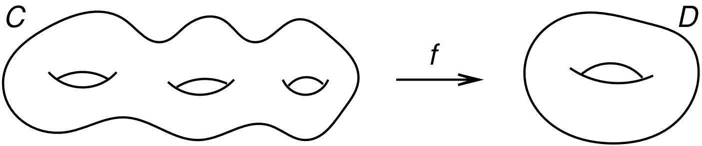
Real Hurwitz numbers
Counting ramified covers with real structures

Hurwitz numbers

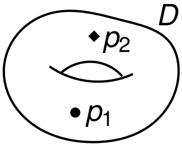
Count ramified covers of Riemann surfaces!



Hurwitz numbers

Count ramified covers of Riemann surfaces!

- Input: Surface D, $p_1, \dots, p_n \in D$, $g \in \mathbb{N}_0$, $d \in \mathbb{N}$,
  $\mu_1, \dots, \mu_n$ partitions of d



$\mu_1 = (3, 2, 1)$

Hurwitz numbers

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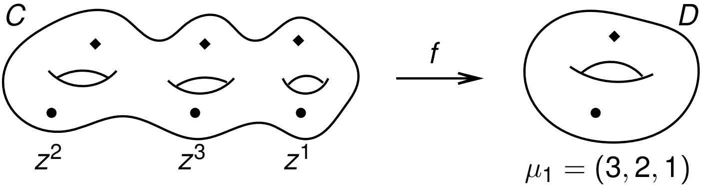
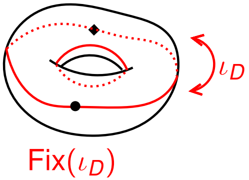
- Input: Surface D, $p_1, \dots, p_n \in D$, $g \in \mathbb{N}_0$, $d \in \mathbb{N}$,
  $\mu_1, \dots, \mu_n$ partitions of d
- Consider $f : C \to D$ with $g(C) = g$ of degree d with ramification profile μ_i over p_i.

$$H_g^{\mathbb{C}}(D, \mu) := \#\{(C, f)\}/\text{isom}.$$



Real Hurwitz numbers

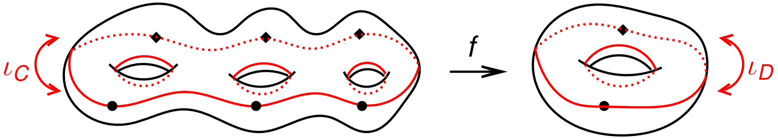
- Fix **real structure**:
  $\iota_D : D \to D$ anti-holomorphic involution ($\iota_D^2 = \text{id}$)



Real Hurwitz numbers

- Fix **real structure**:
  $\iota_D : D \to D$ anti-holomorphic involution ($\iota_D^2 = \mathrm{id}$)
- Consider **real covers** (C, f, ι_C) with

$$\iota_D \circ f = f \circ \iota_C.$$



$$H_g^{\mathbb{R}}((D, \iota_C), \mathcal{P}, \mu) := \#\{(C, f, \iota_C)\}/\text{real isom.}$$

Theorem (Markwig, Guay-Paquet, R., 2014)

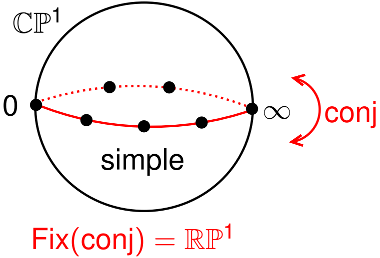
- *Tropical degeneration formula*

$$H_g^{\mathbb{R}}((D,\iota_C),\mathcal{P},\mu) = \sum \text{ products of ``smaller'' } H_g^{\mathbb{R}}(\ldots)$$

- *Explicit combinatorial calculation for real double Hurwitz numbers*
- *Some results on combinatorics, generating functions of these numbers*

Similar results for $H_g^{\mathbb{C}}(D,\mu)$ by [Cavalieri-Johnson-Markwig] and [Bertrand-Brugallé-Mikhalkin]

Double Hurwitz numbers



- $D = \mathbb{CP}^1$, $\iota_D =$ conj, Fix$(\iota_D) = \mathbb{RP}^1$
- $\mathcal{P} = \{0, \infty, s$ points in $\mathbb{R}_+$, remaining points in $\mathbb{R}_-\}$
- ramification profile λ for 0, ν for ∞ (arbitrary)
  simple ramification, i.e. $(21\ldots1)$, at all other points

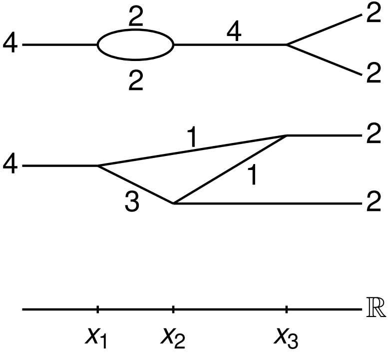
$$H^{\mathbb{R}}_g(s, \lambda, \nu)$$

Tropical ramified covers

Fix $x_i \in \mathbb{R}$.

A **tropical cover** is given by

- a trivalent metric graph Γ,
- a pw. integer-linear map $f : \Gamma \to \mathbb{R}$, i.e. on each edge e $z \mapsto \omega_e z, \omega_e \in \mathbb{Z}$,



here: $g = 1$, $\lambda = (4)$, $\mu = (22)$

Tropical ramified covers

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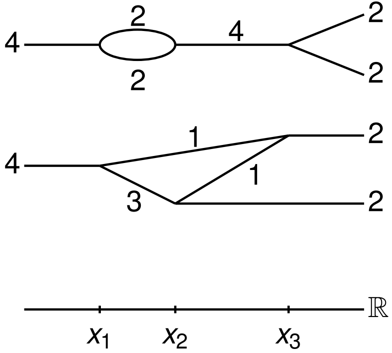
such that

- vertices are mapped one-to-one to marks x_i,
- the balancing condition holds at each vertex.

$$\sum_{e \text{ incoming}} \omega_e = \sum_{e \text{ outgoing}} \omega_e$$

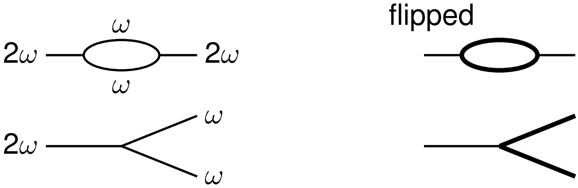


here: $g = 1$, $\lambda = (4)$, $\mu = (22)$

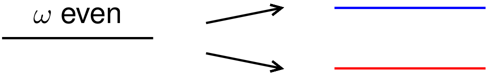
Real tropical covers

A **real coloring** of $f : \Gamma \to \mathbb{R}$.
We can

- **flip** some balanced wieners and forks,



- color (unflipped) even edges in two ways, blue and red,

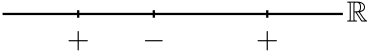


such that blue and red edges do not touch!

Admissible vertices

Choose a number of s of the marks x_i to be **positive**, the others are **negative.**

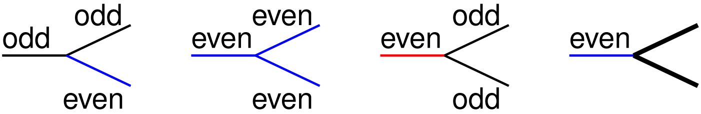
$s = 2$



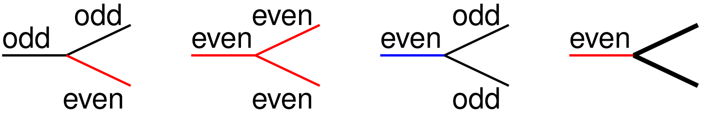
Admissible vertices

Choose a number of s of the marks x_i to be **positive**, the others are **negative.**

Over a positive x_i we only allow:



Over a negative x_i we only allow:



Theorem (Markwig, Guay-Paquet, R., 2014)

We count ***real tropical covers*** *(i.e. tropical covers with real coloring) with the multiplicity*

$$2^{E-B} \cdot \prod_{i=1}^{n} \omega_i,$$

where

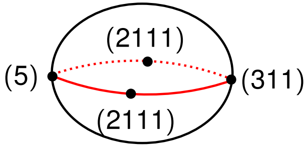
- $E = \#$ *even bounded unflipped edges,*
- $B = \#$ *balanced forks and wieners,*
- *the product runs over all flipped wieners (with edge weight* ω_i*).*

This count equals $H_g^{\mathbb{R}}(s, \lambda, \nu)$*.*

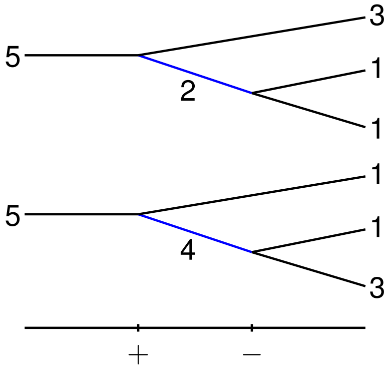
Example

$g = 0, \lambda = (5), \nu = (311)$

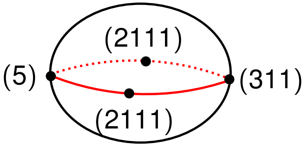
s=1



$$H_g^{\mathbb{R}}(1, \lambda, \nu) = 1 + 2 = 3$$



s=1



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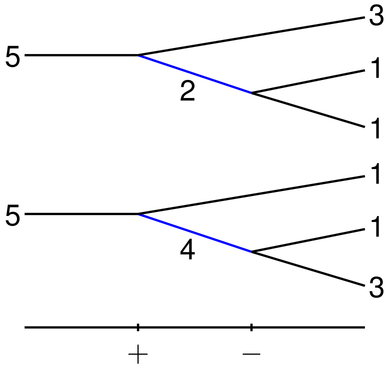

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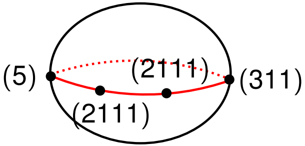
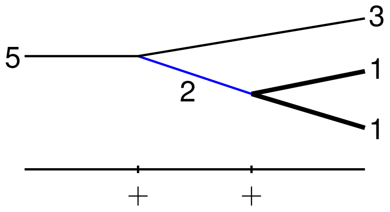
s=2



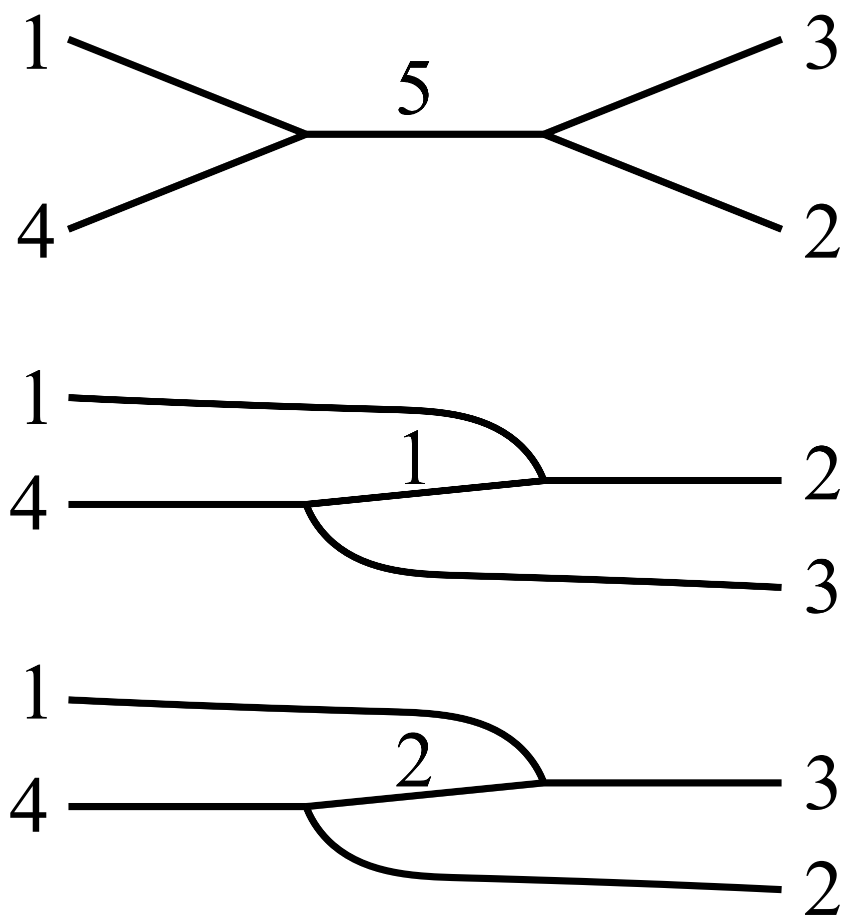
$H_g^{\mathbb{R}}(2, \lambda, \nu) = 1$



Beispiel

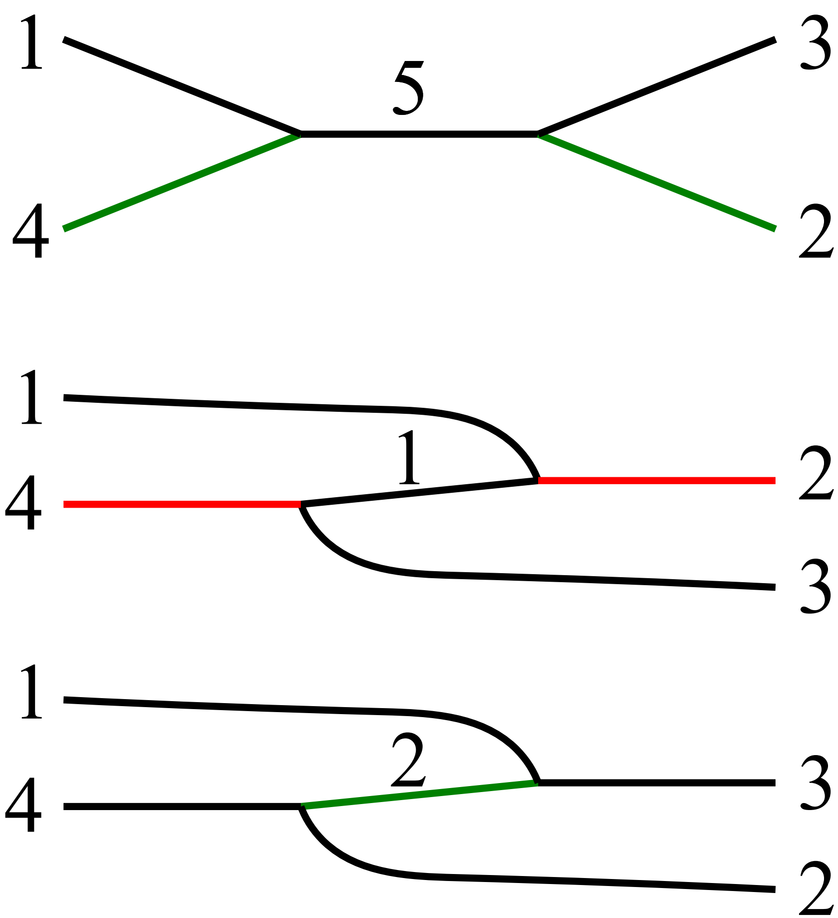
$g = 0, d = 5$

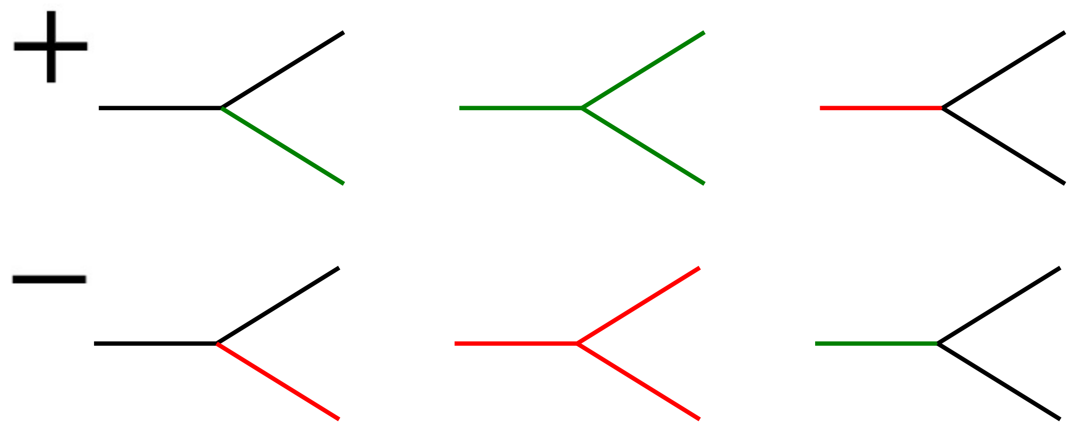
$\lambda=(4,1), \mu = (3,2)$



Beispiel

$$m = 2$$
$$v = (++)$$





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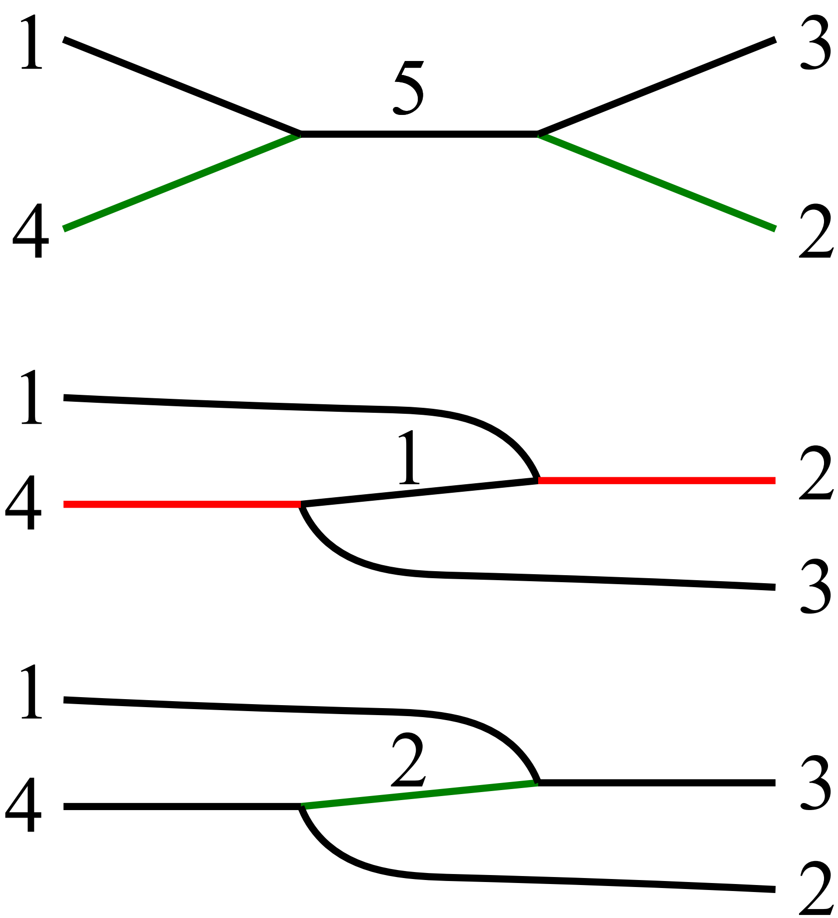
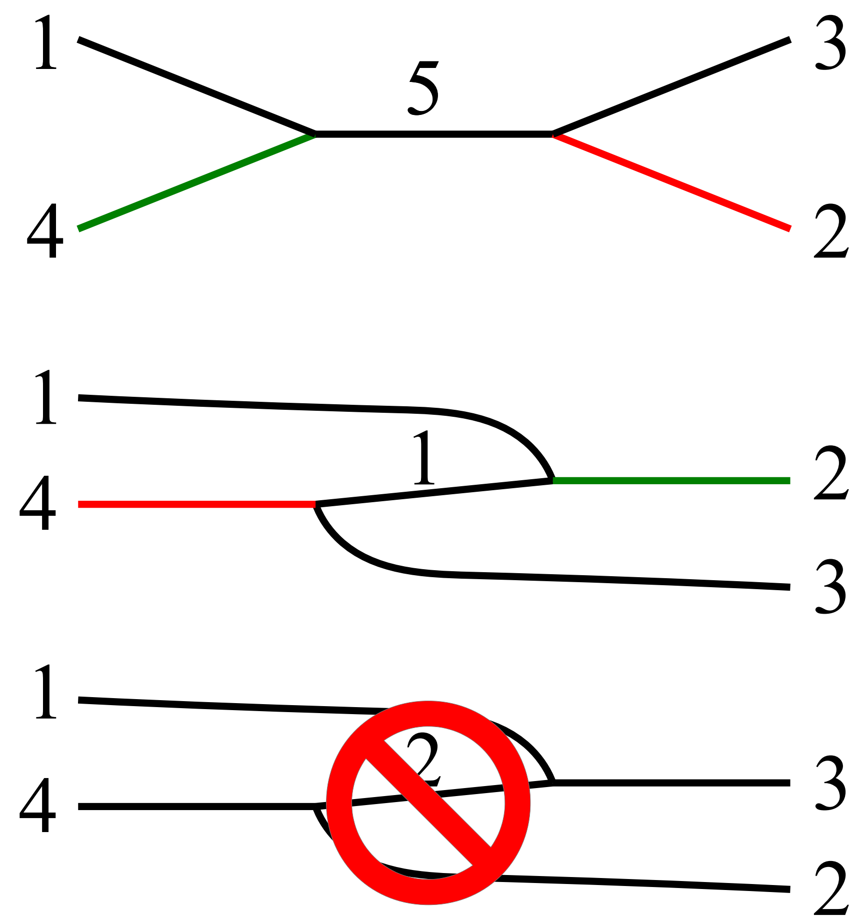


$1 + 1 + 2 = 4$

$m = 1$

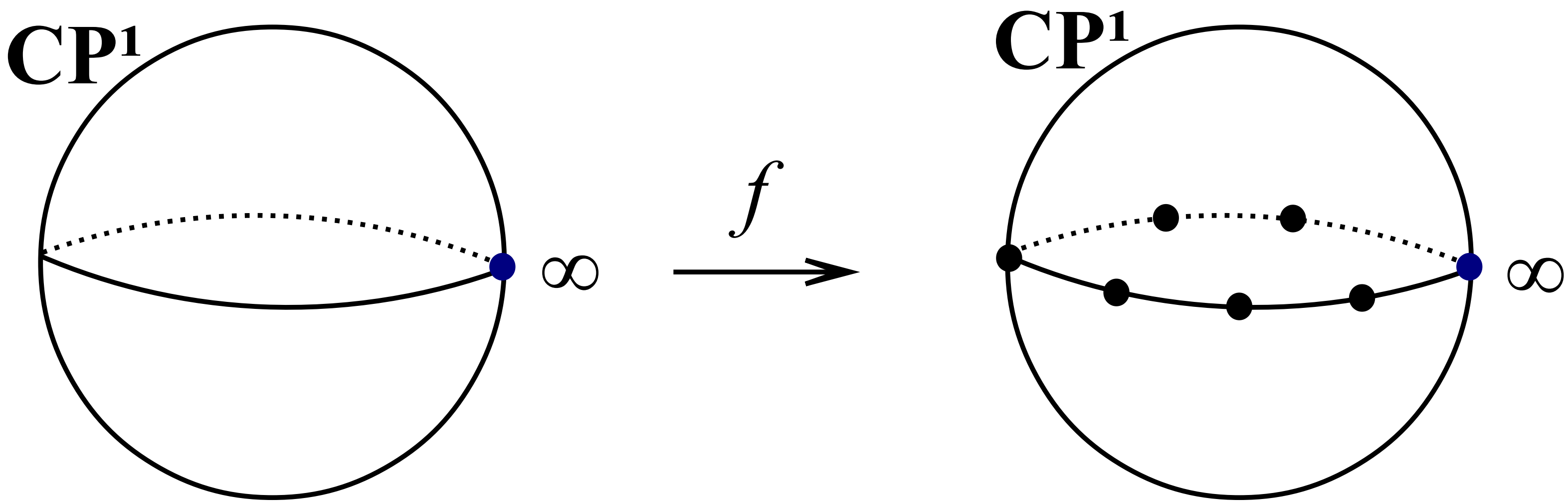
$v = (+-)$



$1 + 1 + 0 = 2$

Spezialfall: Polynome

$g = 0$ und $\lambda_\infty = (d)$ bzw. $f^{-1}(\infty) = \{\infty\}$



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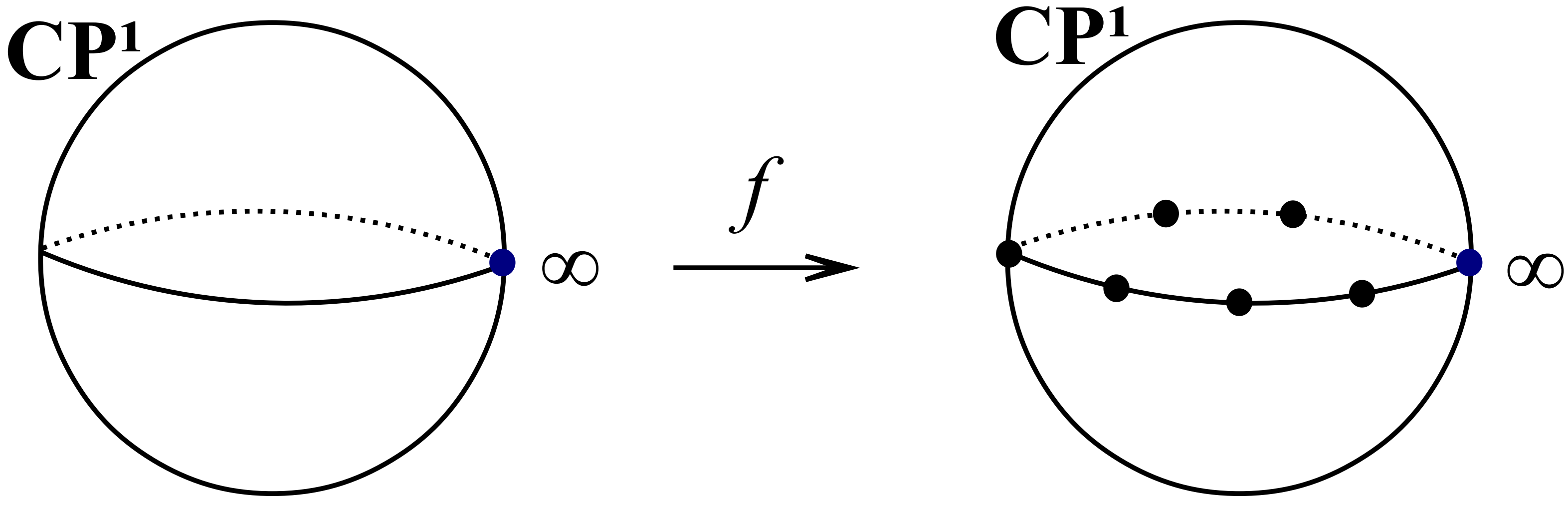

$$\Rightarrow f \in \mathbf{C}[x]$$

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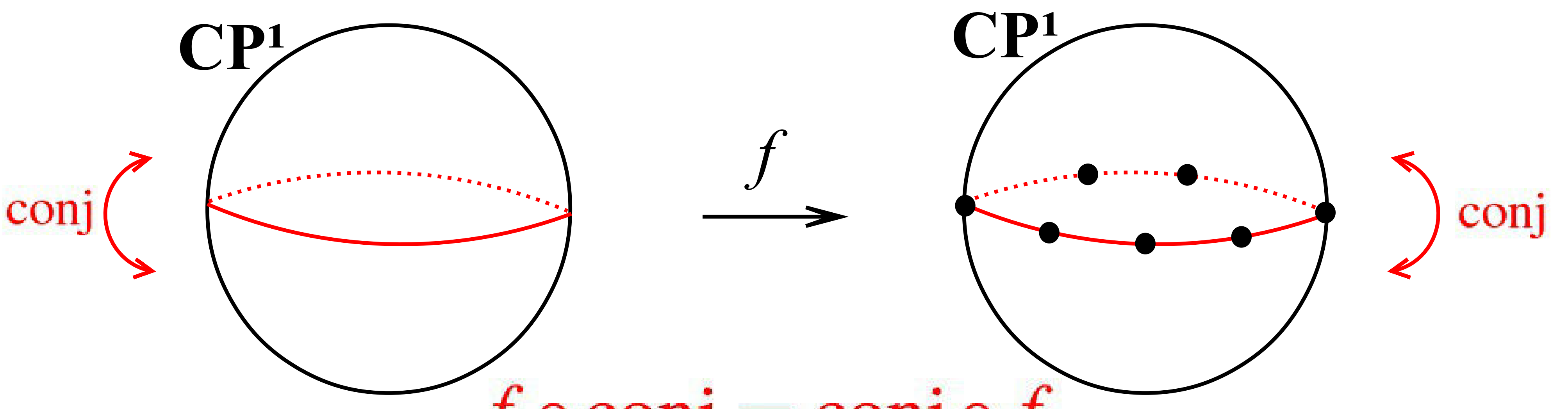

$$f \circ \mathrm{conj} = \mathrm{conj} \circ f$$

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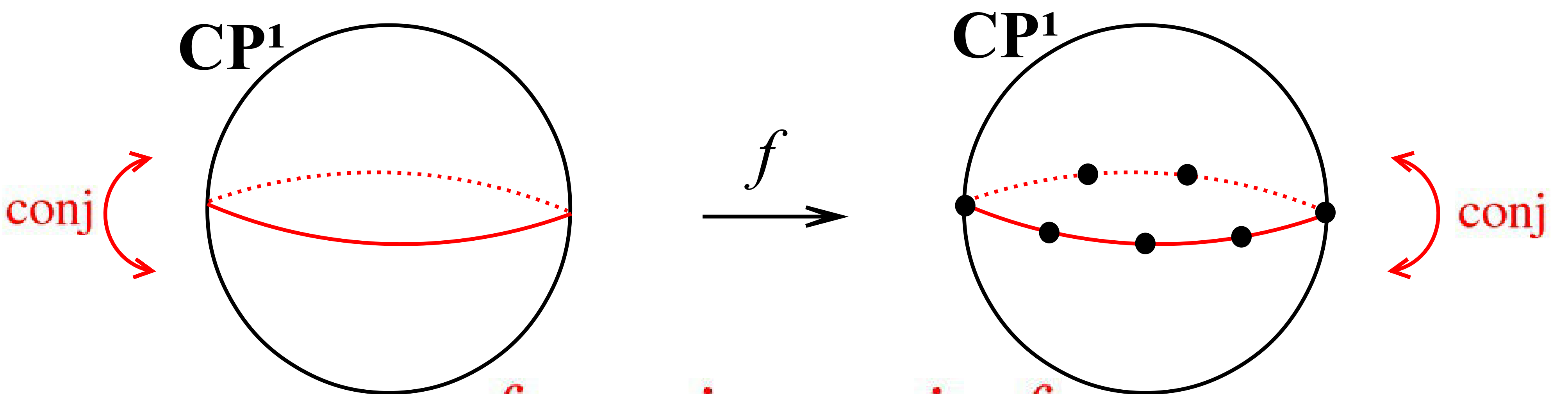

$$f \circ \mathrm{conj} = \mathrm{conj} \circ f$$

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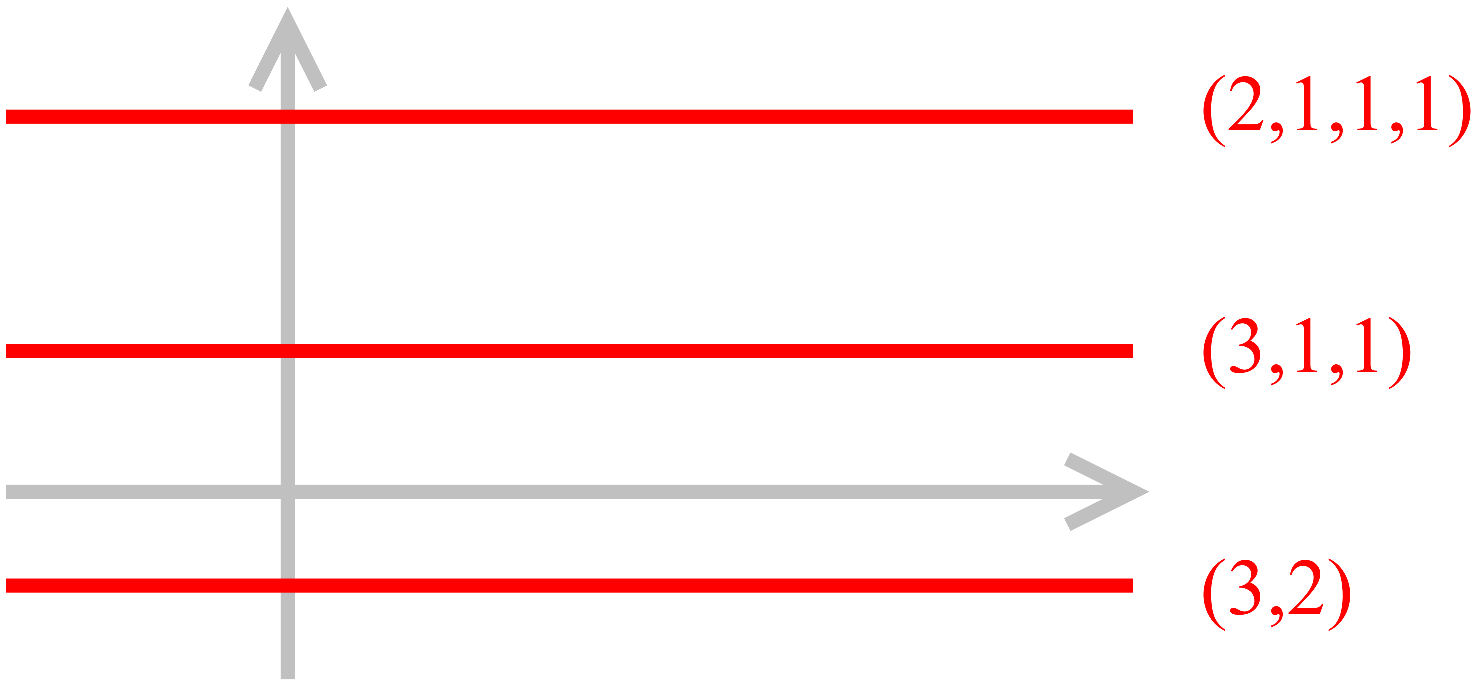
Polynome zählen



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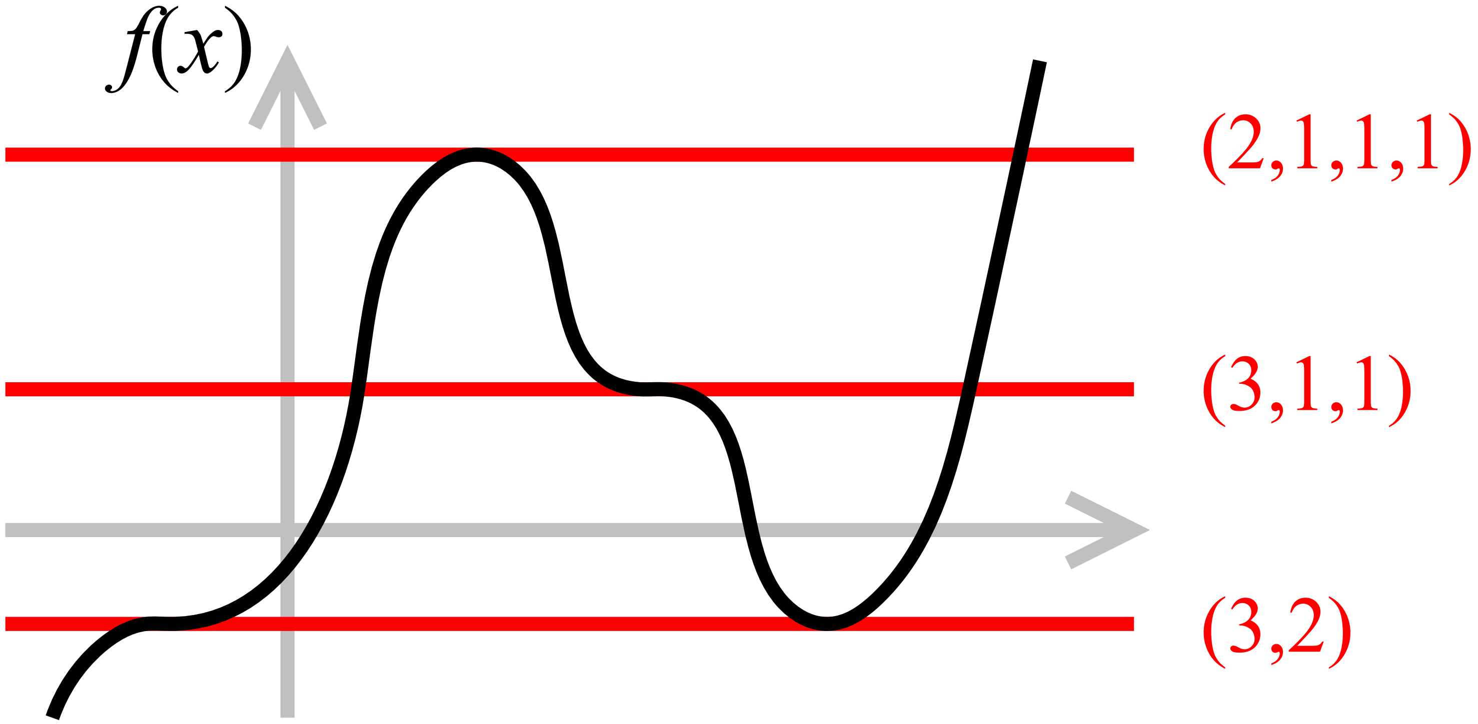
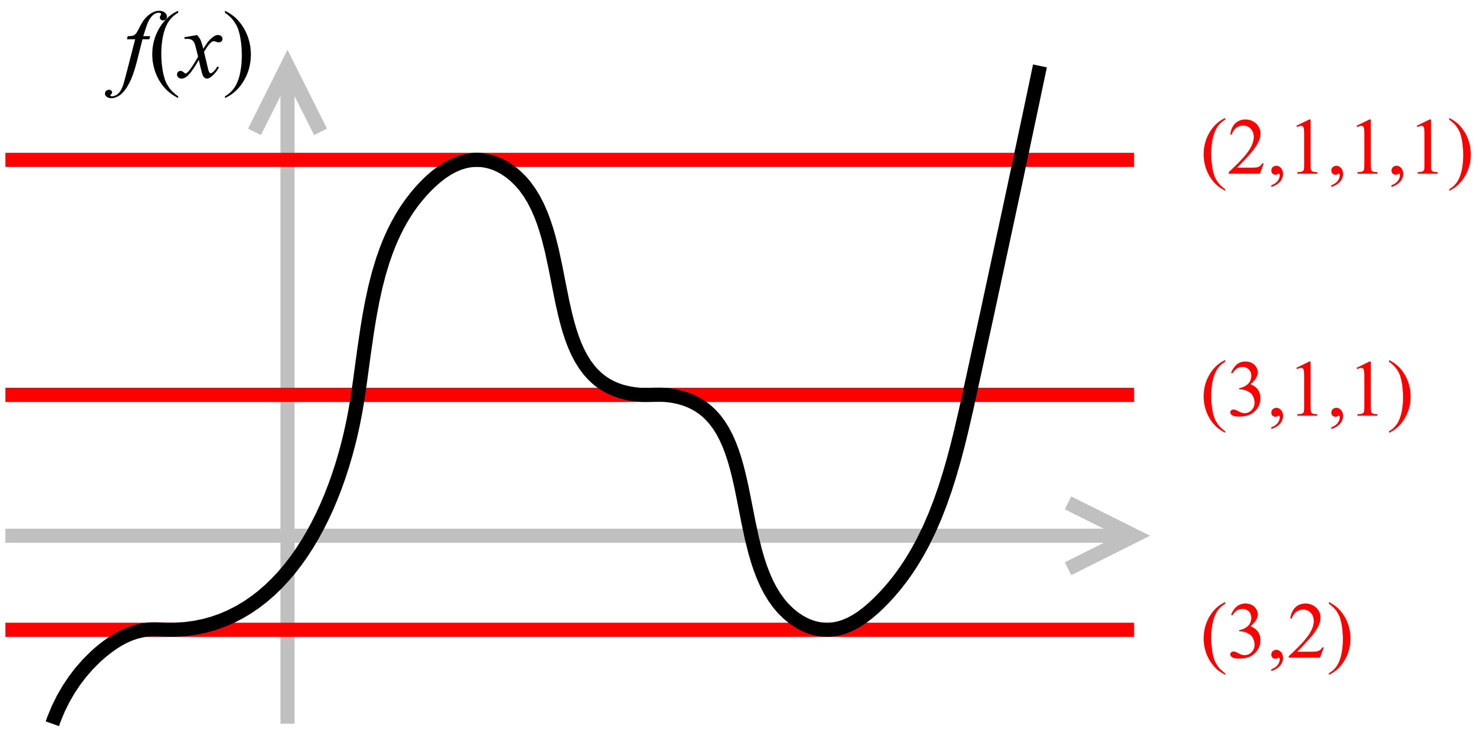


Polynome zählen



Frage

Wie viele solche $f(x)$ in $\mathbf{R}[x]$ gibt es?

Modulo Koordinatenwechsel $x \mapsto ax + b$

Beispiel

Nur Verzweigungsprofil $s = (2,1,\ldots,1)$
„Einfacher Verzweigungspunkt“

Hier außerdem: $\lim_{x \to -\infty} f(x) = -\infty$

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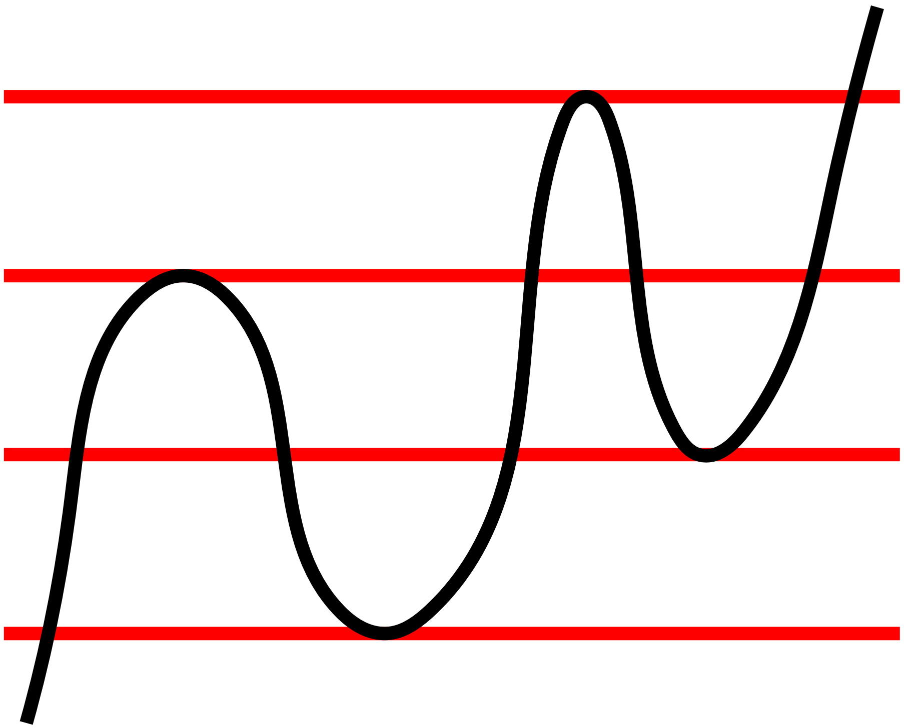
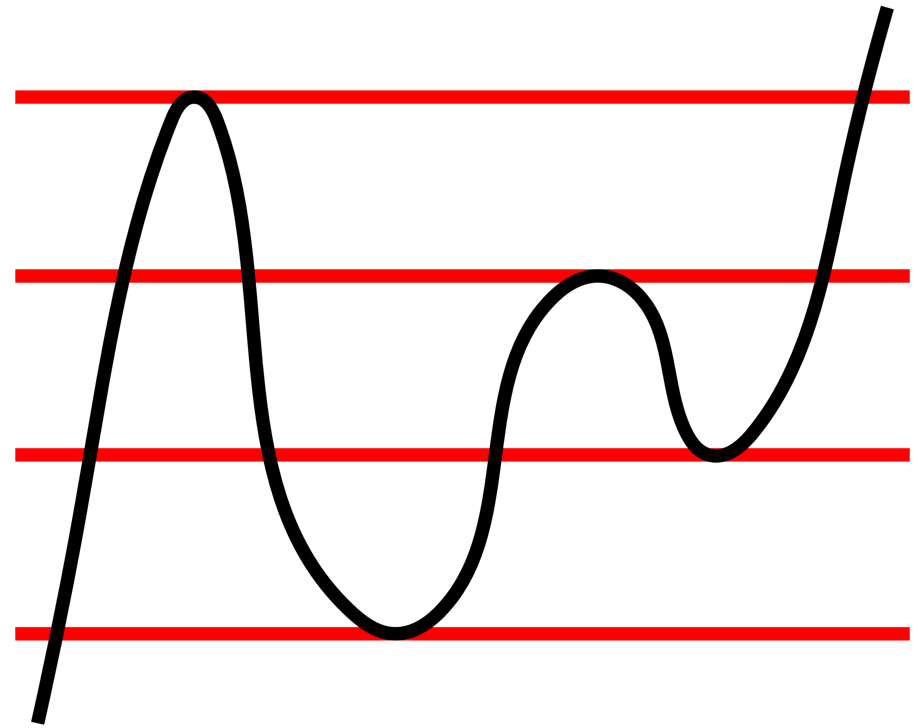


[3,1,4,2]



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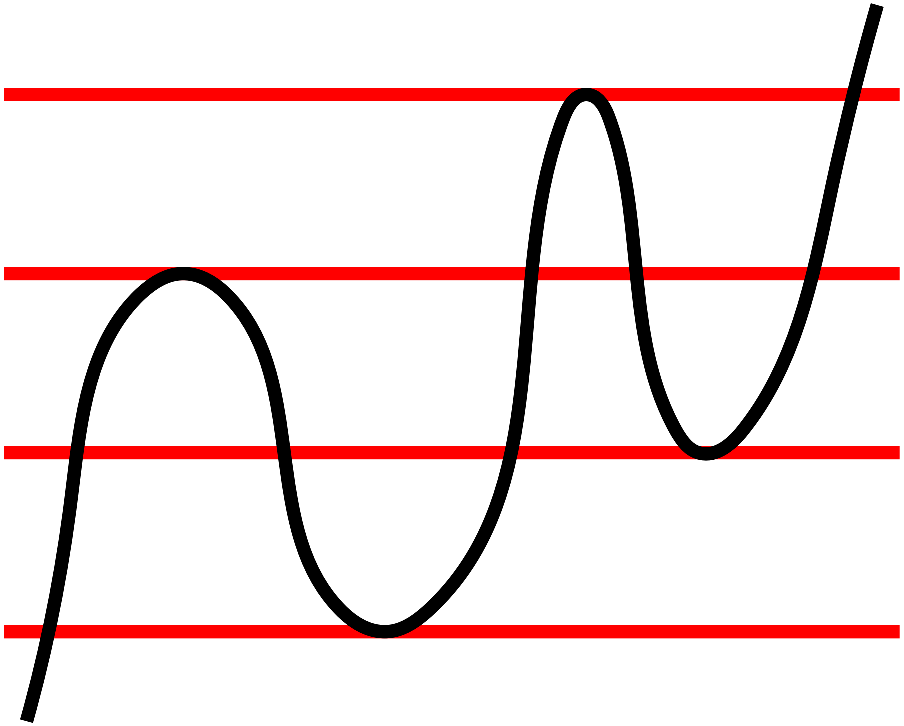
Beispiel



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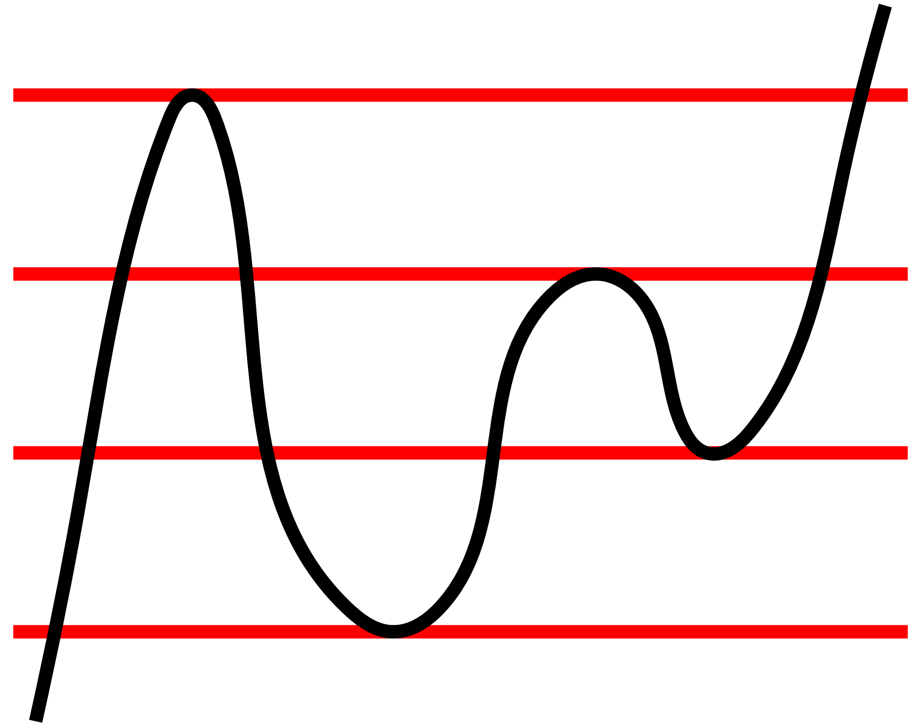

[4,1,3,2]

Alternierende Permutationen
Zick-Zack-Permutationen

$[a(1),\ldots,a(n)]$ mit
$a(1) > a(2) < a(3) > a(4) \ldots$

Euler-Bernoulli-Zahlen

zählen alternierende Permutationen einer bestimmten Länge

					1						1
				0	→	1					1
			1	←	1	←	0				2
		0	→	1	→	2	→	2			5
	5	←	5	←	4	←	2	←	0		16
0	→	5	→	10	→	14	→	16	→	16	61

Satz (Itenberg, Zvonkine, 2016)

Jedem (orientierten) Polynom f lässt sich ein Vorzeichen $s(f) \in \{\pm 1\}$ zuordnen, so dass die Summe

$$\sum_{f \in \mathbf{R}[x]} s(f)$$

unabhängig von der Anordnung der kritischen Werte $p, q, \ldots$ ist.
Asymptotik ≈ Euler-Bernoulli-Zahlen

Methode: Dessins d'enfant

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Satz (Hilany-R., 2017)

Erweiterung auf rationale Funktionen mit einfacher Polstelle.

Satz (Rau, 2018)

- We can define an „tropical“ invariant count for double Hurwitz numbers which provides a lower bound for real Hurwitz numbers.
- We can explicitly describe in which cases this bound is non-zero (existence of real covers).
- If non-zero, the logarithmic growth of this lower bound is equivalent to that of the complex Hurwitz number.