A

Tropical Geometry

Tropical arithmetics

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$$a +_{tr} b = max\{a, b\}$$

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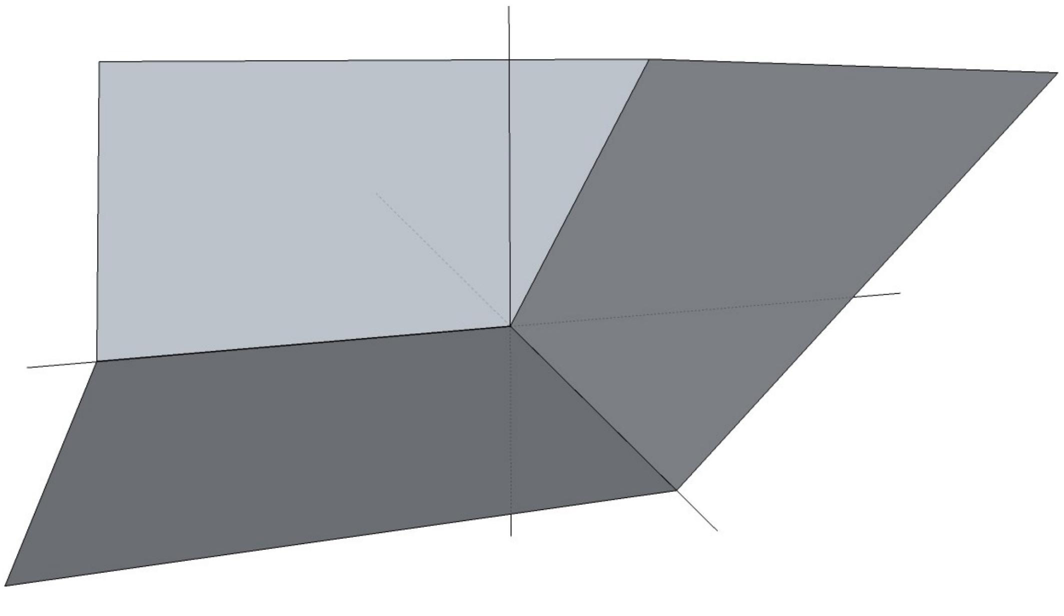
Example

$$f \;=\; 0 \;+_{tr}\; x \;+_{tr}\; y$$

$$\;=\; max\{0, x, y\}$$



Tropical hypersurface

Definition

The tropical hypersurface of a tropical polynomial f is

$$V(f) = \text{corner locus of } f.$$

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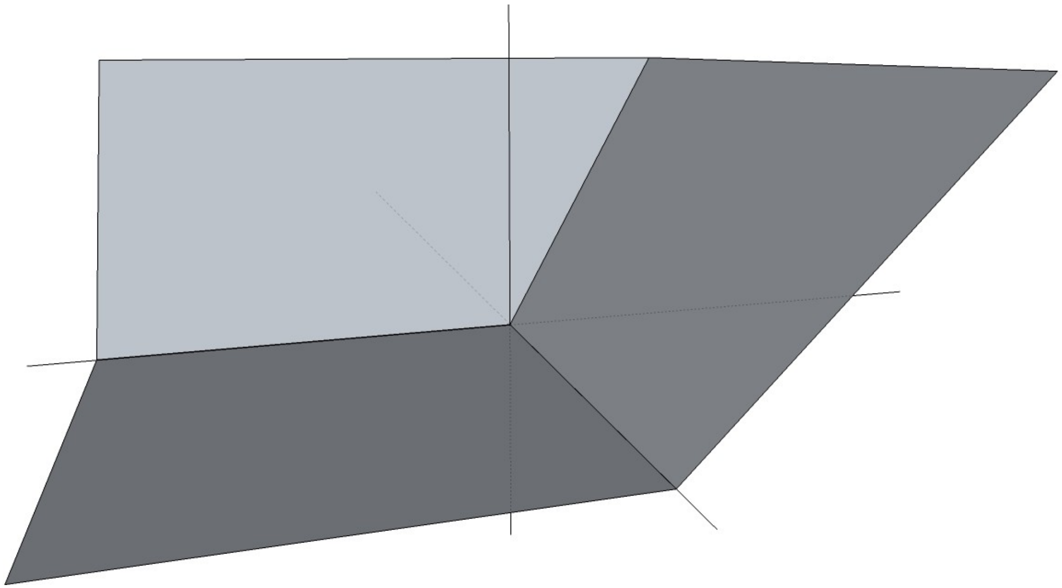
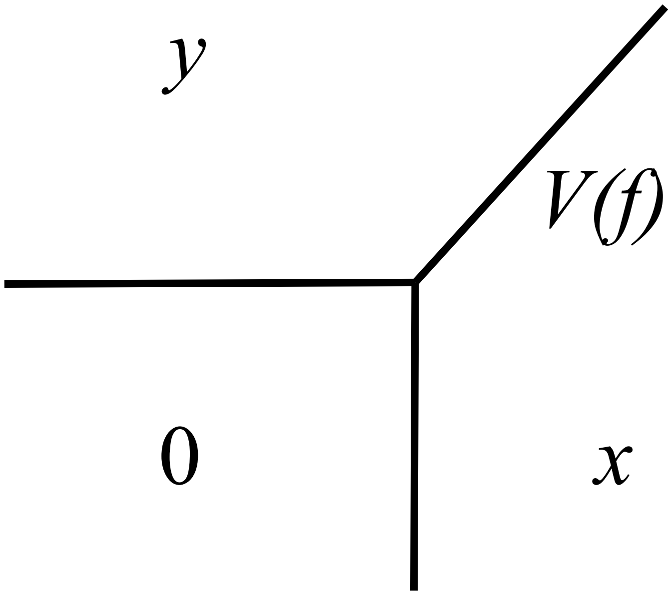
Example – Tropical line

$$f = 0 +_{tr} x +_{tr} y$$





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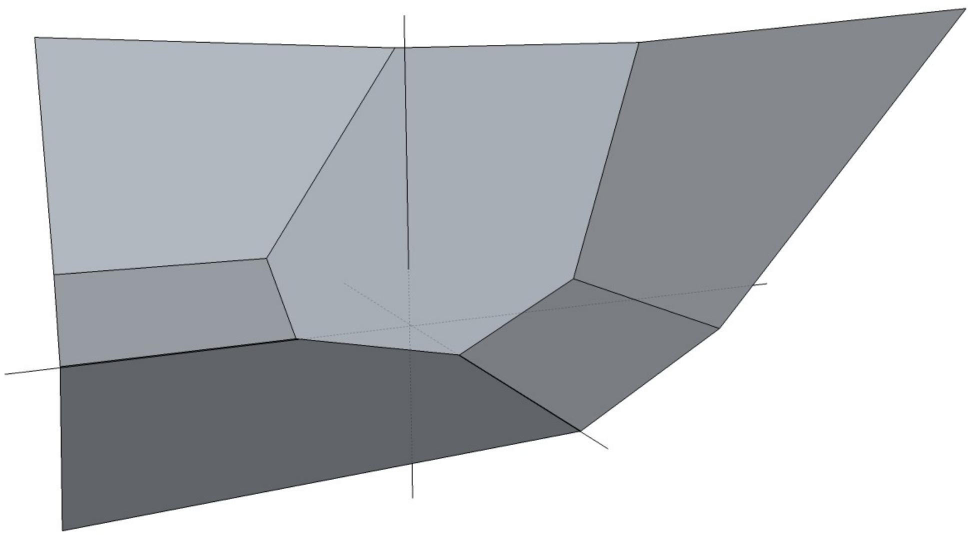
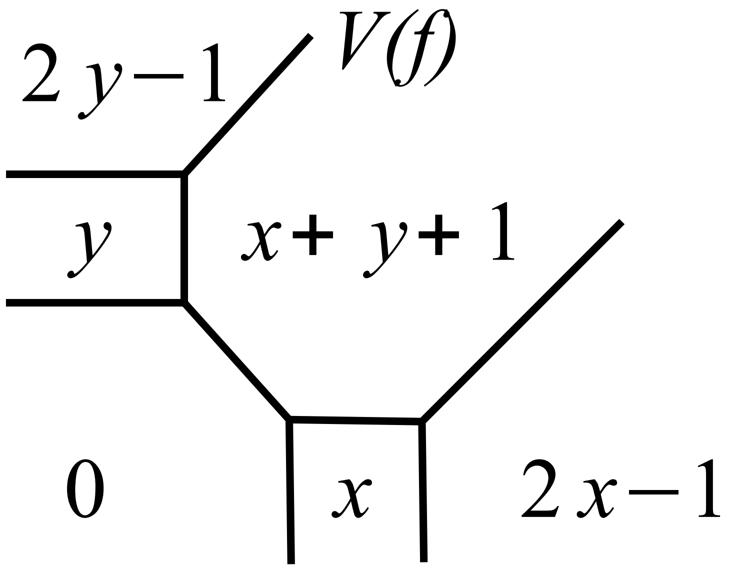
Example – Tropical conic

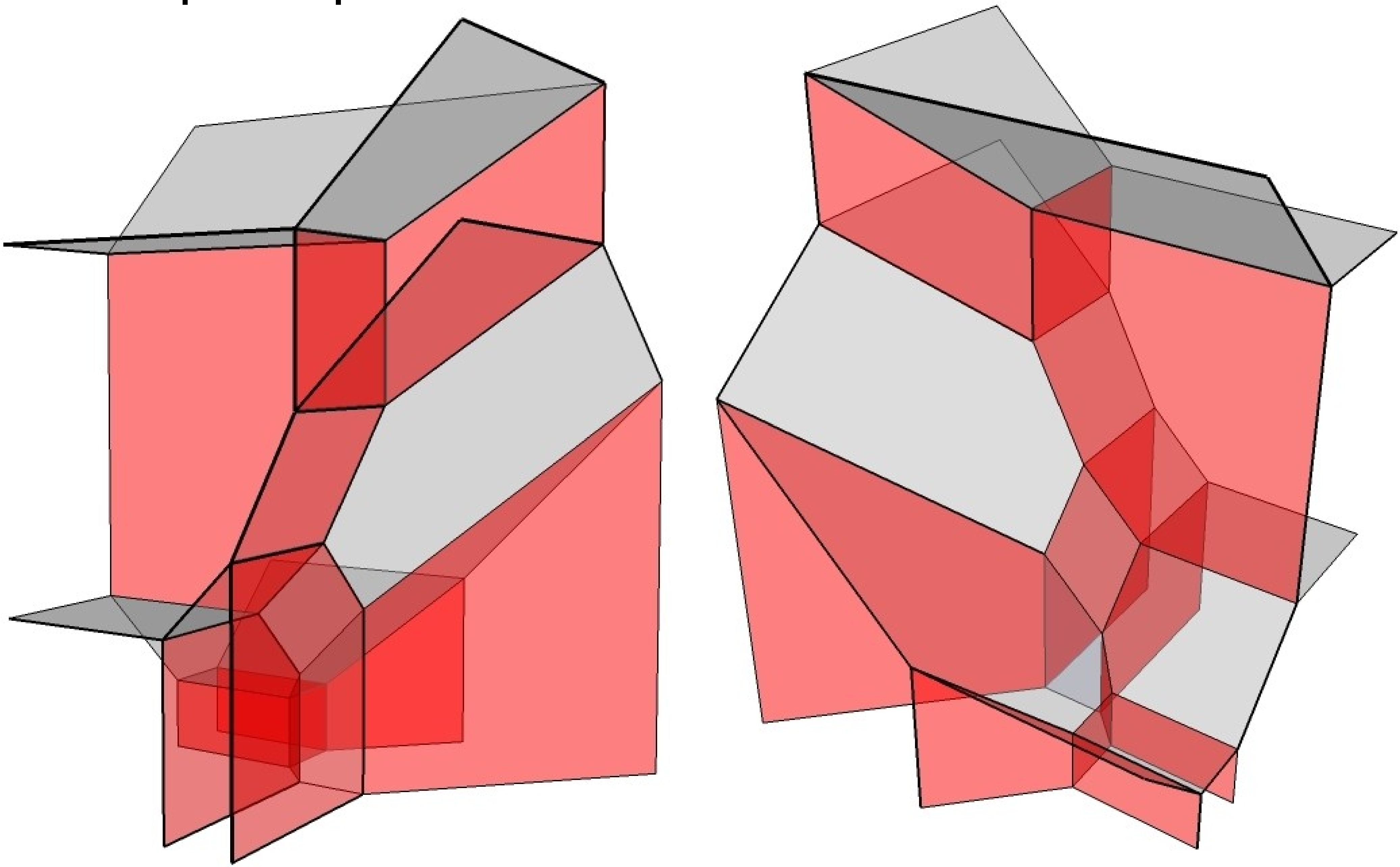
$$f = 0 + x + y + 1xy + (-1)x^2 + (-1)y^2$$





Example

Tropical quartic surface in $\mathbf{R}^3$



Tropical planar curves

$$C = V(f) \subset \mathbf{R}^2$$

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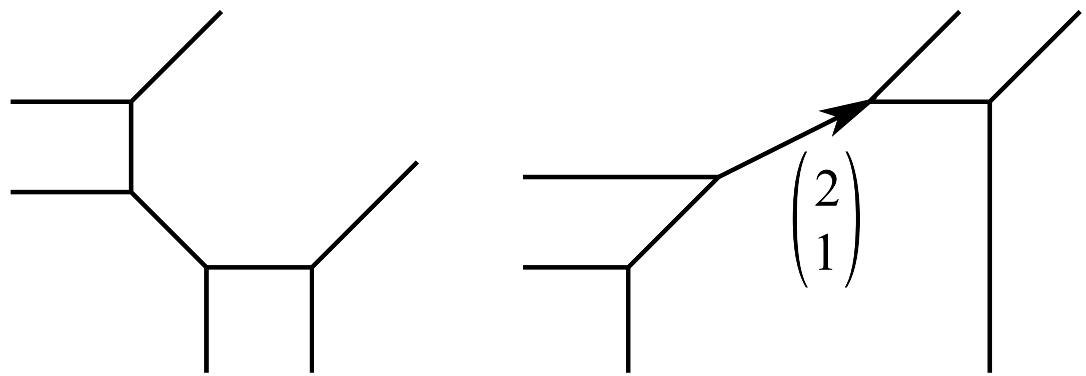
Properties

- Piecewise linearly embedded graph (with rays)
- Edges have rational slopes
  → Direction vectors in $\mathbf{Z}^2$



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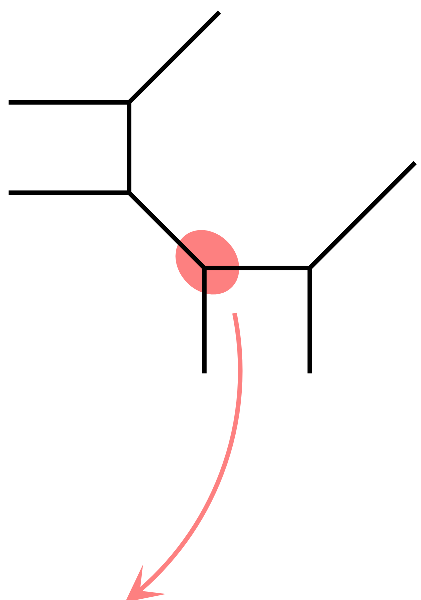
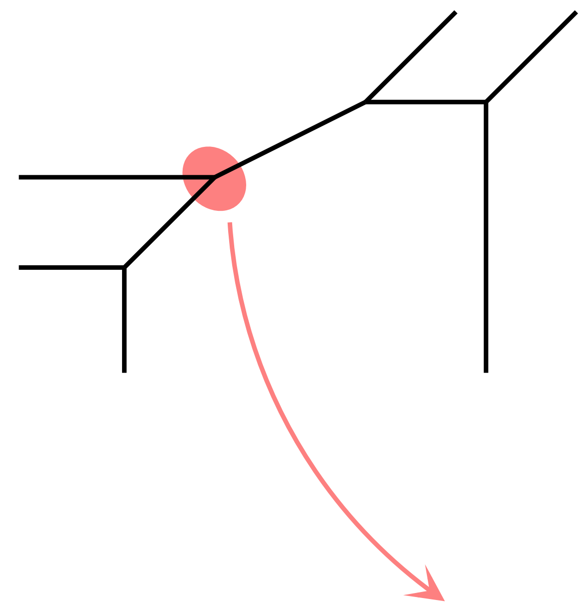
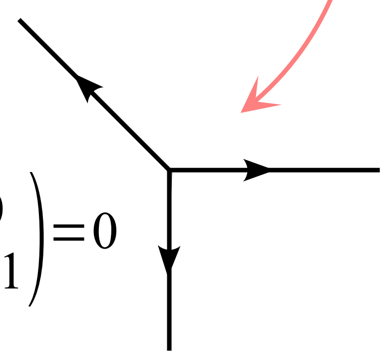
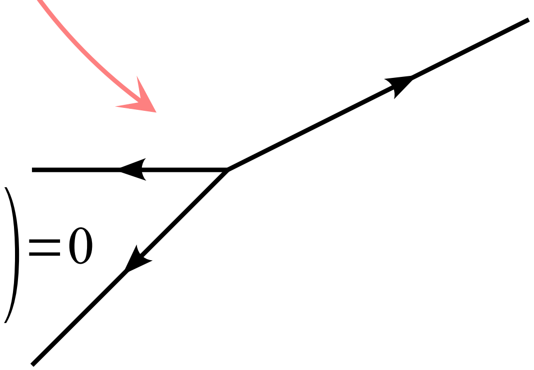




- Balancing condition

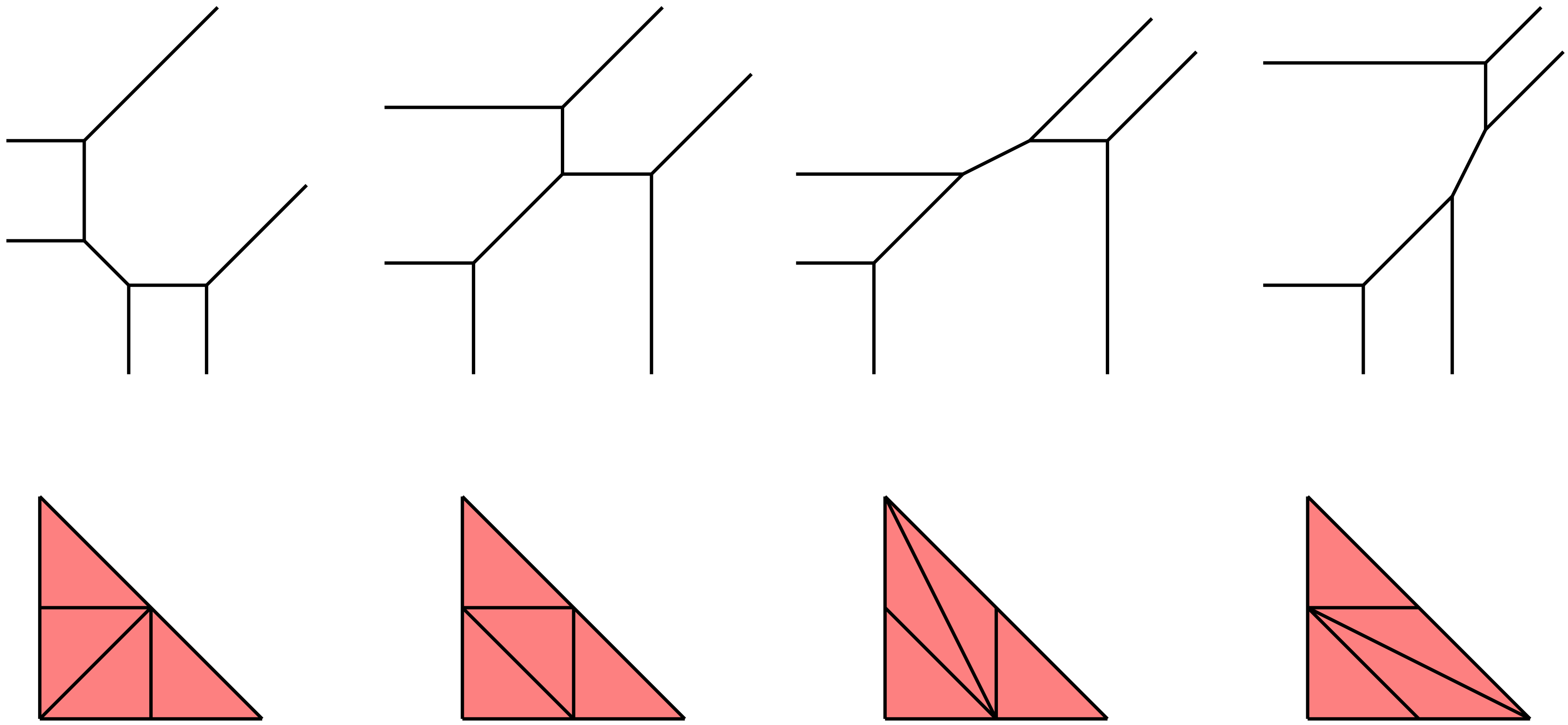


$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0$$

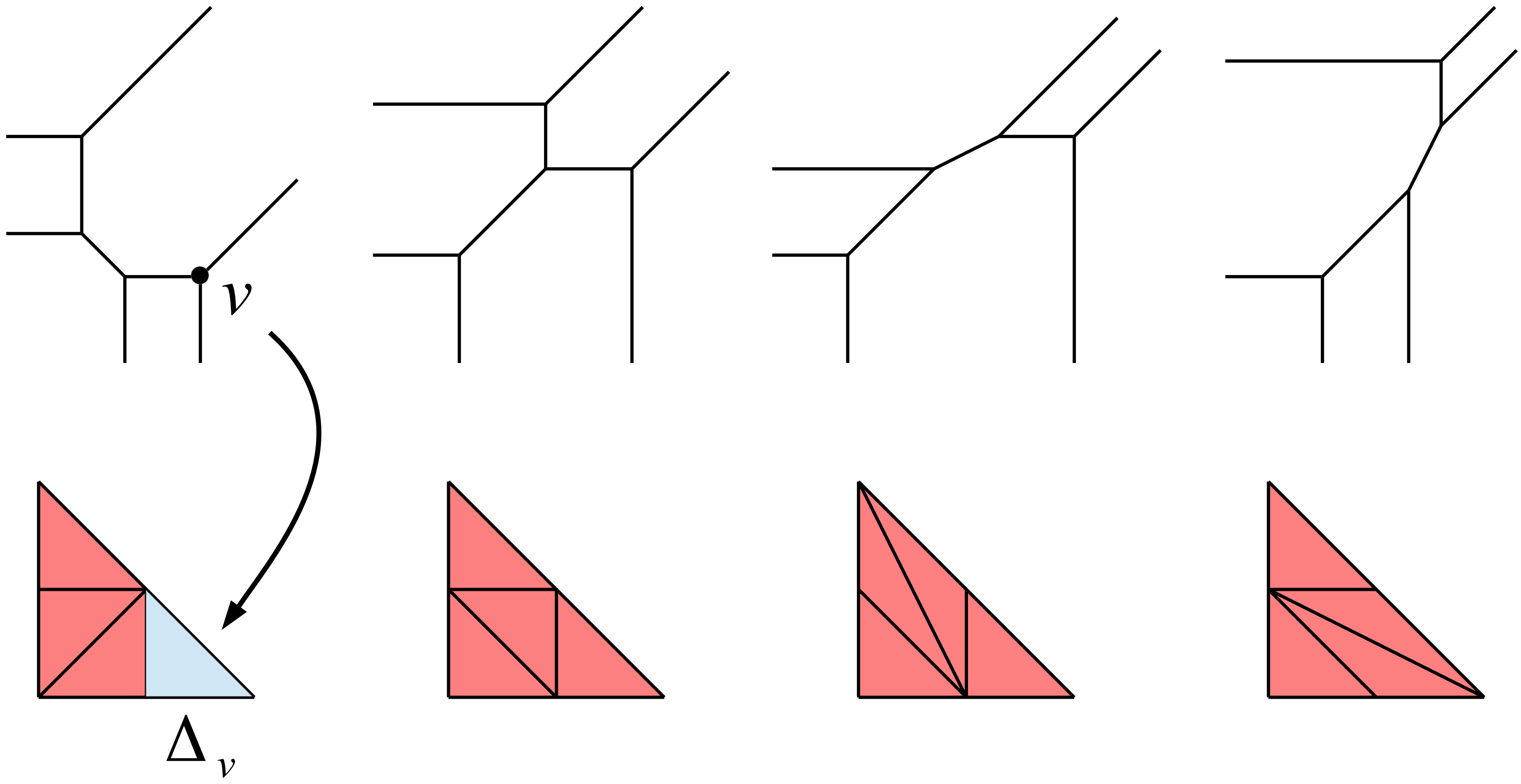


$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0$$

Dual subdivision



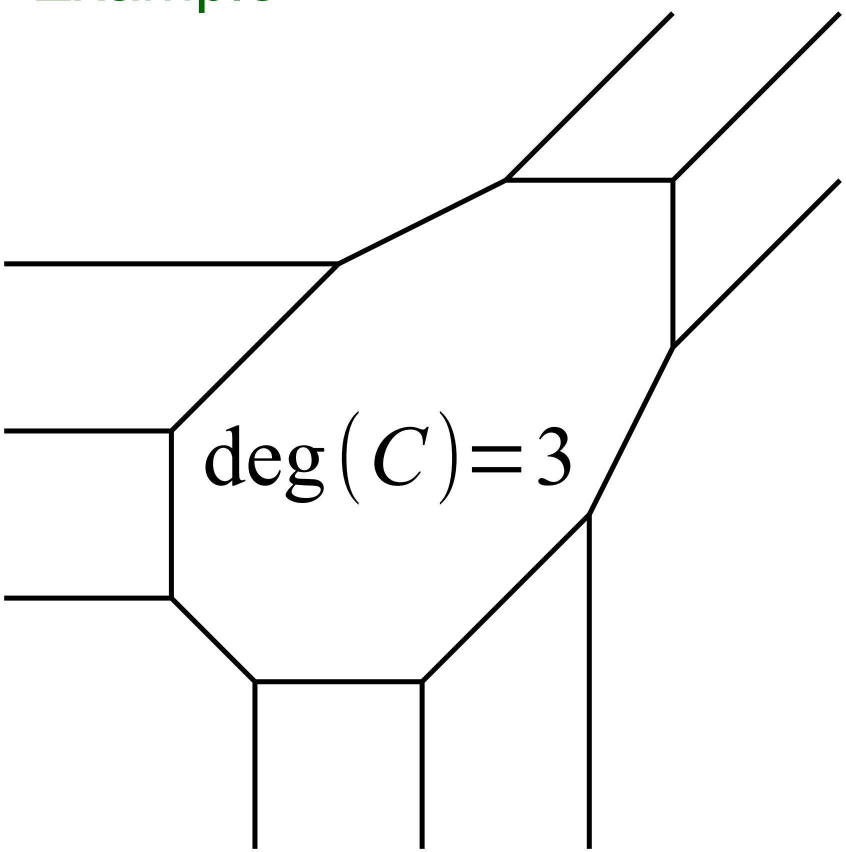
Dual subdivision

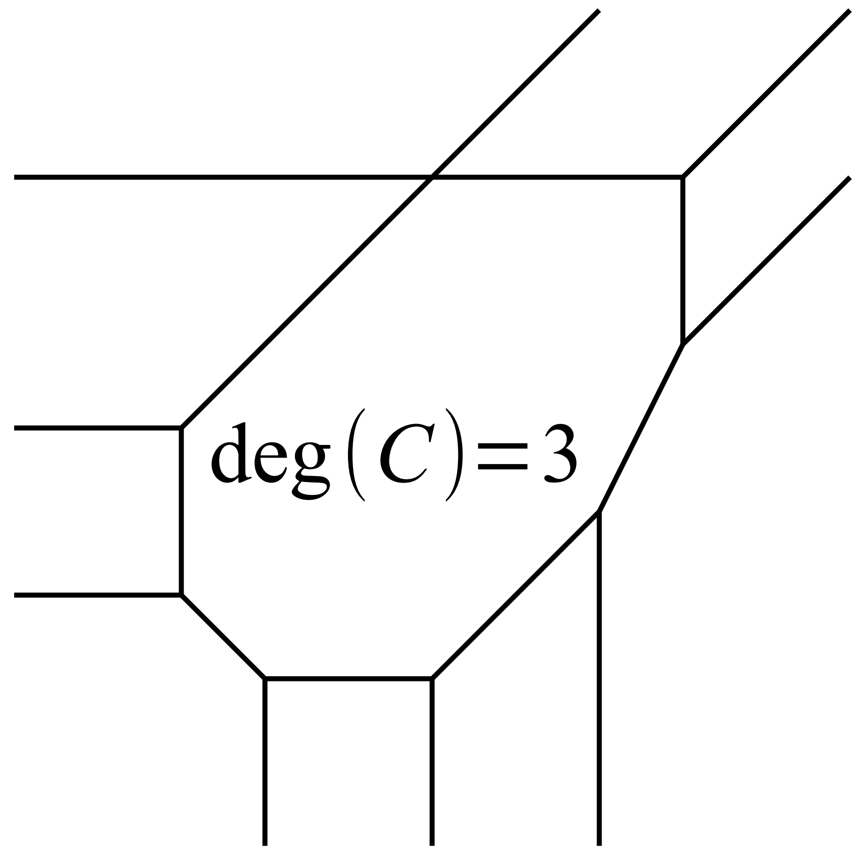


Degree and Genus

$$\deg(C) := \deg(f)$$

Example

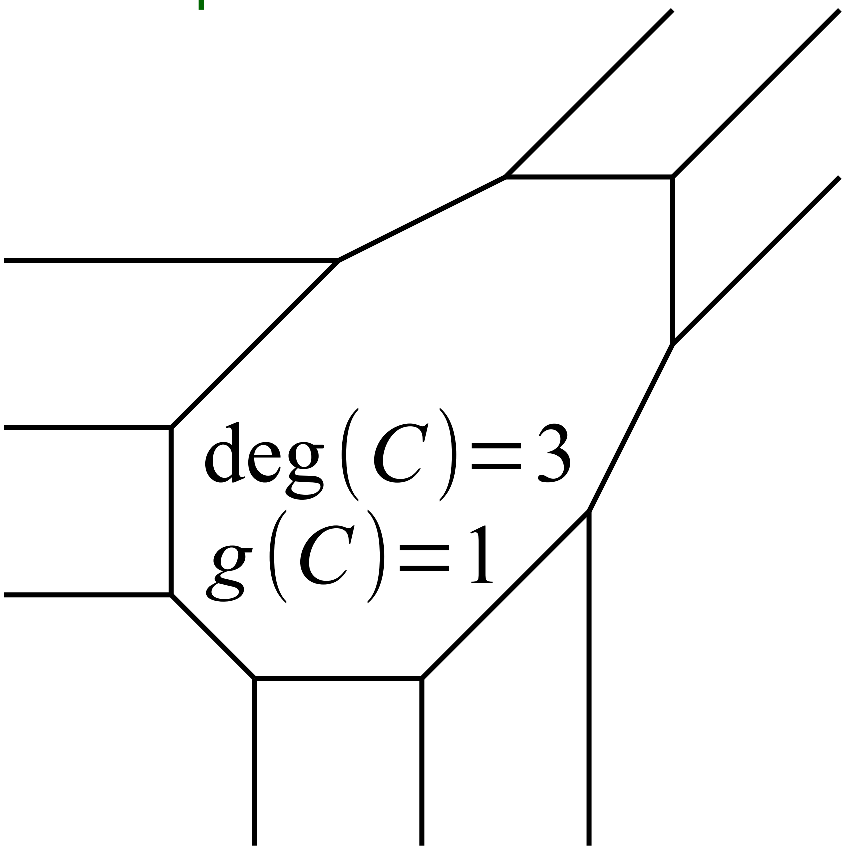


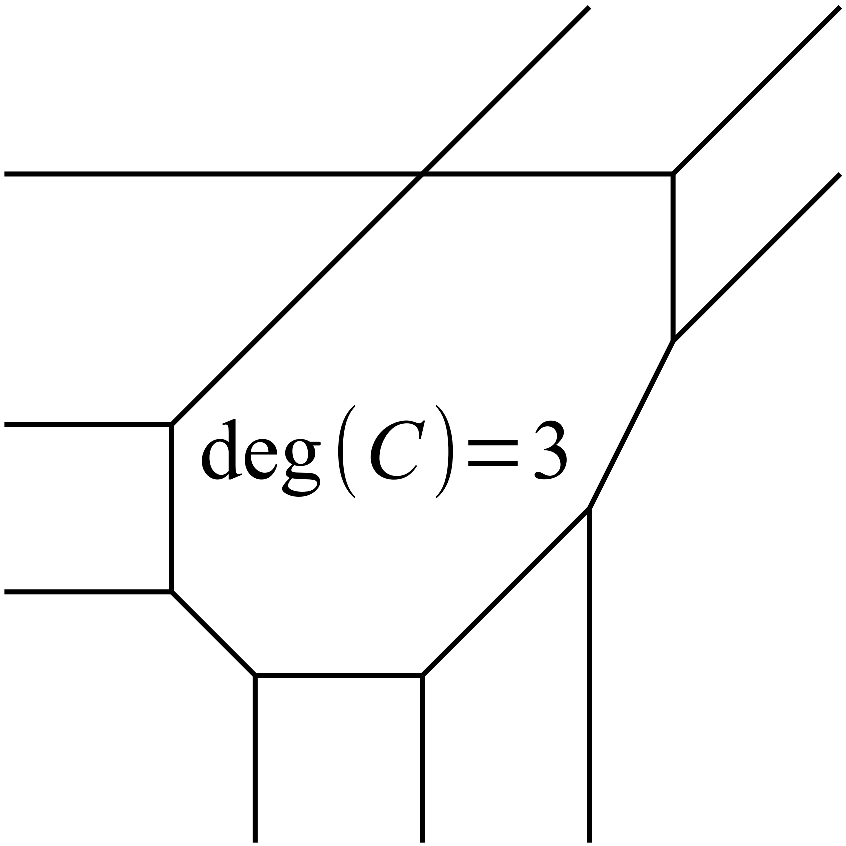


Degree and Genus

$\deg(C) := \deg(f)$ $\qquad$ $g(C) := b_1(C)$

Example





Degree and Genus

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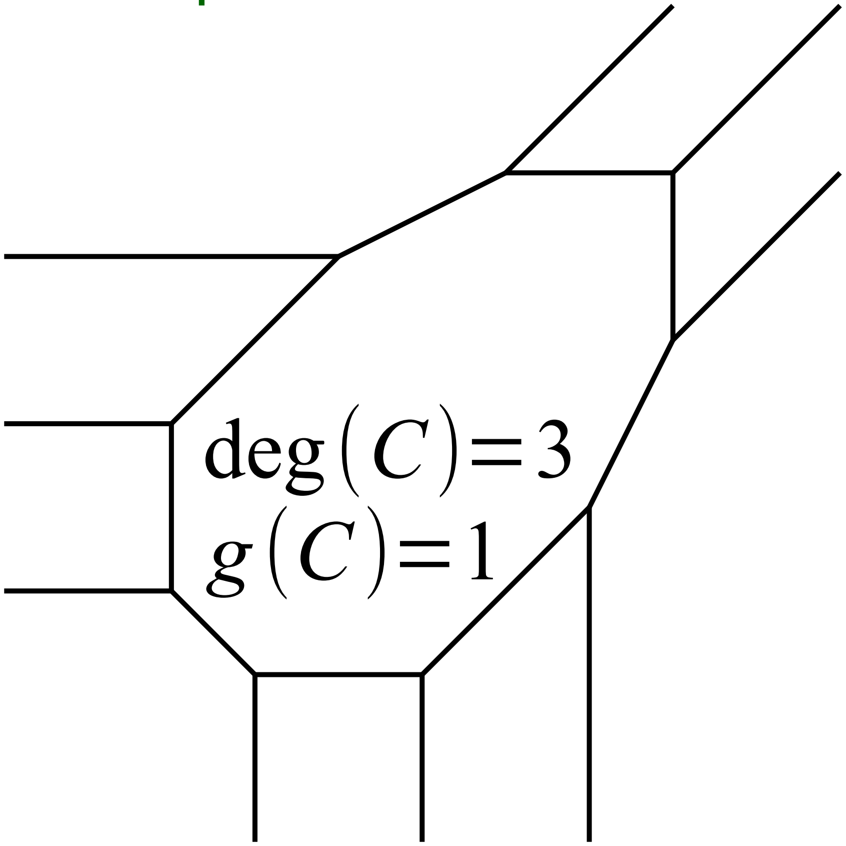
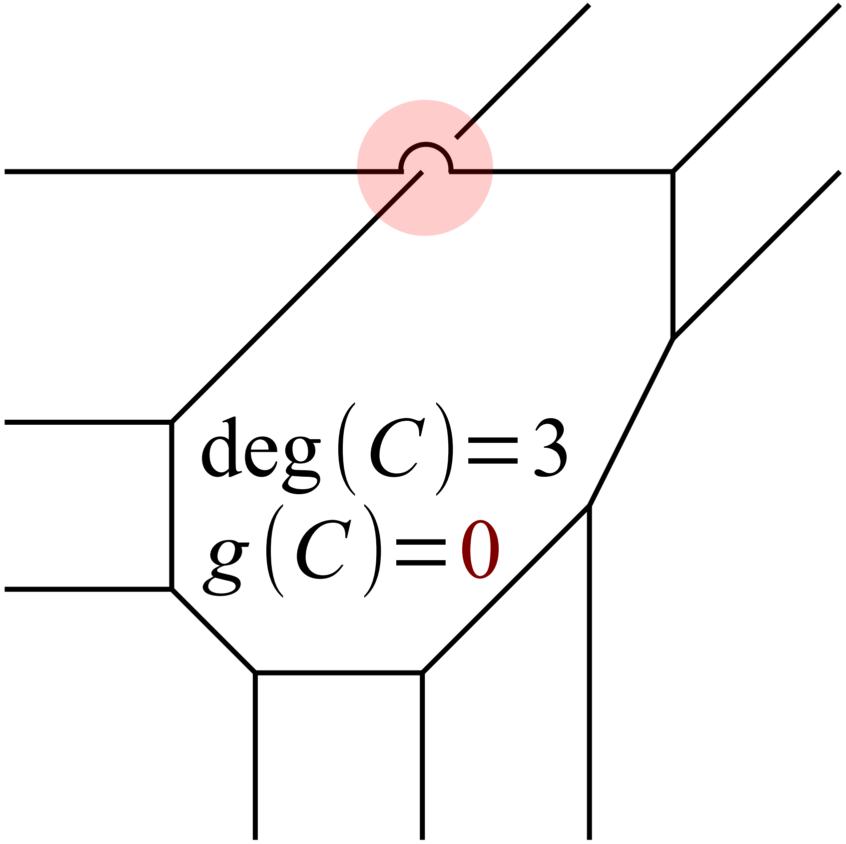
Γ minimal parametrisation

Example





Input $F_t = \sum A_{ij}(t)\, z^i w^j \qquad \in \mathbf{C}[z, w]$

Family (in t) of complex polynomials such that

$$A_{ij}(t) \sim c_{ij} t^{a_{ij}} \qquad t \to \infty .$$

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$$\mathrm{Log}_t : (\mathbf{C}^\times)^2 \to \mathbf{R}^2, (z, w) \to (\log_t |z|, \log_t |w|)$$

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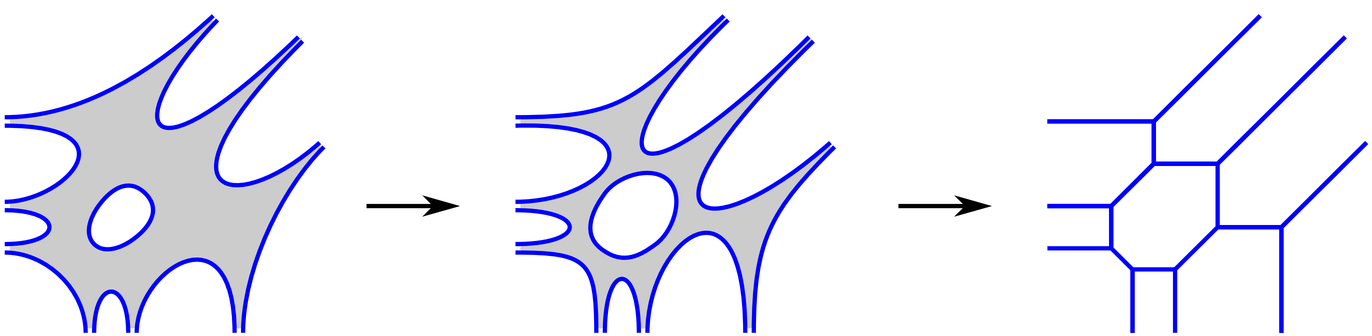
Example

$d = 3$



B

Enumerative Geometry

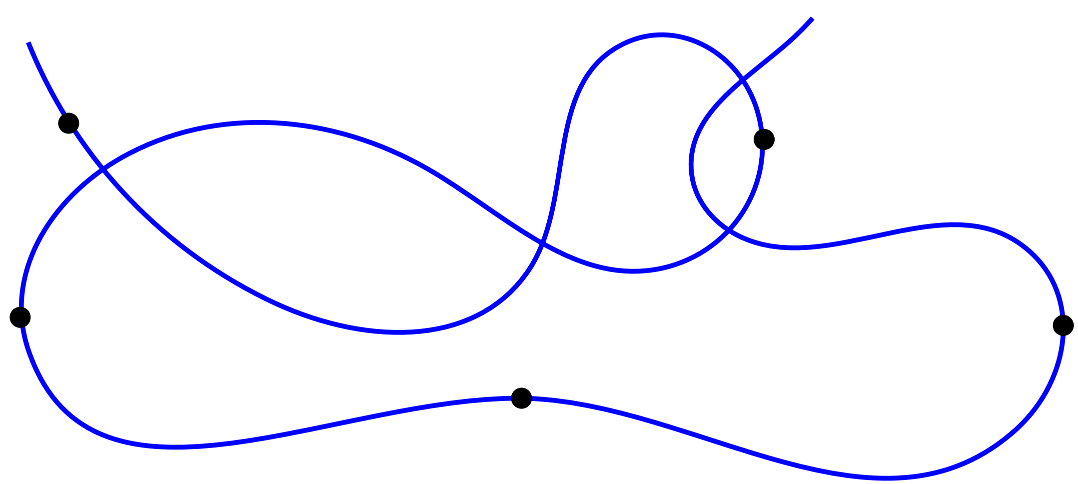
Problem 1

Planar curves passing through points

Fix $d \in \mathbf{N}, \quad n := 3d-1,$

$P_1, \ldots, P_n \in \mathbf{CP}^2$ „generic“ points.

$\mathrm{N}(d) =$ # rational curves in of degree d in $\mathbf{CP}^2$ passing through $P_1, \ldots, P_n$.



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rational

$\leftrightarrow$ Genus g = 0

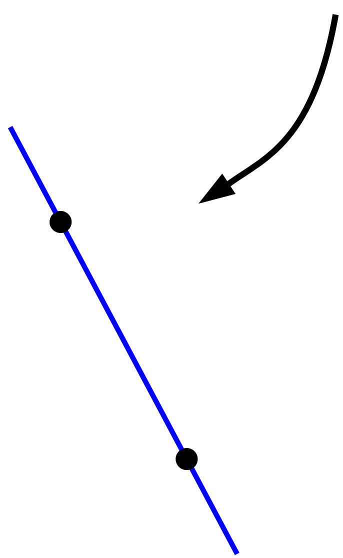
$\leftrightarrow$ parametrised by rational functions $\mathbf{CP}^1 \to \mathbf{CP}^2$.

Problem 1

Planar curves passing through points

$\mathrm{N}(d)$ = # rational curves in of degree d
in $\mathbf{CP}^2$ passing through $3d - 1$ points.

d	1	2	3	4	5	6
$\mathrm{N}(d)$	1					



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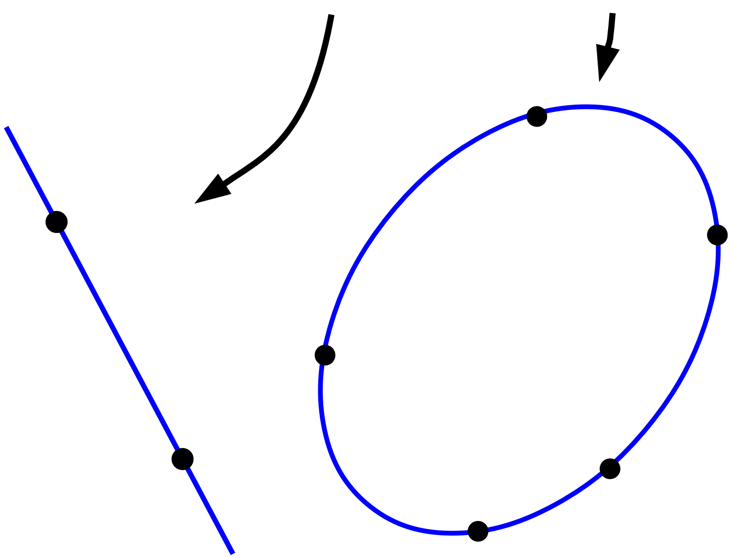

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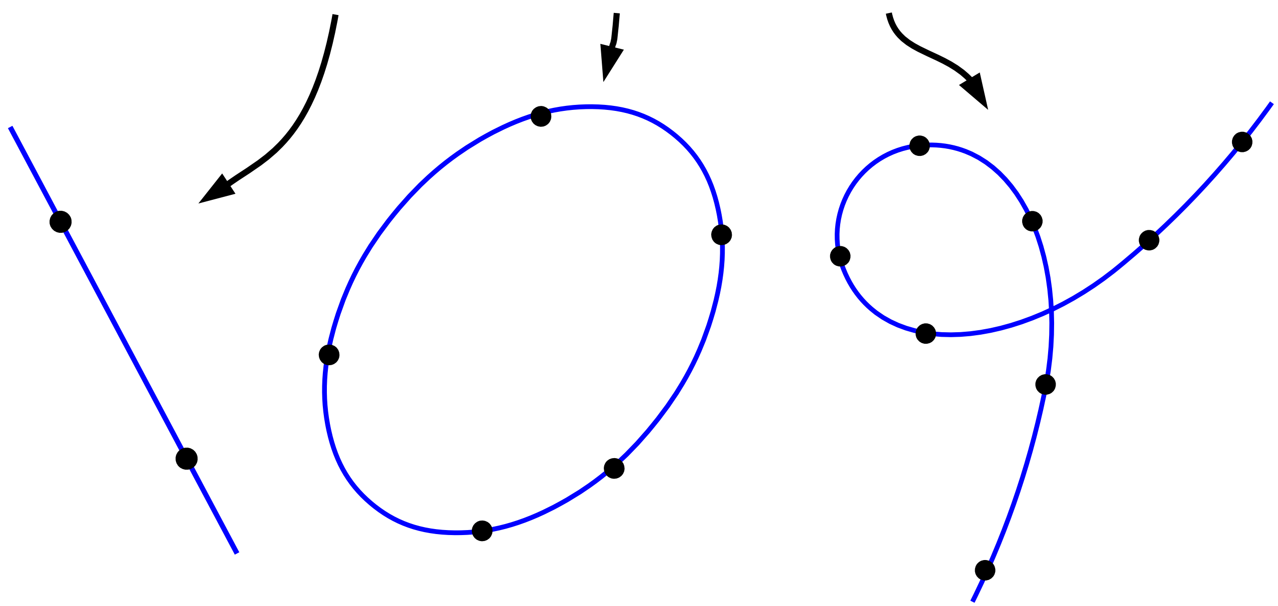
d	1	2	3	4	5	6
$N(d)$	1	1	12	620		



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Planar curves passing through points

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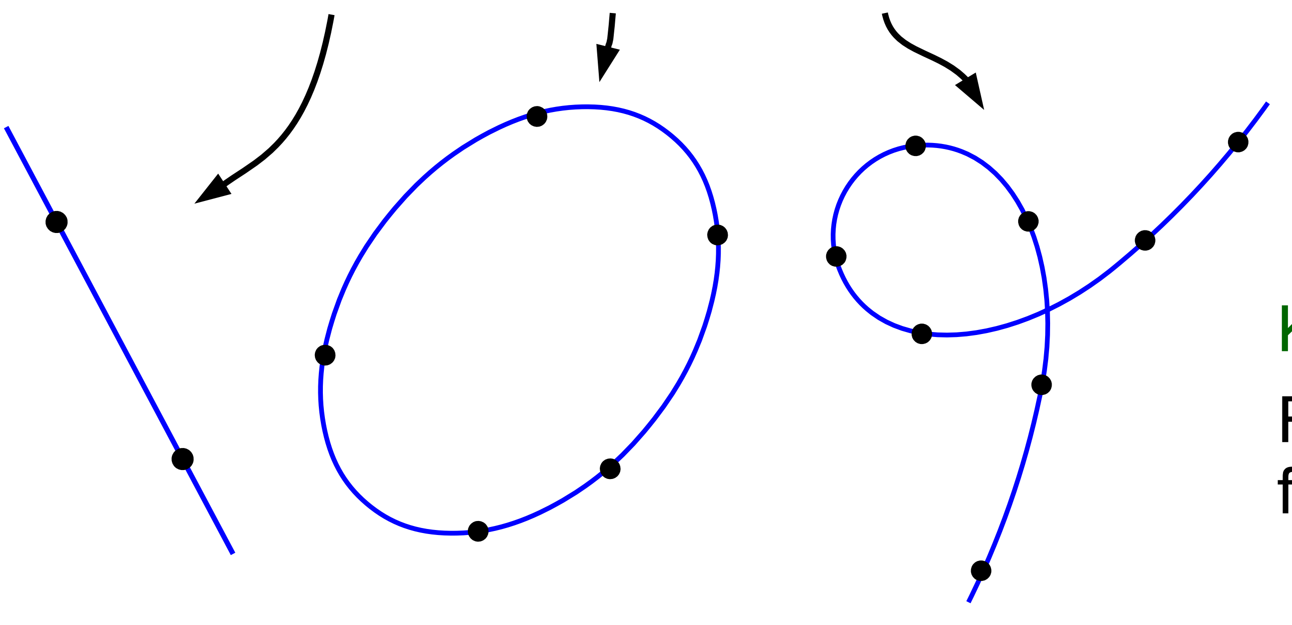
d	1	2	3	4	5	6
$\mathrm{N}(d)$	1	1	12	620	87304	$\sim 26 \cdot 10^6$



Kontsevich '94

Recursive formula for all d.

Problem 2

Gravitational descendants

Fix $d \in \mathbf{N}, k_1, \ldots, k_n \geqslant 0, \quad \sum k_i + 1 = 3d - 1.$

$$\mathrm{N}(d; k_1, \ldots, k_n) = \int_{M_{0,n}(\mathbf{CP}^2, d)} \prod_i \mathrm{ev}_i^*(\mathrm{pt}) \psi_i^{k_i}$$

Psi classes

$\psi_i = c_1(\boldsymbol{L}_i) \in H^2(M_{0,n}(\mathbf{CP}^2, d))$

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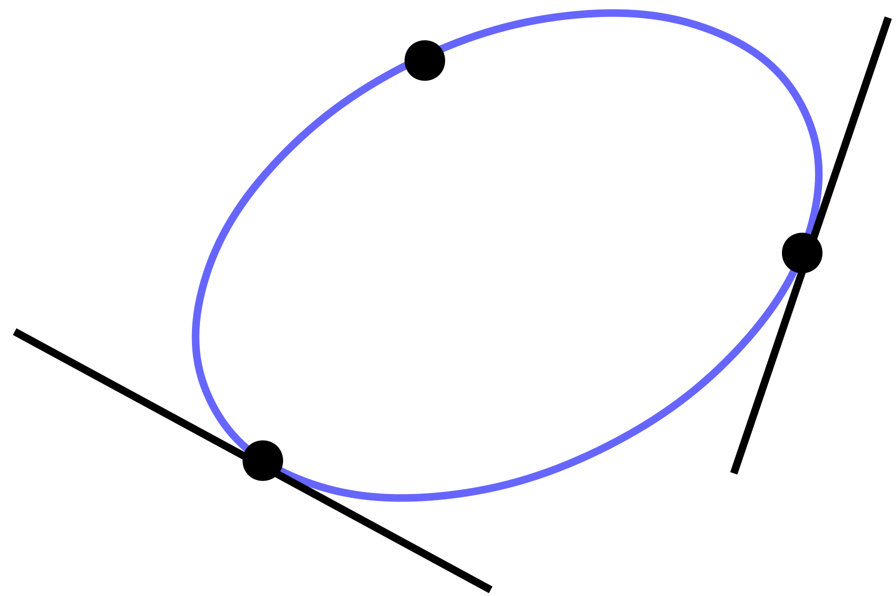
Example

$$N(2; 1, 1, 0) = 1$$



Theorem (Mikhalkin, '02)

$$\mathrm{N}(d) = \sum_{C} \mathrm{mult}(C)$$

- C is tropical curve of degree d, genus 0.
- C passes through generically chosen points $p_1, \ldots, p_n \in \mathbf{R}^2$.
- $\mathrm{mult}(C) = \prod_{v} 2 \cdot \mathrm{Area}(\Delta_v)$

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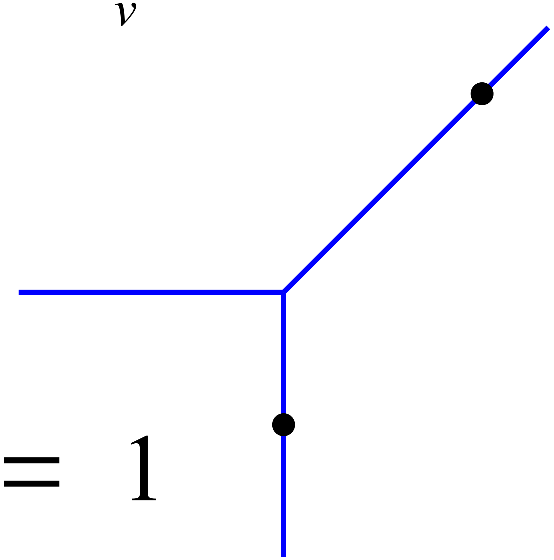
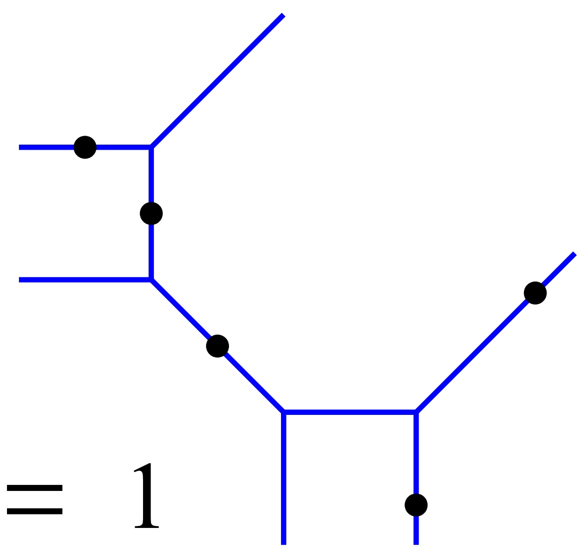


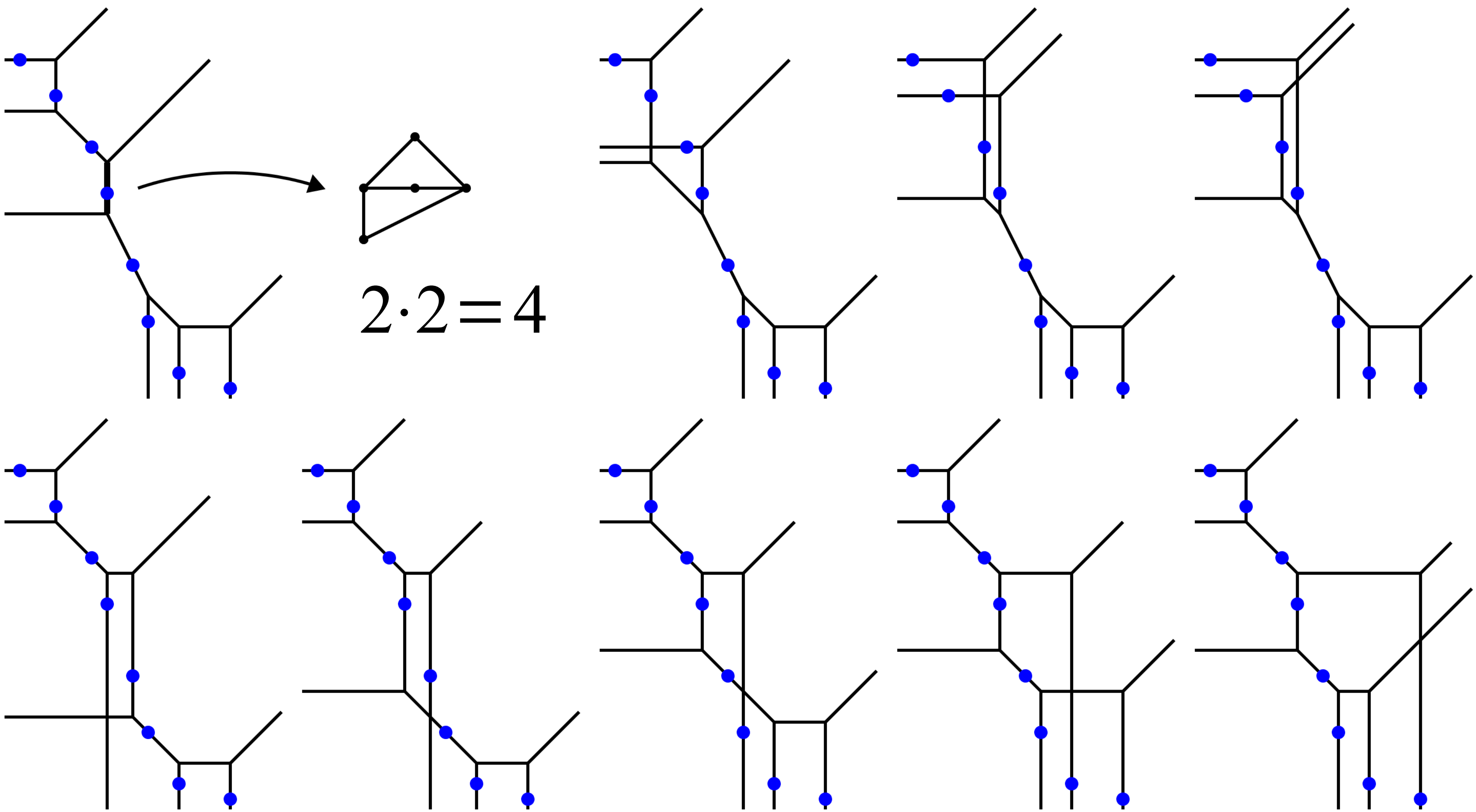
$\mathrm{N}(1) = 1$



$\mathrm{N}(2) = 1$

Example

$$N(3) = 12$$



Theorem (Markwig, R., '08)

$$\mathrm{N}(d; k_1, \ldots, k_n) = \sum_{C} \mathrm{mult}(C)$$

- C is tropical curve of degree d, genus 0.
- C has vertex of valence $k_i + 2$ at p_i.
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Example

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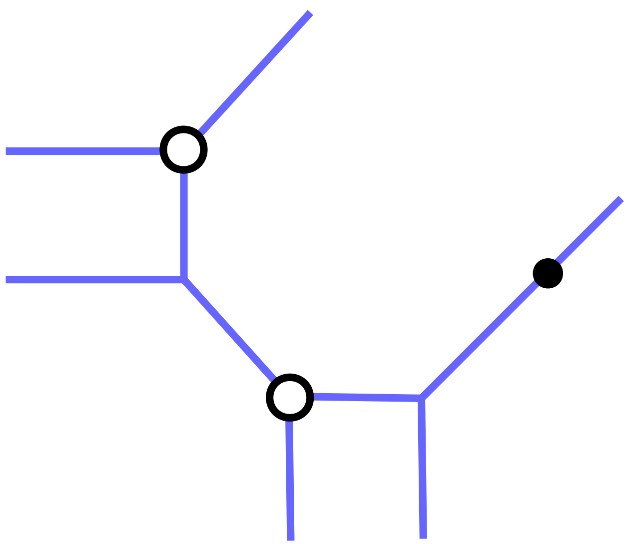

Example

$$N(3;1^2,0^4) = 8$$