^{[num]}$, $n = 1, \ldots, 4$ of [SFC83] and the analytic bounds $\lambda^{[l]}$ and $\lambda^{[u]}$ provided by theorem 4.40 as functions of n for the wave numbers $k = -1$ (left plot) and $k = 0$ (right plot), see table 6.4.

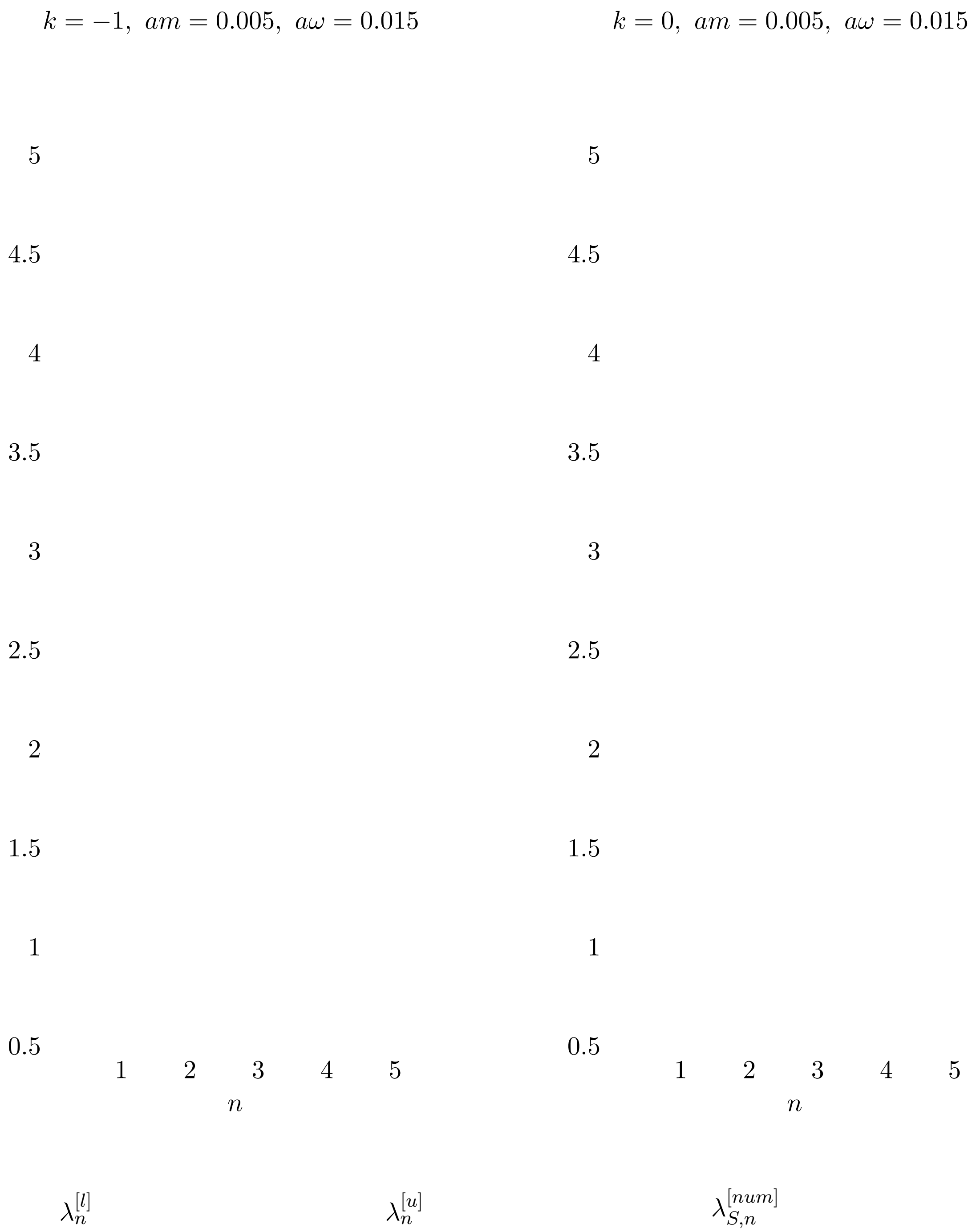

Figure 6.8. Higher eigenvalues. For $am = 0.005$ and $a\omega = 0.015$, the plots show the numerical values $\lambda_{S,-n}^{[num]}$, $n = 1, \ldots, 4$, of [SFC83] and the analytic bounds $\lambda^{[u]}$ and $\lambda^{[l]}$ provided by theorem 4.40 as functions of n in the case $k = -1$ (left plot) and $k = 0$ (right plot), see table 6.5. The analytical bounds $\lambda_n^{[l]}$ and $\lambda_n^{[u]}$ are so close to the numerical values $\lambda_{S,-n}^{[num]}$ that they seem to coincide in this resolution.

Comparison with numerical values in [Cha84]

In [Cha84], Chakrabarti computed numerical values for the squares of the eigenvalues λ of the angular operator $\mathcal{A}$. Observe that, due to the form of the angular equation, the eigenvalues $\lambda_C^{[num]}$ given by Chakrabarti and the eigenvalues obtained in this work differ by sign, i.e., eigenvalues $\lambda_{C,n}^{[num]}$ for fixed a, m, ω and k correspond to $-\lambda_n$ in this work.
The author expands the solution of the angular equation in a series of spin-weighted spherical harmonics which leads to an expansion of λ in terms of $a\omega$ and $\frac{m}{\omega}$. In the article the two smallest positive eigenvalues in the case $k \in \{0, -1\}$ and the smallest eigenvalue in the case $k \in \{-2, 1\}$ are calculated for $a\omega = 0.1,\ 0.2,\ \dots,\ 1.0$ and $\frac{m}{\omega} = 0,\ 0.2,\ \dots, 0.8,\ 1.0$. The resulting numerical values for λ are claimed to be reliable up to at least four digits.

For $a\omega = 0.2$ and $a\omega = 1.0$ the square roots of the original values $\lambda_C^{[num]}$ of [Cha84] are presented in tables 6.6 and 6.7, together with the numerical values $\lambda^{[num]}$ that have been obtained by solving the continued fractions equation for λ (see the preceding discussion concerning the numerical values in [SFC83]).

For $k = 0$ and $a\omega > 0$, we have $\Omega_- = 2a\omega(k + \frac{1}{2}) - |a\omega| = 0$. Hence it follows from theorem 4.39 for $k = 0$ and all parameters am and $a\omega$ under consideration that

$$\|B^{-1}\|^{-1} \;=\; \sqrt{\nu_1} \;\geq\; \sqrt{(|k+\tfrac{1}{2}| - \tfrac{1}{2} + 1)^2 + \Omega_-} \;=\; 1.$$

In table 6.6 we have $|am| \leq 0.2$ so that $n_0 = m_+ = 0$ by lemma 4.34 (ii).
For table 6.6 this conclusion holds for $|am| \leq 0.2$ only. However, for all am of the table it follows from theorem 4.41 that $\lambda_1 \geq \|B^{-1}\|^{-1} - |am| \geq 0$ and that $\lambda_{-1} \leq -\|B^{-1}\|^{-1} - |am| \leq 0$. Moreover, theorem 5.10 implies that $(-\lambda_Q, \lambda_Q) \cap \sigma(\mathcal{A}) = \emptyset$ with $\lambda_Q = \sqrt{a\omega(k + \frac{1}{2})} \approx 1.41121 > |am|$. Hence for all am under consideration the first positive eigenvalue is greater than $|am|$ and it is the analytic continuation of the first positive eigenvalue in the case $a = 0$. Although we could show that $m_+ = 0$ we cannot prove that also $n_0 = 0$; therefore we use the bounds $\lambda^{[l,\mathrm{SPT}]}$ and $\lambda^{[u,\mathrm{SPT}]}$ in table 6.7 for $a\omega = 1.0$ since there is no need to determine the index shift n_0 for these bounds. Note that in table 6.6 we have $\lambda_n^{[l]} = \lambda_n^{[l,\mathrm{SPT}]}$ and $\lambda_n^{[u]} = \lambda_n^{[u,\mathrm{SPT}]}$.

From the tables and from their graphical representation in figure 6.8 it can be seen that Chakrabarti's values are still within the analytical bounds. However, they differ significantly from the numerical values that we have calculated with the help of the recursion formula for the eigenvalues given in [SFC83]; for example, for $am = a\omega = 1.0$, the values differ even in the leading digit.
It should be mentioned that in the case $am = a\omega$ the numerical values for λ have been computed according to an exact formula, see [SFC83].

$k = 0, \quad a\omega = 0.2$								
	$n=1$				$n=2$			
am	$\lambda_1^{[l]}$	$\lambda_{C,1}^{[num]}$	$\lambda_1^{[num]}$	$\lambda_1^{[u]}$	$\lambda_2^{[l]}$	$\lambda_{C,2}^{[num]}$	$\lambda_2^{[num]}$	$\lambda_2^{[u]}$
0.00	1.000000	1.136135	1.136116	1.183216	2.000000	2.058001	2.058002	2.097618
0.04	0.960000	1.124034	1.148441	1.223216	1.960000	2.055153	2.061228	2.137618
0.08	0.920000	1.112156	1.160994	1.263216	1.920000	2.052685	2.064837	2.177618
0.12	0.880000	1.100499	1.173773	1.303216	1.880000	2.050594	2.068830	2.217618
0.16	0.840000	1.089061	1.186776	1.343216	1.840000	2.048877	2.073209	2.257618
0.20	0.800000	1.077840	1.200000	1.383216	1.800000	2.047530	2.077973	2.297618

Table 6.6. The table shows the analytic bounds $\lambda^{[l]}$ and $\lambda^{[u]}$ for the two lowest positive eigenvalues and the numerical value $\lambda^{[num]}$ obtained by solving the continued fractions relation for λ. The numerical values $\lambda_C^{[num]}$ are taken from [Cha84].

$k = 0, \quad a\omega = 1.0$								
	$n=1$				$n=2$			
am	$\lambda_1^{[l,\mathrm{SPT}]}$	$\lambda_{C,1}^{[num]}$	$\lambda_1^{[num]}$	$\lambda_1^{[u,\mathrm{SPT}]}$	$\lambda_2^{[l,\mathrm{SPT}]}$	$\lambda_{C,2}^{[num]}$	$\lambda_2^{[num]}$	$\lambda_2^{[u,\mathrm{SPT}]}$
0.00	1.000000	1.720028	1.720243	1.802776	2.000000	2.362204	2.364111	2.500000
0.20	0.800000	1.677631	1.766714	2.002776	1.800000	2.347623	2.386963	2.700000
0.40	0.600000	1.639603	1.818185	2.202776	1.600000	2.341359	2.418308	2.900000
0.60	0.400000	1.605706	1.874453	2.402776	1.400000	2.343150	2.458034	3.100000
0.80	0.200000	1.575667	1.935199	2.602776	1.200000	2.352702	2.505900	3.300000
1.00	0.000000	1.549193	2.000000	2.802776	1.000000	2.369680	2.561553	3.500000

Table 6.7. The table shows the analytic bounds $\lambda^{[l,\mathrm{SPT}]}$ and $\lambda^{[u,\mathrm{SPT}]}$ for the two lowest positive eigenvalues and the numerical value $\lambda^{[num]}$ obtained by solving the continued fractions relation for λ. The numerical values $\lambda_C^{[num]}$ are taken from [Cha84].

$k = 0, \quad a\omega = 0.2$

$n = 1$

3

2

1

0

0 0.00 0.08 0.12 0.16 0.20

am

$n = 2$

3

2

1

0

0 0.00 0.08 0.12 0.16 0.20

am

$k = 0, \quad a\omega = 1.0$

$n = 1$

3

2

1

0

0 0.2 0.4 0.6 0.8 1.0

am

$n = 2$

3

2

1

0

0 0.2 0.4 0.6 0.8 1.0

am

λ_G $\quad \lambda_n^{[l,\mathrm{SPT}]},\ n = 1, 2$ $\quad \lambda_n^{[num]},\ n = 1, 2$

λ_Q $\quad \lambda_n^{[u,\mathrm{SPT}]},\ n = 1, 2$ $\quad \lambda_{C,n}^{[num]},\ n = 1, 2$

Table 6.8. The plots show the numerical values $\lambda_{C,n}^{[num]}$, $n = 1, 2$, given in [Cha84] and the numerical values $\lambda_n^{[num]}$ obtained by solving the continued fractions relation for $k = 0$ and $a\omega = 0.2$ and $a\omega = 1.0$, respectively. In addition, the analytical bounds $\lambda_n^{[l,\mathrm{SPT}]}$ and $\lambda_n^{[u,\mathrm{SPT}]}$ are plotted. Since λ_G is a lower bound for the modulus of eigenvalues of $\mathcal{A}$, it is displayed only where it is nonnegativ. For increasing $|am|$, the bound λ_Q becomes the sharpest lower bound of all analytic bounds under consideration.

Appendix A

The variational principle of [EL04]

The results in chapter 4 are based on a general variational principle for selfadjoint operator functions proved by Eschwé and Langer in [EL04]. For convenience, we state here their main theorem.

First, let us fix the notation used in the theorem. Let $\mathcal{H}$ be a Hilbert space, $\Delta \subseteq \mathbb{R}$ an open, half-open or closed interval with endpoints $-\infty \leq \alpha < \beta \leq \infty$ and

$$S : \Delta \longrightarrow \mathscr{L}(\mathcal{H})$$

a selfadjoint operator function, i.e., for every $\lambda \in \Delta$, the operator $S(\lambda)$ with domain $\mathcal{D}(S(\lambda))$ is a selfadjoint operator in the Hilbert space $\mathcal{H}$. Let

$$\lambda_e \;:=\; \begin{cases} \inf \sigma_{ess}(S) & \text{if} \quad \sigma_{ess}(S) \neq \emptyset, \\ \beta & \text{if} \quad \sigma_{ess}(S) = \emptyset \end{cases}$$

and define $\Delta' := \{\lambda \in \Delta \,:\, \lambda < \lambda_e\}$. To the operator valued function S we associate a function $\mathfrak{s}$ with values in the sesquilinear forms on $\mathcal{H}$

$$\mathfrak{s}(\lambda)[u, v] \;:=\; (u,\, S(\lambda)v), \qquad \mathcal{D}(\mathfrak{s}(\lambda)) \;:=\; \mathcal{D}(S(\lambda)).$$

In theorem A.1, the following conditions are used.

(i) Either $\mathcal{D}(S(\lambda))$ does not depend on λ or for all $\lambda \in \Delta$ the form $\mathfrak{s}(\lambda)$ is closable with closure $\widetilde{\mathfrak{s}}(\lambda)$ and there exists a linear manifold $\mathcal{D}$ of $\mathcal{H}$ such that

$$\mathcal{D}(S(\lambda)) \;\subseteq\; \mathcal{D} \;\subseteq\; \mathcal{D}(\widetilde{\mathfrak{s}}(\lambda)), \qquad \lambda \in \Delta.$$

If condition (i) holds, then, for fixed $x \in \mathcal{D}$, we define

$$\sigma^x : \Delta \longrightarrow \mathbb{R}, \qquad \sigma^x(\lambda) \;:=\; \widetilde{\mathfrak{s}}(\lambda)[x].$$

(ii) The function S is continuous in the generalised sense, i.e., in norm resolvent topology. Further, the function σ^x is continuous for all $x \in \mathcal{D}$.

(iii) For every $x \in \mathcal{D} \setminus \{0\}$, the function σ^x is decreasing at value zero on Δ.

(iv) There exists a $\lambda_0 \in \Delta$ such that $\dim \mathcal{L}_{(-\infty,0)} S(\lambda_0) < \infty$.

If conditions (i)–(iii) hold, then, for fixed $x \in \mathcal{D}$, there is at most one $p(x) \in \Delta$ with $\sigma^x(p(x)) = 0$. If σ^x has no zero, we set

$$p(x) := \begin{cases} \infty & \text{if } \sigma^x(\lambda) > 0 \text{ for all } \lambda \in \Delta, \\ -\infty & \text{if } \sigma^x(\lambda) < 0 \text{ for all } \lambda \in \Delta. \end{cases}$$

For $\lambda \in \Delta$, let $n(\lambda) := \dim \mathcal{L}_{(-\infty,0)} S(\lambda)$ and $\mathcal{N}_\lambda := \{x \in \mathcal{D} : \mathfrak{s}(\lambda)[x] < 0\} \cup \{0\}$. It can be shown ([EL04, lemma 2.5]) that $n(\lambda)$ is equal to the dimension of any maximal subspace of $\mathcal{N}_\lambda$.
If $\beta \notin \Delta$, we define $\mathcal{N}_\beta := \bigcup_{\lambda \in \Delta} \mathcal{N}_\lambda$ and denote by $n(\beta)$ the dimension of maximal subspaces of $\mathcal{N}_\beta$.
For $n \in \mathbb{N}$ set

$$\mu_n := \inf_{\substack{L \subseteq \mathcal{D} \\ \dim L = n}} \sup_{x \in L^\times} p(x).$$

As usually let $L^\times = L \setminus \{0\}$. Finally, for a supplementary result, the following condition is used.

(v) For every $\lambda \in \Delta$ and $\varepsilon > 0$ with $\lambda + \varepsilon < \beta$, there exists a $\delta = \delta(\lambda, \varepsilon) > 0$ such that we have the implication

$$0 < \sigma^x(\lambda) < \delta \quad \text{for an } x \in \mathcal{D} \text{ with } \|x\| = 1 \qquad \Longrightarrow \qquad \sigma^x(\lambda + \varepsilon) < 0.$$

Theorem A.1 ([EL04, theorem 2.1]). *Assume that the conditions (i)–(iv) hold and suppose that Δ' is not empty. If $\alpha \in \Delta'$, then we set $n_0 := n(\alpha)$; otherwise there exists an $\alpha' \in \Delta'$ such that $(\alpha, \alpha') \subseteq \rho(S)$ and we set $n_0 := n(\alpha')$. In both cases, n_0 is a finite number.*
The spectrum of S in Δ', $\sigma(S) \cap \Delta'$, consists only of a finite or infinite sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$, counted according to their multiplicity, with $N \in \mathbb{N} \cup \{\infty\}$ given by

$$N = \begin{cases} n(\beta) - n_0 + \dim\ker S(\beta) & \text{if } \beta \in \Delta \text{ and } \sigma_{ess}(S) = \emptyset, \\ n(\lambda_e) - n_0 & \text{otherwise.} \end{cases}$$

Then the eigenvalues λ_n of S in Δ' are given by

$$\lambda_n = \mu_{n+n_0} = \min_{\substack{L \subseteq \mathcal{D} \\ \dim L = n+n_0}} \max_{x \in L^\times} p(x).$$

If $N = \infty$, then $\lim_{n\to\infty} \lambda_n = \lambda_e$.
If $N < \infty$ and $\sigma_{ess}(S) = \emptyset$, then $\mu_n = \infty$ for $n > n_0 + N$.
If $N < \infty$, $\lambda_e < \beta$ and assumption (v) is fulfilled, then $\mu_n = \lambda_e$ for $n > n_0 + N$.

Appendix B

The Schur complements of $\mathcal{A}$

In section 4.1 we gave formulae for the eigenvalues λ in some right half plane of selfadjoint block operator matrices $\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}$ with $\mathcal{D}(\mathcal{T}) = \mathcal{D}(T_{12}^*) \oplus \mathcal{D}(T_{12}) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ satisfying certain assumptions on the entries T_{ij}. The main tool was to associate with $\mathcal{T}$ the operator valued function S_1, the so-called Schur complement, such that the spectrum of S_1 and that of $\mathcal{T}$ coincide in some right half plane. To the function S_1 we then applied the variational principle [EL04, theorem 2.1]. To obtain the Schur complement, we have first introduced the minimal Schur complement

$$\begin{aligned} \mathcal{D}(S_1^{[\min]}(\lambda)) &= \{x \in \mathcal{D}(T_{12}^*) : (T_{22} - \lambda)^{-1} T_{12}^* x \in \mathcal{D}(T_{12})\}, \\ S_1^{[\min]}(\lambda) x &= \big(T_{11} - \lambda - T_{12}(T_{22} - \lambda)^{-1} T_{12}^*\big) x \end{aligned}$$

for $\lambda \in (c_2, \infty) \subseteq \rho(T_{22})$ and then used the theory of sesquilinear forms to construct selfadjoint extensions $S_1(\lambda)$, the so-called Friedrichs extensions.

Observe that in the special case of the angular operator the above mentioned construction leads to uniquely defined selfadjoint Schur complements $S_1(\lambda)$, $\lambda \in (|am|, \infty)$, for all wave numbers $k \in \mathbb{R}$ although in section 2.1.2 we have shown that the minimal angular operator $\mathcal{A}^{\min}$ with domain $\mathcal{D}(\mathcal{A}^{\min}) = \mathcal{C}_0^\infty(0,\pi)^2$ is essentially selfadjoint only for $k \in \mathbb{R} \setminus (-1, 0)$.

In this section we investigate the Schur complement of $\mathcal{A}$ from the point of view of spectral theory of differential operators. To this end, we consider the formal differential expression $\mathfrak{S}_1(\lambda)$ on $(0,\pi)$ associated with the Schur complement; for $\lambda > |am|$ it is defined by

$$\begin{aligned} \mathfrak{S}_1(\lambda) &:= -\mathfrak{D} - \lambda - \mathfrak{B}_+ (\mathfrak{D} - \lambda)^{-1} \mathfrak{B}_- \\ &= -am\cos\vartheta - \lambda - \Big(\tfrac{\mathrm{d}}{\mathrm{d}\vartheta} + \tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)(am\cos\vartheta - \lambda)^{-1}\Big(-\tfrac{\mathrm{d}}{\mathrm{d}\vartheta} + \tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big) \end{aligned} \tag{B.1}$$

where $\mathfrak{D}$ is the formal operator of multiplication by $am\cos\vartheta$ on $(0,\pi)$. To $\mathfrak{S}_1(\lambda)$ we associate the following operators and forms that are minimal from the point of view of the theory for differential operators:

$$\mathcal{D}(S_1^{\min}(\lambda)) := \mathcal{C}_0^\infty(0,\pi), \qquad S_1^{\min}(\lambda) f := \mathfrak{S}_1(\lambda) f, \tag{B.2}$$

$$\begin{aligned} \mathcal{D}(\mathfrak{s}_1^{\min}(\lambda)) := \mathcal{C}_0^\infty(0,\pi), \qquad \mathfrak{s}_1^{\min}(\lambda)[f,g] &:= \big(f,\, S_1^{\min} g\big) \\ &= \big(f,\, (-D-\lambda)g\big) - \big(B^* f,\, (D-\lambda)^{-1} B^* g\big). \end{aligned} \tag{B.3}$$

Observe that $S_1^{\min}(\lambda) \subsetneq S_1^{[\min]}(\lambda)$. In lemma B.4 we show that the Friedrichs extensions of both operators are equal. From the theory of linear differential operators it is well known that all selfadjoint extensions of $S_1^{\min}(\lambda)$ are given by restrictions of the maximal operator $S_1^{\max}(\lambda) =$

$S_1^{\min}(\lambda)^*$ in terms of boundary conditions. In the second part of this chapter we identify the boundary conditions that correspond to the Friedrichs extension of $S_1^{\min}(\lambda)$. We show that for $k \in \mathbb{R} \setminus (-2,1)$ there is no need to specify boundary conditions since in this case $S_1^{\min}(\lambda)$ is essentially selfadjoint. For $k \in (-2,-1] \cup [0,1)$, one boundary condition is necessary while for $k \in (-1,0)$ we need two coupled boundary conditions to obtain the Schur complement $S_1(\lambda)$, cf. lemma B.7.

The minimal Schur complement $S_1^{\min}$

The formal differential expression $\mathfrak{S}_1$ associated with the Schur complement of $\mathcal{A}$ has already been given in (B.1). In the subsequent calculations, however, we often use the following differential expression, defined for $\lambda > |am|$ on the interval $(0,\pi)$:

$$\begin{aligned} \mathfrak{T}_1(\lambda) &:= \mathfrak{S}_1(\lambda) + \mathfrak{D} + \lambda = -\mathfrak{B}_+(\mathfrak{D}-\lambda)\mathfrak{B}_- \\ &= -\Big(-\tfrac{\mathrm{d}}{\mathrm{d}\vartheta} + \tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)(am\cos\vartheta - \lambda)^{-1}\Big(\tfrac{\mathrm{d}}{\mathrm{d}\vartheta} + \tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big) \end{aligned} \tag{B.4}$$

with the associated minimal operator and sesquilinear form

$$\mathcal{D}(T_1^{\min}(\lambda)) := \mathcal{C}_0^\infty(0,\pi), \qquad T_1^{\min}(\lambda)f := \mathfrak{T}_1(\lambda)f = S_1^{\min}(\lambda)f + (D+\lambda)f, \tag{B.5}$$

$$\begin{aligned} \mathcal{D}(\mathfrak{t}_1^{\min}(\lambda)) := \mathcal{C}_0^\infty(0,\pi), \qquad \mathfrak{t}_1^{\min}(\lambda)[f,g] &:= -\big(B^*f,\,(D-\lambda)^{-1}B^*g\big) \\ &= \mathfrak{s}_1^{\min}(\lambda)[f,g] + \big(f,\,(D+\lambda)g\big). \end{aligned} \tag{B.6}$$

The operators $S_1^{\min}$ and $T_1^{\min}$ are symmetric and their adjoint operators are the maximal operators

$$\begin{aligned} \mathcal{D}(S_1^{\max}(\lambda)) &= \mathcal{D}(T_1^{\max}(\lambda)) \\ &= \{f \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) \;:\; f,\, f' \text{ absolutely continuous},\ \mathfrak{S}_1(\lambda)f \in \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)\}, \end{aligned}$$

$$S_1^{\max}(\lambda)f = \mathfrak{S}_1(\lambda)f, \qquad T_1^{\max}(\lambda)f = \mathfrak{T}_1(\lambda)f,$$

see [Wei87, chap. 3].
Obviously, we have $S_1^{\min}(\lambda) \subseteq S_1^{[\min]}(\lambda) \subseteq S_1(\lambda)$ and $T_1^{\min}(\lambda) \subseteq T_1^{[\min]}(\lambda) \subseteq T_1(\lambda)$, $\lambda > |am|$. In the following we study the question whether the minimal operator $S_1^{\min}(\lambda)$ is already essentially selfadjoint. Since the operator $-(D+\lambda)$ is bounded, it suffices to consider the operator $T_1^{\min}(\lambda)$ instead of $S_1^{\min}(\lambda)$.

Remark B.1. The formal differential expressions $\mathfrak{S}_1(\lambda)$ and $\mathfrak{T}_1(\lambda)$ are of the form (2.13) with

$$r(\vartheta) = 1, \qquad p_1(\vartheta) = (am\cos\vartheta - \lambda)^{-1},$$

$$p_0(\vartheta) = \tfrac{\mathrm{d}}{\mathrm{d}\vartheta}\Big((am\cos\vartheta-\lambda)\Big(\tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)\Big) + (am\cos\vartheta-\lambda)\Big(\tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)^2 + \widetilde{p}_0(\vartheta),$$

where $\widetilde{p}_0(\vartheta) = 0$ for $\mathfrak{T}_1(\lambda)$ and $\widetilde{p}_0(\vartheta) = -am\cos\vartheta - \lambda$ for $\mathfrak{S}_1(\lambda)$. ◊

As in section 2.1.2, we use Weyl's alternative to investigate the selfadjoint realisations of the differential expression $\mathfrak{T}_1(\lambda)$.
The (essential) selfadjointness of $T_1^{\min}(\lambda)$ depends on the behaviour of the solutions of $\mathfrak{T}_1(\lambda)u = 0$ at the points 0 and π, so the next lemma is crucial for our question.

Lemma B.2. *For $\lambda \in \rho(D) \cap \mathbb{R} = \mathbb{R} \setminus [-|am|, |am|]$ a fundamental system of the differential equation $\mathfrak{T}_1(\lambda)u = 0$ is given by $\{\psi, h\}$, where $\psi(\vartheta) = \mathrm{e}^{-a\omega\cos\vartheta}(\tan\frac{\vartheta}{2})^{k+\frac12}$ is a solution of $\mathfrak{B}_-u = 0$, cf. lemmata 2.8 and 3.21, and h solves the differential equation*

$$(\mathfrak{D}-\lambda)^{-1}\mathfrak{B}_-h = \varphi,$$

with $\varphi(\vartheta) = \mathrm{e}^{a\omega\cos\vartheta}(\tan\frac{\vartheta}{2})^{-(k+\frac{1}{2})}$ *being a solution of* $\mathfrak{B}_+ u = 0$ (see lemmata 2.8 and 3.21). For ψ and h the following equivalences hold:

$$\begin{array}{llcl}
\psi \text{ lies left} & \text{in } \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) & \iff & k > -1,\\
\psi \text{ lies right} & \text{in } \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) & \iff & k < 0,\\
h \text{ lies left} & \text{in } \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) & \iff & k < 1,\\
h \text{ lies right} & \text{in } \mathscr{L}^2((0,\pi),\mathrm{d}\vartheta) & \iff & k > -2.
\end{array}$$

Proof. It is clear that ψ and h are solutions of $\mathfrak{T}_1(\lambda)u = 0$. They are linearly independent because $\mathfrak{B}_- h \neq 0$ but φ lies in the kernel of $\mathfrak{B}_-$. The assertions concerning ψ have been shown in the proof of lemma 2.8, so we give a proof only for the behaviour of h. By definition, h solves the differential equation

$$\Big(-\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta\Big)h(\vartheta) \;=\; (am\cos\vartheta - \lambda)\varphi(\vartheta), \qquad \vartheta \in (0,\pi). \tag{B.7}$$

If we apply the ansatz $h(\vartheta) = c(\vartheta)\varphi(\vartheta)^{-1}$ and use the relation $\varphi(\vartheta)^{-1} = \psi(\vartheta)$, we obtain the differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}\vartheta}c(\vartheta) \;=\; (am\cos\vartheta - \lambda)\varphi^2(\vartheta), \qquad \vartheta \in (0,\pi)$$

for the function c. In the following, we consider the behaviour of h at the point 0. We have to distinguish several cases.
Case 1. Let $k \in (-\infty, -\frac{1}{2}]$; then a solution of (B.7) is given by

$$h(\vartheta) \;=\; -\varphi(\vartheta)^{-1}\int_0^{\vartheta}(am\cos t - \lambda)\varphi^2(t)\,\mathrm{d}t, \qquad \vartheta \in (0,\pi). \tag{B.8}$$

There is an $M_1 > 0$ such that $|am\cos t - \lambda|\,\mathrm{e}^{a\omega(2\cos t - \cos\vartheta)} \le M_1$ for all $t,\ \vartheta \in (0,\pi)$. For $k \le -\frac{1}{2}$ and $t \in (0,\pi)$ the function $(\tan\frac{t}{2})^{-2k-1}$ is nondecreasing, hence $0 < (\tan\frac{t}{2})^{-2k-1} \le (\tan\frac{\vartheta}{2})^{-2k-1}$ holds for $0 \le t \le \vartheta < \pi$. This shows that for any $d \in (0,\pi)$

$$\begin{aligned}
\int_0^d |h(\vartheta)|^2\,\mathrm{d}\vartheta \;&=\; \int_0^d \varphi(\vartheta)^{-2}\bigg(\int_0^{\vartheta}(am\cos t - \lambda)\varphi^2(t)\,\mathrm{d}t\bigg)^2\mathrm{d}\vartheta\\
&\le\; M_1^2\int_0^d\Big(\tan\frac{\vartheta}{2}\Big)^{2k+1}\bigg(\int_0^{\vartheta}\Big(\tan\frac{t}{2}\Big)^{-2k-1}\,\mathrm{d}t\bigg)^2\mathrm{d}\vartheta\\
&\le\; M_1^2\int_0^d\Big(\tan\frac{\vartheta}{2}\Big)^{-2k-1}\bigg(\int_0^{\vartheta}1\,\mathrm{d}t\bigg)^2\mathrm{d}\vartheta \;<\; \infty
\end{aligned}$$

and, consequently, h lies left in $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$.

Case 2. Let $k \in (-\frac{1}{2}, 1)$, hence $2k+1 > 0$; then a solution of (B.7) is given by

$$h(\vartheta) \;=\; \varphi(\vartheta)^{-1}\int_{\vartheta}^{\pi}(am\cos t - \lambda)\varphi^2(t)\,\mathrm{d}t, \qquad \vartheta \in (0,\pi). \tag{B.9}$$

Now, if we fix $d \in (0, \pi)$ and use the inequalities $0 < \tan\frac{\vartheta}{2} = \frac{\sin\frac{\vartheta}{2}}{\cos\frac{\vartheta}{2}} \le \frac{\vartheta}{2\cos\frac{d}{2}}$ for $0 < \vartheta \le d < \pi$ and $0 < (\tan\frac{t}{2})^{-1} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \le \frac{\pi}{t}$ for $t \in (0, \pi)$, we obtain for $k \neq 0$

$$\begin{aligned}
\int_0^d |h(\vartheta)|^2 \,\mathrm{d}\vartheta &\le M_1^2 \int_0^d \Big(\tan\frac{\vartheta}{2}\Big)^{2k+1} \bigg(\int_\vartheta^\pi \Big(\tan\frac{t}{2}\Big)^{-2k-1} \,\mathrm{d}t \bigg)^2 \mathrm{d}\vartheta \\
&\le \frac{M_1^2\, \pi^{2(2k+1)}}{(2\cos\frac{d}{2})^{2k+1}} \int_0^d \vartheta^{2k+1} \bigg(\int_\vartheta^\pi t^{-2k-1} \,\mathrm{d}t \bigg)^2 \mathrm{d}\vartheta \\
&= \frac{M_1^2\, \pi^{2(2k+1)}}{(2k)^2(2\cos\frac{d}{2})^{2k+1}} \int_0^d \vartheta^{2k+1} (\pi^{-4k} - 2\pi^{-2k}\vartheta^{-2k} + \vartheta^{-4k}) \,\mathrm{d}\vartheta \\
&= \frac{M_1^2\, \pi^{2(2k+1)}}{(2k)^2(2\cos\frac{d}{2})^{2k+1}} \bigg[\frac{\pi^{-4k}}{2k+2}\, \vartheta^{2k+2} - \pi^{-2k}\, \vartheta^2 + \frac{1}{-2k+2}\, \vartheta^{-2k+2} \bigg]_0^d \;<\; \infty.
\end{aligned}$$

The case $k = 0$ furnishes technical problems only. We can show by a direct calculation involving the logarithm that the integral on the left hand side is finite also in this case. Hence, for every $k \in (-\frac{1}{2}, 1)$, the function h lies left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$.

Case 3. Let $k \in [1, \infty)$; also in this case (B.9) is a solution of (B.7). Now we show that the function h does not lie left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. To this end we fix some $d_0 \in (0, \pi)$. Since $|\lambda| > |am|$ by assumption, there is a constant $M_2 > 0$ such that $|am\cos t - \lambda|\, e^{a\omega(2\cos t - \cos\vartheta)} \ge M_2$ for all $t, \vartheta \in (0, \pi)$. Therefore we can estimate

$$\begin{aligned}
\int_0^{d_0} |h(\vartheta)|^2 \,\mathrm{d}\vartheta &= \int_0^{d_0} \varphi(\vartheta)^{-2} \bigg(\int_\vartheta^\pi (am\cos t - \lambda)\varphi^2(t) \,\mathrm{d}t \bigg)^2 \mathrm{d}\vartheta \\
&\ge M_2^2 \int_0^{d_0} (\tan\tfrac{\vartheta}{2})^{2k+1} \bigg(\int_\vartheta^{d_0} (\tan\tfrac{t}{2})^{-2k-1} \,\mathrm{d}t \bigg)^2 \mathrm{d}\vartheta.
\end{aligned}$$

In lemma 3.24 we have seen that the function $(0, \pi) \to \mathbb{R}$, $x \mapsto \frac{\tan\frac{x}{2}}{x}$ is monotonously increasing, hence $\big(\tan\frac{t}{2}\big)^{-(2k+1)} \ge \Big(d_0^{-1} \tan\frac{d_0}{2}\Big)^{-(2k+1)} t^{-(2k+1)}$ for all $t \in (0, d_0)$. It is well known that the function $(0,\pi) \to \mathbb{R}$, $x \mapsto \frac{\sin x}{x}$ is continuous and converges to 1 for $x \to 0$. Hence there exists a $d \in (0, d_0)$ such that

$$\frac{\tan\frac{\vartheta}{2}}{\vartheta} \;\ge\; \frac{1}{2}\, \frac{\sin\frac{\vartheta}{2}}{\frac{\vartheta}{2}} \;\ge\; \frac{1}{4}, \qquad \vartheta \in (0, d).$$

If we set $M = M_2^2 \left(d_0^{-1} \tan \frac{d_0}{2}\right)^{-2(2k+1)}$ we obtain

$$\begin{aligned}
\int_0^{d_0} |h(\vartheta)|^2 \,\mathrm{d}\vartheta \;&\geq\; M \int_0^{d} (\tan \tfrac{\vartheta}{2})^{2k+1} \left(\int_\vartheta^d t^{-2k-1} \,\mathrm{d}t \right)^2 \mathrm{d}\vartheta \\
&= \frac{M}{(2k)^2} \int_0^d (\tan \tfrac{\vartheta}{2})^{2k+1} (d^{-2k} - \vartheta^{-2k})^2 \,\mathrm{d}\vartheta \\
&= \frac{M}{(2k)^2} \int_0^d (\tan \tfrac{\vartheta}{2})^{2k+1} \vartheta^{-4k} - 2d^{-2k} \left(\frac{\tan \frac{\vartheta}{2}}{\vartheta} \right)^{-2k} \tan \tfrac{\vartheta}{2} - d^{-4k} (\tan \tfrac{\vartheta}{2})^{2k+1} \,\mathrm{d}\vartheta.
\end{aligned}$$

Obviously, the integral over the last two terms converges since they are bounded in $(0, d)$, whereas the integral over the first summand diverges, which implies that the function h does not lie left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$.

It remains to consider the behaviour of solutions of h at the point π. To this end we use symmetry properties of the function h. We attach subscripts k and a to the functions so that we can distinguish between solutions for different wave numbers k and Kerr parameters a. As already pointed out in remark 3.29, for $k \in \mathbb{R}$ the function φ satisfies

$$\varphi_{k,a}(\vartheta) \;=\; \varphi_{-k-1,-a}(\pi - \vartheta), \qquad\qquad \vartheta \in (0, \pi).$$

In the case $k \leq -\frac{1}{2}$ this yields

$$\begin{aligned}
h_{k,a}(\pi - \vartheta) \;&=\; -\varphi_{k,a}(\pi - \vartheta)^{-1} \int_0^{\pi - \vartheta} (am \cos t - \lambda) \varphi_{k,a}^2(t) \,\mathrm{d}t \\
&=\; -\varphi_{k,a}(\pi - \vartheta)^{-1} \int_\vartheta^{\pi} (am \cos(\pi - t) - \lambda) \varphi_{k,a}^2(\pi - t) \,\mathrm{d}t \\
&=\; -\varphi_{-k-1,-a}(\vartheta)^{-1} \int_\vartheta^{\pi} (-am \cos t - \lambda) \varphi_{-k-1,-a}^2(t) \,\mathrm{d}t \;=\; h_{-k-1,-a}(\vartheta).
\end{aligned}$$

A similar computation shows that $h_{k,a}(\pi - \,\cdot\,) = h_{-k-1,-a}$ holds also for $k > -\frac{1}{2}$. Consequently, $h_{k,a}$ is square integrable at π if and only if $h_{-k-1,-a}$ is square integrable at 0. Since the square integrability does not depend on the value of a, it follows from the above considerations that h lies right in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ if and only if $k \geq -2$. □

The foregoing lemma shows that $h \in \mathcal{D}(T_1^{\max}(\lambda))$ for all $k \in (-2, 1)$ and that $\psi \in \mathcal{D}(T_1^{\max}(\lambda))$ for all $k \in (-1, 0)$. It also provides information about selfadjoint extensions of the minimal operators $S_1^{\min}(\lambda)$ and $T_1^{\min}(\lambda)$.

Lemma B.3. *Let $\lambda \in \rho(D) \cap \mathbb{R} = \mathbb{R} \setminus [\,-|am|, |am|\,]$. For $k \in \mathbb{R} \setminus (-2, 1)$ the operators $T_1^{\min}(\lambda)$ and $S_1^{\min}(\lambda)$ are essentially selfadjoint. For $k \in (-2, -1\,] \cup [\,0, 1)$ all selfadjoint extensions of the operators $T_1^{\min}(\lambda)$ and $S_1^{\min}(\lambda)$ are one-dimensional restrictions of $T_1^{\max}(\lambda)$ and $S_1^{\max}(\lambda)$, respectively, while for $k \in (-1, 0)$ they are two-dimensional restrictions.*

Proof. It follows from lemma B.2 that the differential expression $\mathfrak{T}_1(\lambda)$ is in the limit point case at 0 if and only if $k \in \mathbb{R} \setminus (-1, 1)$ and that it is in the limit point case at π if and only if $k \in \mathbb{R} \setminus (-2, 0)$. Thus the assertions are direct consequences of [Wei87, theorem 5.7]. □

Recall that $S_1(\lambda)$ is the Friedrichs extension of $S_1^{[\min]}(\lambda)$ for $\lambda > |am|$. Although $S_1^{\min}(\lambda) \subsetneq S_1^{[\min]}(\lambda)$, the next lemma shows that their Friedrichs extensions coincide.

Lemma B.4. *The Friedrichs extensions of $S_1^{\min}(\lambda)$ and of the Schur complement $S_1(\lambda)$ are equal.*

Proof. First we show that for $\lambda \in \rho(D) \cap \mathbb{R} = \rho(-D) \cap \mathbb{R}$ the forms $\mathfrak{s}_1^{\min}(\lambda)$ and $\mathfrak{t}_1^{\min}(\lambda)$ are symmetric, semibounded and closable. The symmetry of $S_1^{\min}(\lambda)$ implies that

$$\mathfrak{s}_1^{\min}(\lambda)[u,v] \;=\; (u,\ S_1^{\min}(\lambda)v) \;=\; (S_1^{\min}(\lambda)u,\ v) \;=\; \overline{(v,\ S_1^{\min}(\lambda)u)} \;=\; \overline{\mathfrak{s}_1^{\min}(\lambda)[v,u]}$$

for all $u, v \in \mathcal{D}(\mathfrak{s}_1^{\min}(\lambda))$, hence the form is symmetric. Next we show that the form $\operatorname{sign}(\lambda)\mathfrak{s}_1^{\min}(\lambda)$ is bounded from below. Since $|\lambda| > \|D\| = |am|$ by assumption, it follows that $(|\lambda| - \operatorname{sign}(\lambda)D)^{-1}$ is a positive operator, thus we have for all $u \in \mathcal{D}(\mathfrak{s}_1^{\min}(\lambda))$

$$\begin{aligned}
\operatorname{sign}(\lambda)\mathfrak{s}_1^{\min}(\lambda)[u] \;&=\; \operatorname{sign}(\lambda)\big(u,\,(-D-\lambda)u\big) - \operatorname{sign}(\lambda)\big(u,\,B(D-\lambda)^{-1}B^*u\big)\\
&=\; -\big(u,\,(\operatorname{sign}(\lambda)D + |\lambda|)\,u\big) + \big(B^*u,\,(|\lambda| - \operatorname{sign}(\lambda)D)^{-1}B^*u\big)\\
&\geq\; -(|\lambda| + \|D\|)\,\|u\|^2.
\end{aligned}$$

Since the above calculation also implies that the operator $S_1^{\min}(\lambda)$ is semibounded, it follows that the form $\mathfrak{s}_1^{\min}(\lambda)$ is closable, see [Kat80, chap. VI, corollary 1.28]. The corresponding assertions for the form $\mathfrak{t}_1^{\min}(\lambda)$ can be shown analogously.
In the following we denote the closures of $\mathfrak{s}_1^{\min}(\lambda)$ and $\mathfrak{t}_1^{\min}(\lambda)$ by $\mathfrak{s}_1(\lambda)$ and $\mathfrak{t}_1(\lambda)$, respectively. Obviously, we have $\mathcal{D}(\mathfrak{t}_1(\lambda)) = \mathcal{D}(\mathfrak{s}_1(\lambda))$. Now we show that the domains of the closed forms are given by

$$\mathcal{D}(\mathfrak{s}_1(\lambda)) \;=\; \mathcal{D}(\mathfrak{t}_1(\lambda)) \;=\; \mathcal{D}(B^*).$$

Since $\mathfrak{s}_1(\lambda)$ is the closure of $\mathfrak{s}_1^{\min}(\lambda)$, an element $u \in \mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$ lies in $\mathcal{D}(\mathfrak{s}_1(\lambda))$ if and only if there is a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathfrak{s}_1^{\min}(\lambda)) = \mathcal{C}_0^\infty(0,\pi)$ such that $u_n \to u$ and $\mathfrak{s}_1^{\min}(\lambda)[u_n - u_m] \to 0$ for $m, n \to \infty$. In $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$, the operators $-D-\lambda$ and $(D-\lambda)^{-1}$ are bounded and either strictly positive or strictly negative for $\lambda \in \mathbb{R} \setminus [-|am|, |am|\,]$. Hence it follows that

$$\mathfrak{s}_1(\lambda)[u_n - u_m] = \big(u_n - u_m,\,(-D-\lambda)(u_n - u_m)\big) - \big(B^*(u_n - u_m),\,(D-\lambda)^{-1}B^*(u_n - u_m)\big) \longrightarrow 0$$

for $m, n \to \infty$ is equivalent to

$$\big(B^*(u_n - u_m),\,B^*(u_n - u_m)\big) \;=\; \|B^*(u_n - u_m)\|^2 \;\longrightarrow\; 0, \qquad m, n \to \infty.$$

Since B^* is closed, this is equivalent to $u \in \mathcal{D}(B^*)$.
Comparing with lemma 4.30, we find that $\mathfrak{s}_1(\lambda) = \mathfrak{s}_1^{[\min]}(\lambda)$. Since the Schur complement $S_1(\lambda)$ is defined as the selfadjoint operator associated with the closure of $\mathfrak{s}_1^{[\min]}(\lambda)$, it follows that $S_1(\lambda)$ is also the selfadjoint operator associated with the closure $\mathfrak{s}_1(\lambda)$ of $\mathfrak{s}_1^{\min}(\lambda)$. Hence the Friedrichs extension of $S_1^{\min}(\lambda)$ is equal to the Schur complement $S_1(\lambda)$. □

The lemma shows that the domain of the forms is independent of λ and thus justifies the definitions

$$\mathcal{D}(\mathfrak{t}_1) \;:=\; \mathcal{D}(\mathfrak{s}_1) \;:=\; \mathcal{D}(\mathfrak{t}_1(\lambda)) \;=\; \mathcal{D}(\mathfrak{s}_1(\lambda)) \;=\; \mathcal{D}(B^*).$$

Of course, assertions analogous to those from lemmata B.3 and B.4 hold for the minimal operator $S_2^{\min}(\lambda)$. They are summarised in the following lemma.

Lemma B.5. *For $\lambda \in \rho(-D) \cap \mathbb{R} = \mathbb{R} \setminus [-|am|, |am|\,]$, the formal differential expression*

$$\mathfrak{S}_2(\lambda) \;=\; \mathfrak{D} - \lambda - \mathfrak{B}_-(-\mathfrak{D} - \lambda)^{-1}\mathfrak{B}_+$$

is in the limit point case at 0 if and only if $k \in \mathbb{R} \setminus (-2, 0)$; it is in the limit point case at π if and only if $k \in \mathbb{R} \setminus (-1, 1)$. Therefore, the operator

$$\mathcal{D}(S_2^{\min}(\lambda)) := \mathcal{C}_0^\infty(0,\pi), \qquad S_2^{\min}(\lambda)f \;=\; \mathfrak{S}_2(\lambda)f \tag{B.10}$$

is essentially selfadjoint if and only if $k \in \mathbb{R} \setminus (-2, 1)$.
The Schur complement $S_2(\lambda)$ and the Friedrichs extension of $S_2^{\min}(\lambda)$ coincide. The domain of the corresponding closed form is given by

$$\mathcal{D}(\mathfrak{s}_2) \;:=\; \mathcal{D}(\mathfrak{t}_2) \;:=\; \mathcal{D}(\mathfrak{t}_2(\lambda)) \;=\; \mathcal{D}(\mathfrak{s}_2(\lambda)) \;=\; \mathcal{D}(B).$$

Remark B.6. For $k \in (-2, -1\,] \cup [\,0, 1)$ the block operator matrices

$$\begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D - \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -D - \lambda & 0 \\ 0 & S_2(\lambda) \end{pmatrix}$$

with domain $\mathcal{C}_0^\infty(0,\pi)^2$ are symmetric, but not essentially selfadjoint; this means that their domains are not large enough. However, the products

$$\begin{pmatrix} I & B(D-\lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D - \lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (D-\lambda)^{-1}B^* & I \end{pmatrix}, \tag{B.11}$$

$$\begin{pmatrix} I & 0 \\ B^*(-D-\lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} -D - \lambda & 0 \\ 0 & S_1(\lambda) \end{pmatrix} \begin{pmatrix} I & (-D-\lambda)^{-1}B \\ 0 & I \end{pmatrix} \tag{B.12}$$

with domain $\mathcal{C}_0^\infty(0,\pi)^2$ are essentially selfadjoint since they are equal to the minimal operator $\mathcal{A}^{\min}$. This corresponds to the fact that for $k \in (-2,-1] \cup [0,1)$ the ranges of B and B^* are large enough to guarantee the essential selfadjointness of the products. For example, fix some $k \in [0,1)$. Then there are two linearly independent solutions ξ_j, $j = 1, 2$, of $\mathfrak{S}_1(\lambda)u = 0$ which lie both left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. Formally, we can define the vectors

$$\Xi_j \;:=\; \begin{pmatrix} \xi_j \\ -(\mathfrak{D} - \lambda)^{-1}\mathfrak{B}_-\xi_j \end{pmatrix}, \qquad j = 1, 2,$$

which satisfy the differential equation

$$(\mathfrak{A} - \lambda)\Xi_j \;=\; \begin{pmatrix} I & \mathfrak{B}(\mathfrak{D}-\lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathfrak{S}_1(\lambda) & 0 \\ 0 & \mathfrak{D} - \lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (\mathfrak{D}-\lambda)^{-1}\mathfrak{B}_- & I \end{pmatrix} \Xi_j \;=\; 0.$$

Since $\mathcal{A}$ is in the limit point case at 0, it follows that at least for one $j \in \{1, 2\}$ the function Ξ_j cannot lie left in $\mathscr{L}^2((0,\pi), \mathrm{d}\vartheta)$. $\Diamond$

Boundary conditions

If $k \in (-2, 1)$, then the operators $S_1^{\min}(\lambda)$, $\lambda > |am|$, are not essentially selfadjoint, hence it follows from the theory of linear differential operators that there are infinitely many selfadjoint extensions each given as a restriction of $S_1^{\max}(\lambda)$ in terms of boundary conditions.
For simplicity we work with the differential expression $\mathfrak{T}_1(\lambda)$ instead of $\mathfrak{S}_1(\lambda)$. Then all results are easily carried over to the Schur complement since $T_1(\lambda)$ is a selfadjoint extension of $T_1^{\min}(\lambda)$ if and

only if $S_1(\lambda) := -(D+\lambda) + T_1(\lambda)$ is a selfadjoint extension of $S_1^{\min}(\lambda)$. Also the corresponding boundary conditions are the same.

For $\vartheta \in [0,\pi]$ and functions $u, v \in \mathcal{D}(T_1^{\max}(\lambda))$ let

$$[u,\, v]_\vartheta \;:=\; (D-\lambda)^{-1}(uv' - u'v)(\vartheta) \;=\; (am\cos\vartheta - \lambda)^{-1}\big(u(\vartheta)v'(\vartheta) - u'(\vartheta)v(\vartheta)\big), \quad \vartheta \in (0,\pi).$$

For $\vartheta = 0$ and $\vartheta = \pi$ this definition has to be understood as the limit of $[u,\, v]_\vartheta$ for $\vartheta \to 0$ and $\vartheta \to \pi$ respectively. By [Wei87, theorem 3.10], this limit always exists. Furthermore, let

$$[u,\, v]_\vartheta^\tau \;:=\; [u,\, v]_\tau - [u,\, v]_\vartheta, \qquad \vartheta,\, \tau \in [0,\, \pi].$$

For $u,\, v \in \mathcal{D}(T_1^{\max}(\lambda))$ Green's formula

$$[u,\, v]_0^\pi \;=\; (T_1^{\max}(\lambda)u,\, v) - (u,\, T_1^{\max}(\lambda)v)$$

holds. These square bracket expressions contain information about the behaviour of functions when subject to integration by parts and therefore we can use them to characterise selfadjoint extensions of the minimal operator $T_1^{\min}(\lambda)$. By [Wei87, theorem 5.8], all selfadjoint extensions of $T_1^{\min}(\lambda)$ are given by

$$\begin{aligned}
\mathcal{D}(T_1^{\eta_\pi}(\lambda)) &= \{u \in \mathcal{D}(T_1^{\max}(\lambda)) \,:\, [\eta_\pi,\, u]_\pi = 0\} && \text{if } k \in (-2,-1],\\
\mathcal{D}(T_1^{\eta_0}(\lambda)) &= \{u \in \mathcal{D}(T_1^{\max}(\lambda)) \,:\, [\eta_0,\, u]_0 = 0\} && \text{if } k \in [0,1),\\
\mathcal{D}(T_1^{\eta_0^1,\eta_0^2,\eta_\pi^1,\eta_\pi^2}(\lambda)) &= \{u \in \mathcal{D}(T_1^{\max}(\lambda)) \,:\, [\eta_0^j,\, u]_0 - [\eta_\pi^j,\, u]_\pi = 0,\ j = 1,2\} && \text{if } k \in (-1,0)
\end{aligned}$$

where η_0, η_π are non-vanishing real solutions of $\mathfrak{T}_1(\lambda)u = 0$ and η_0^j, η_π^j, $j = 1,2$, are solutions of $\mathfrak{T}_1(\lambda)u = 0$ such that $\eta_0^1 + \eta_\pi^1$ and $\eta_0^2 + \eta_\pi^2$ are linearly independent modulo $\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ and that

$$[\eta_\pi^i,\, \eta_\pi^j]_\pi - [\eta_0^i,\, \eta_0^j]_0 \;=\; 0, \qquad i,j = 1,2.$$

Lemma B.7. *The Friedrichs extension $S_1(\lambda)$ of $S_1^{\min}(\lambda)$ is given by*

(i) $S_1(\lambda) \;=\; S_1^{\eta_\pi}(\lambda) \;=\; T_1^{\eta_\pi}(\lambda) - D - \lambda$ *with $\eta_\pi = h$ if $k \in (-2,-1\,]$,*

(ii) $S_1(\lambda) \;=\; S_1^{\eta_0}(\lambda) \;=\; T_1^{\eta_0}(\lambda) - D - \lambda$ *with $\eta_0 = h$ if $k \in [0,1)$,*

(iii) $S_1(\lambda) \;=\; S_1^{\eta_0^1,\eta_0^2,\eta_\pi^1,\eta_\pi^2}(\lambda) \;=\; T_1^{\eta_0^1,\eta_0^2,\eta_\pi^1,\eta_\pi^2}(\lambda) - D - \lambda$ *with $\eta_0^1 = \eta_\pi^1 = \psi$ and $\eta_0^2 = \eta_\pi^2 = h$ if $k \in (-1,0)$.*

Here ψ and h are the functions from lemma B.3, i.e., $\psi(\vartheta) = \mathrm{e}^{-a\omega\cos\vartheta}(\tan\frac{\vartheta}{2})^{k+\frac12}$ is a solution of $\mathfrak{B}_- u = 0$ and h is solution of $(\mathfrak{D}-\lambda)^{-1}\mathfrak{B}_- u = \varphi$.

Proof. We show that $T_1^{\eta_\pi}(\lambda)$, $T_1^{\eta_0}(\lambda)$ and $T_1^{\eta_0^1,\eta_0^2,\eta_\pi^1,\eta_\pi^2}(\lambda)$ are the Friedrichs extensions of $T_1^{\min}(\lambda)$. Then the corresponding assertions for $S_1^{\min}(\lambda)$ follow from the boundedness of $-D-\lambda$. Lemma B.2 shows that the differential expression $\mathfrak{T}_1(\lambda)$ is quasi-regular at 0 if and only if $k \in (-1,1)$ and that it is quasi-regular at π if and only if $k \in (-2,0)$. Moreover, we know that $h \in \mathcal{D}(T_1^{\max}(\lambda))$ for all $k \in (-2,1)$ and $\psi \in \mathcal{D}(T_1^{\max}(\lambda))$ for all $k \in (-1,0)$. From $\mathfrak{B}_-\psi = 0$ and the definition of h in (B.7) it follows that

$$\begin{aligned}
\psi'(\vartheta) &= -\mathfrak{B}_-\psi + \Big(\tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)\psi(\vartheta) \;=\; \Big(\tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)\psi(\vartheta),\\
h'(\vartheta) &= -\mathfrak{B}_- h(\vartheta) + \Big(\tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)h(\vartheta) \;=\; -(am\cos\vartheta - \lambda)\varphi + \Big(\tfrac{k+\frac12}{\sin\vartheta} + a\omega\sin\vartheta\Big)h(\vartheta).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
[h,\psi]_\vartheta &= (am\cos\vartheta-\lambda)^{-1}(h(\vartheta)\psi'(\vartheta)-h'(\vartheta)\psi(\vartheta)) \\
&= (am\cos\vartheta-\lambda)^{-1}\Big(\Big(\tfrac{k+\frac12}{\sin\vartheta}+a\omega\sin\vartheta\Big)h(\vartheta)\psi(\vartheta) \\
&\qquad + (am\cos\vartheta-\lambda)\varphi(\vartheta)\psi(\vartheta)-\Big(\tfrac{k+\frac12}{\sin\vartheta}+a\omega\sin\vartheta\Big)h(\vartheta)\psi(\vartheta)\Big) \\
&= 1
\end{aligned}$$

for all $\vartheta\in(0,\pi)$ and then also for $\vartheta=0,\ \pi$. Now we show that the functions h and ψ do not lie in the domain of the closure of $T_1^{\min}(\lambda)$. Note that for $k\in\mathbb{R}\setminus(-2,1)$ we have $h\notin\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ and that for $k\in\mathbb{R}\setminus(-1,0)$ we have $\psi\notin\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$, hence $\psi,\ h\notin\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ in these cases. For the remaining cases, we slightly modify the proof of [Wei87, theorem 5.4]. Let $k\in(-1,1)$ and fix a function $\psi_0\in\mathcal{D}(T_1(\lambda)^{\max})$ such that $\psi_0=\psi$ in a neighbourhood of 0 and $\psi_0=0$ in a neighbourhood of π. Such a function exists since ψ lies left in $\mathscr{L}^2((0,\pi),\mathrm{d}\vartheta)$ and it follows that

$$[h,\ \psi_0]_\vartheta=1 \qquad\text{and}\qquad [h,\ \psi_0]_\tau=0$$

for ϑ in a neighbourhood of 0 and τ in a neighbourhood of π. If we assume $h\in\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ then Green's formula and the fact that $T_1^{\max}(\lambda)^*=\overline{T_1^{\min}(\lambda)}\subseteq T_1^{\max}(\lambda)$ show that

$$\begin{aligned}
-1 &= -[h,\ \psi]_0 = [h,\ \psi_0]_0^\pi = \big(T_1^{\max}(\lambda)h,\ \psi_0\big)-\big(h,\ T_1^{\max}(\lambda)\psi_0\big) \\
&= \big(T_1^{\max}(\lambda),\ h\big)\psi_0-\big(T_1^{\max}(\lambda)^*h,\ \psi_0\big) = 0.
\end{aligned}$$

a contradiction. If $k\in(-2,0)$, we use a function $\psi_\pi\in\mathcal{D}(T_1^{\max}(\lambda))$ such that $\psi_\pi=0$ in a neighbourhood of 0 and $\psi_\pi=\psi$ in a neighbourhood of π to obtain a contradiction as above. Analogous considerations show that $\psi\notin\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ for $k\in(-1,0)$. Now we consider the three cases of the lemma.

(i) Assume $k\in(-2,1]$. Since all selfadjoint extensions are one-dimensional restrictions of $\overline{T_1^{\max}(\lambda)}$ and since we have already shown that $h\notin\mathcal{D}(\overline{T_1^{\min}(\lambda)})$, it suffices to show that h lies in the domain of both selfadjoint extensions $T_1(\lambda)$ and $T_1^h(\lambda)$. The latter inclusion is obvious. To show the first one, we note that $h\in\mathcal{D}(\mathfrak{t}_1(\lambda))$. Since for all $u\in\mathcal{D}(\mathfrak{t}_1(\lambda))$ we find

$$\mathfrak{t}_1(\lambda)[u,\ h] = (u,\ \mathfrak{T}_1(\lambda)h) = 0,$$

the function h lies in the domain of the Friedrichs extension $T_1(\lambda)$ and $T_1(\lambda)h=0$ holds.

(ii) For $k\in[0,1)$ the assertion follows analogously.

(iii) In the case $k\in(-1,0)$ the selfadjoint extensions of $\overline{T_1^{\min}(\lambda)}$ are two-dimensional restrictions of $\overline{T_1^{\max}(\lambda)}$. We have already shown that $h,\ \psi\notin\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ and that the functions h and ψ are linearly independent. As in the first and second case, it follows that these functions lie in $\mathcal{D}(T_1(\lambda))$; further, they also lie in $\mathcal{D}(T_1^{\psi,h,\psi,h}(\lambda))$. Hence it remains to prove that h and ψ are linearly independent modulo $\mathcal{D}(\overline{T_1^{\min}(\lambda)})$. Just as in the beginning of the proof we can show that no linear combination of these two functions lies in $\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ which completes the proof. □

Remark B.8. For $k\in(-2,-1\,]\cup[\,0,1)$ we can show $h\notin\mathcal{D}(\overline{T_1^{\min}(\lambda)})$ also without using Green's formula. We have seen in lemma 3.30 that for $k\in\mathbb{R}\setminus(-1,0)$ there are $C(\omega)\neq0$ and $\delta(k,\omega)\neq0$ such that the inequalities $\|Bf\|\geq C(\omega)\delta(k,\omega)\|f\|$, $f\in\mathcal{D}(B)$, and $\|Bg\|\geq C(\omega)\delta(k,\omega)\|g\|$, $g\in\mathcal{D}(B^*)$, hold. Now assume $h\in\mathcal{D}(\overline{T_1^{\min}(\lambda)})$. Because of $\overline{T_1^{\min}(\lambda)}\subseteq T_1(\lambda)$ it follows that h lies in the kernel of $\overline{T_1^{\min}(\lambda)}$. Hence there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T_1^{\min}(\lambda)) = \mathcal{C}_0^\infty(0,\pi)$ such that $u_n \to h$ and $T_1^{\min}(\lambda)u_n \to 0$. Obviously, $((D-\lambda)^{-1}B^*u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_0^\infty(0,\pi) \subseteq \mathcal{D}(B^*)$. This leads to the contradiction

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|T_1^{\min}(\lambda)u_n\| \;=\; \lim_{n\to\infty} \|B(D-\lambda)^{-1}B^*u_n\| \\ &\geq C^2(\omega)\,\delta(k,\omega)^2\,\|D-\lambda\|^{-1} \lim_{n\to\infty}\|u_n\| \;=\; C^2(\omega)\,\delta(k,\omega)^2\,\|D-\lambda\|^{-1}\,\|h\| \;>0. \end{aligned}$$

$\Diamond$

It should be observed that in the case $k \in (-1,0)$ the Friedrichs extension is given by coupled boundary conditions, although there exist selfadjoint extensions of $S_1^{\min}(\lambda)$ with separated boundary conditions.

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Notation

The following table lists some frequently used symbols and their usual meaning together with the page of their first occurrence.

General notation

$\overline{z}$	complex conjugation
$[x]$	Gauß bracket; $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$
$\langle \cdot, \cdot \rangle$	scalar product in $\mathbb{C}^m$ with the convention $\langle x, y \rangle = \sum_{j=1}^m \overline{x}_j y_j$ for $x, y \in \mathbb{C}^m$,
$(\cdot, \cdot)$	scalar product in Hilbert spaces; for $\mathscr{L}^2$-spaces of $\mathbb{C}^m$-valued functions it is defined as usual by $(f, g) = \int \langle f, g \rangle (x) \mathrm{d}x$
$\mathcal{A},\ \mathcal{B}, \dots$	operator matrices
$A,\ B,\ \dots$	linear operators
A^*	adjoint operator
$A^{\min}$	minimal operator associated with a formal differential expression
$\mathfrak{A},\ \mathfrak{B},\ \dots$	formal differential operators
$\mathfrak{a}, \mathfrak{b},\ \dots$	sesquilinear forms or formal differential expressions
$\mathcal{C}_0^\infty(0, \pi)$	space of smooth functions with compact support in $(0, \pi)$
$\mathscr{C}(\mathcal{H})$	space of all closed operators on a Hilbert space $\mathcal{H}$
$\mathcal{D}(A)$	domain of the operator A
$\ker(A)$	kernel of the operator A
$\mathcal{H}$	general (usually complex) Hilbert space
$I,\ I_n$	identity operator
$\mathcal{L}^\times$	$:= \mathcal{L} \setminus \{0\}$ for linear spaces $\mathcal{L}$
$\mathscr{L}(\mathcal{H})$	space of all linear operators on the Hilbert space $\mathcal{H}$
$\mathscr{L}^2((0,1), \mathrm{d}x)$	space of square integrable functions on $(0, 1)$
$\mathbb{N}$	$= \{1,\ 2,\ 3, \dots\}$, the natural numbers
$o,\ \mathcal{O}$	Landau symbols; a function f is of order $o(g)$ for $x \to x_0$ if $f(x)/g(x) \to 0$ for $x \to x_0$; a function f is of order $\mathcal{O}(g)$ for $x \to x_0$ if $f(x)/g(x)$ is bounded for $x \to x_0$
$\mathrm{rg}(A)$	range of the operator A
$W(A),\ W(\mathfrak{a})$	numerical range of the operator A or the sesquilinear form $\mathfrak{a}$
$W^2(\mathcal{A})$	quadratic numerical range of a block operator matrix $\mathcal{A}$
$\rho(A)$	resolvent set of the operator A
$\sigma(A)$	spectrum of the operator A
$\sigma_d(A)$, $\sigma_{ess}(A)$	discrete spectrum, essential spectrum of the linear operator A
$\sigma_p(A)$	point spectrum of the linear operator A

The angular operator and its associated operators

$\widehat{\mathfrak{A}}$	untransformed angular part of the coupled system in the Kerr-Newman background	10
$\widehat{\mathfrak{A}}_s$	$= V_0\,\widehat{\mathfrak{A}}$	11
$\mathfrak{A}^{(d)}$	untransformed angular part of the decoupled Dirac equation in the Kerr-Newman background	11
$\mathfrak{A}$	transformed angular part of the decoupled Dirac equation in the Kerr-Newman background	17
$\mathfrak{A}^0, \mathfrak{A}^\pi$	restriction of $\mathfrak{A}$ to $(0,c]$ and $[c,\pi)$ respectively	20
$\mathfrak{A}^0_U,\ \mathfrak{A}^\pi_U$	formal differential expression, unitarily equivalent to $\mathfrak{A}^0$, $\mathfrak{A}^\pi$ respectively	22
$\mathfrak{A}_b$	bounded part of $\mathfrak{A}$	17
$\mathfrak{A}_u$	$= \mathfrak{A} - \mathfrak{A}_b$, singular part of $\mathfrak{A}$	17
$\mathcal{A}^{\min}$	minimal operator with domain $\mathcal{D}(\mathcal{A}^{\min}) = \mathcal{C}_0^\infty(0,\pi)$ associated with $\mathfrak{A}$	19
$\mathcal{A}_u^{\min}$	minimal operator associated with $\mathfrak{A}_u$	17
$\mathcal{A}^{0\min},\ \mathcal{A}^{\pi\min}$	minimal operators associated with $\mathfrak{A}^0$, $\mathfrak{A}^\pi$	20
$\mathcal{A}$	the angular operator, defined as the closure of $\mathcal{A}^{\min}$ and maximal operator associated with $\mathfrak{A}$, $\mathcal{A} = \begin{pmatrix} -D & B \\ B^* & D \end{pmatrix}$	19
$\mathcal{A}_u$	$= \mathcal{A} - \mathcal{A}_b$, singular part of $\mathcal{A}$	17
$\mathcal{A}_b$	bounded part of $\mathcal{A}$	19
$\mathcal{A}_u^{\max}$	maximal operator associated with $\mathfrak{A}_u$	19
$\mathcal{A}^0, \mathcal{A}^\pi$	selfadjoint extensions of $\mathcal{A}^{0\min}$ and $\mathcal{A}^{\pi\min}$, respectively	21
$\mathcal{A}_U$	operator unitarily equivalent to $\mathcal{A}$	107
$\mathfrak{B}_\mu$	$= \begin{pmatrix} 0 & \mathfrak{B}_+ - \mu \\ \mathfrak{B}_- - \mu & 0 \end{pmatrix}$	46
$\mathcal{B}_\mu$	selfadjoint realisation of $\mathfrak{B}_\mu$	46
$\mathcal{B}$	$= \mathcal{B}_0$	20, 46

Other operators, functions and parameters

$\mathfrak{B}_\pm$	formal differential operators, formally adjoint to each other, $\mathfrak{B}_\pm = \pm\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta$	20
$B,\ B_- = B^*$	closed differential operators, adjoint to each other, $B = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta$	20
B_U	entry of the block operator matrix $\mathcal{A}_U$, $B_U = \frac{\mathrm{d}}{\mathrm{d}\vartheta} + am\cos\vartheta$	107
$\mathfrak{b}$	formal differential expression associated with BB^*	95
$\mathfrak{b}_U$	sesquilinear form, $\mathfrak{b}_U[u,\ v] = (u, B_U v)$	108
$c(k)$		102
$c_\pm$		51
$C(\omega)$		53

$\mathfrak{D}$	formal multiplication operator, $\mathfrak{D} = am\cos\vartheta$	10
D	multiplication operator, $D = am\cos\vartheta$	27
D_U	entry in $\mathcal{A}_U$, $D_U = -\left(\frac{k+\frac{1}{2}}{\sin\vartheta} + a\omega\sin\vartheta\right)$	107
$\mathfrak{d}_U$	sesquilinear form, $\mathfrak{d}_U[u,\, v] = (u, D_U v)$	108
H_D	$= \vec{\alpha}\cdot\vec{p} + \beta m$, Dirac operator in flat spacetime	28
$\widehat{\mathfrak{H}}_s$	$= \widehat{\mathfrak{R}}_s + \widehat{\mathfrak{A}}_s$, Dirac operator in the Kerr-Newman metric	11
$\vec{\mathfrak{J}}$	$= \vec{L} + \vec{S}$, total angular momentum operator	29
$\mathfrak{K}$	spin-orbit operator	29
$\widehat{\mathfrak{K}}$	block spin-orbit operator	29
$\vec{L}$	angular momentum operator	28
$\mathfrak{L}_{\pm}^{t,\varphi}$, $\mathfrak{L}_{\pm}$	entries of $\widehat{\mathfrak{A}}$, $\mathfrak{A}$	10, 11
m_+	index shift	90
n_0	index shift	79
P	parity operator	31
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j_z	eigenvalue of $\mathfrak{J}_z$	31
k	wave number	11
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m	mass of the fermion	10
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