Hilbert in the Tropics

1. Section

A brief story about real algebraic curves

Real algebraic curves in the plane

$$\left\{(x, y) \in \mathbf{R}^2 : f(x, y) = 0\right\}, \qquad f \in \mathbf{R}[x, y]$$

Which shapes are possible?

Real algebraic curves in the plane

$$\{(x,y)\in\mathbf{R}^2 : f(x,y)=0\}, \qquad f\in\mathbf{R}[x,y]$$

Which shapes are possible?

→ Topological classification

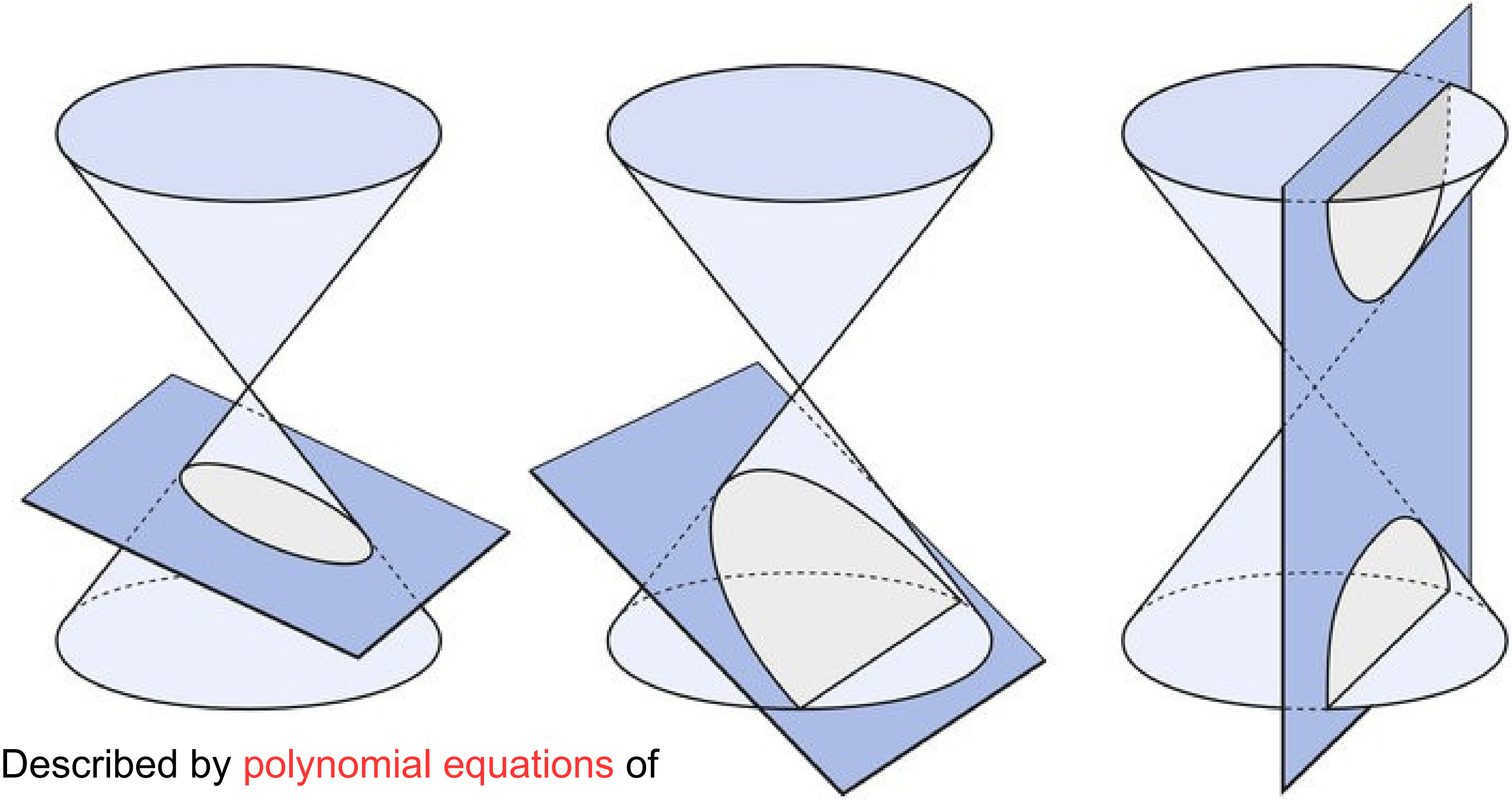
- How many components? Open or closed?
- Arrangements?

But NOT: metric properties

- curvature?
- Bitangents? Inflection points?

Conic sections

Menaichmos, Euklid, Apollonius



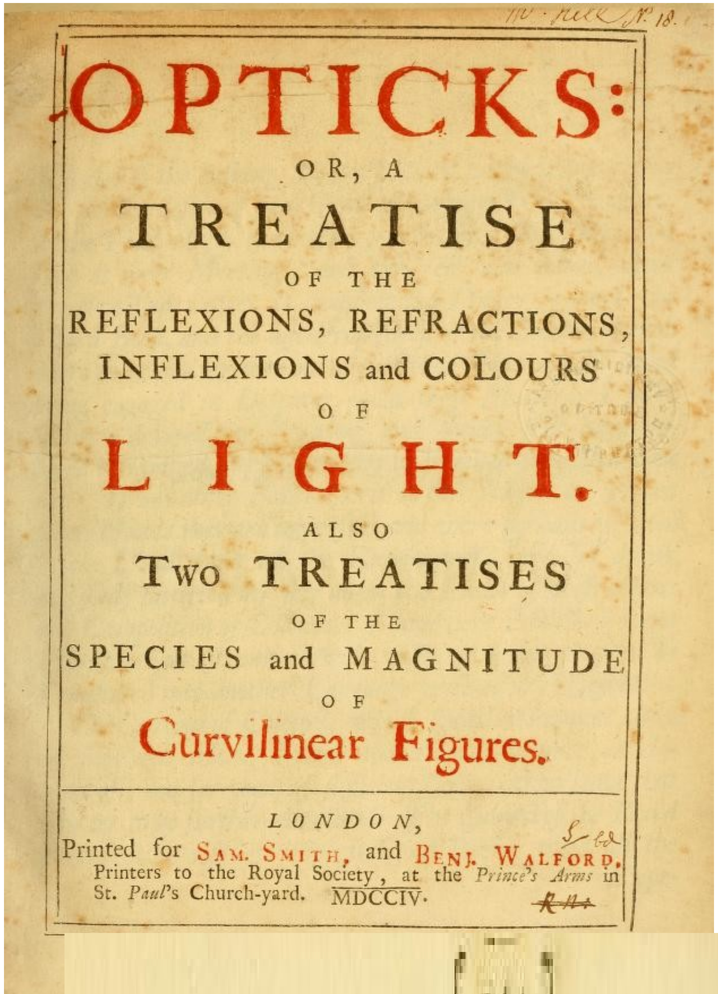
Described by polynomial equations of

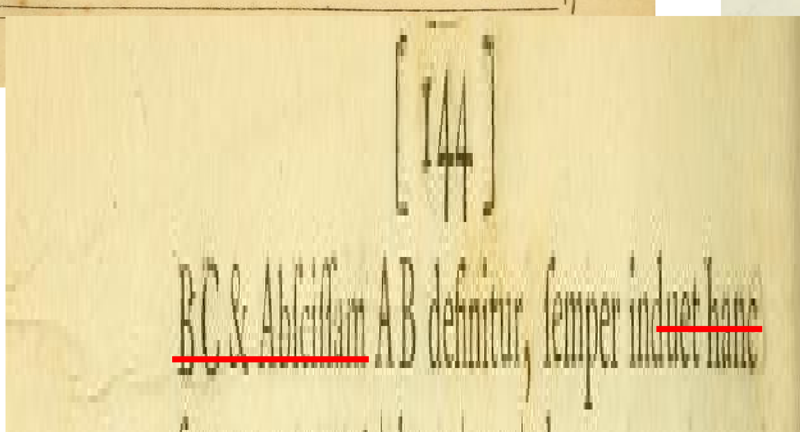
Degree 2

$$x^2 + 3y^2 = 1 \qquad x^2 + 4 = y \qquad x^2 - 3y^2 = 1$$

Cubics

Newton (1704)

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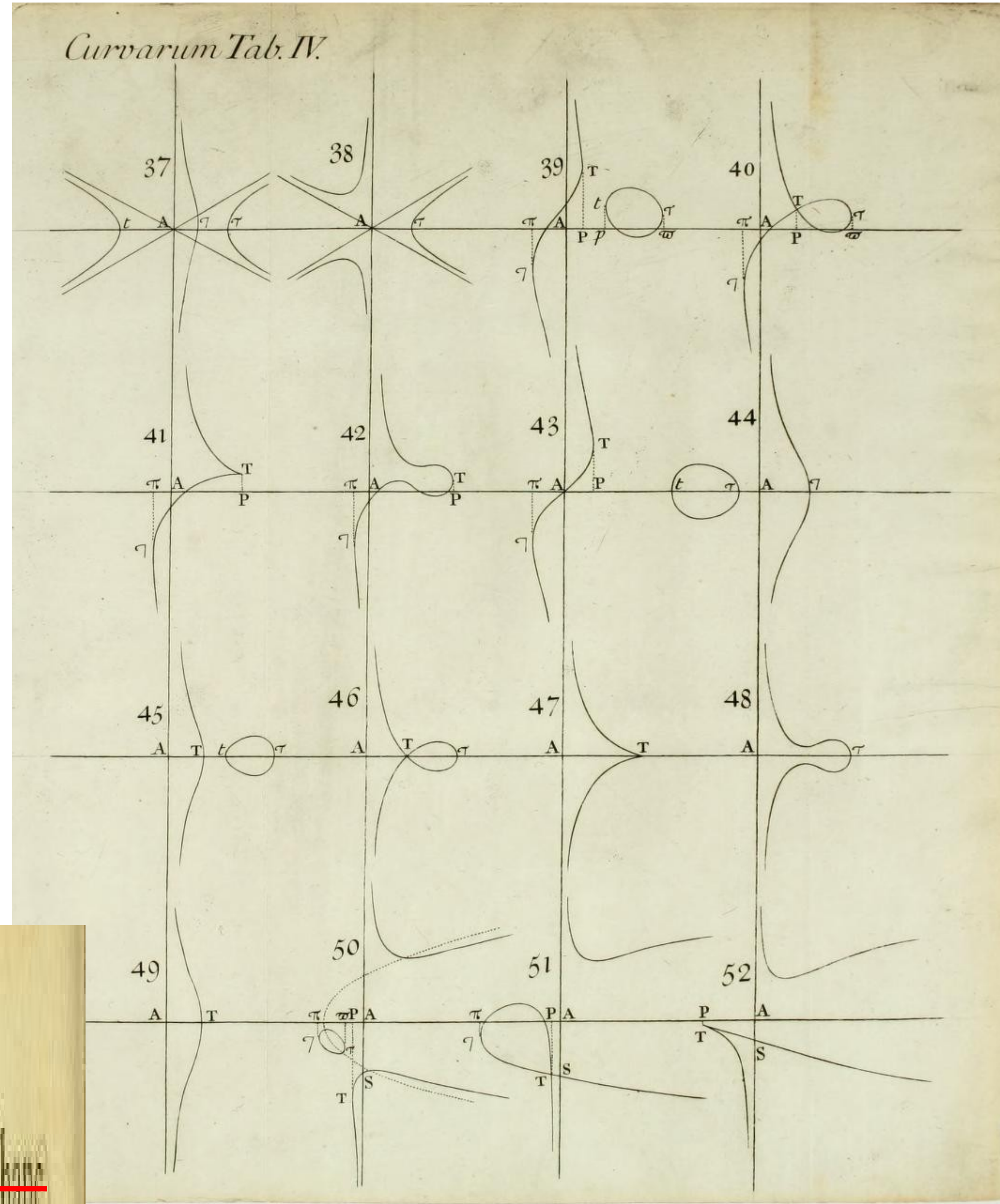


Degree 3

Example Generic curves of degree 3

$$a_1 x^3 + a_2 x^2 y + a_2 xy^2 + a_4 y^3 + a_5 x^2 + a_6 xy + a_7 y^2 + a_8 x + a_9 y + a_{10} = 0$$

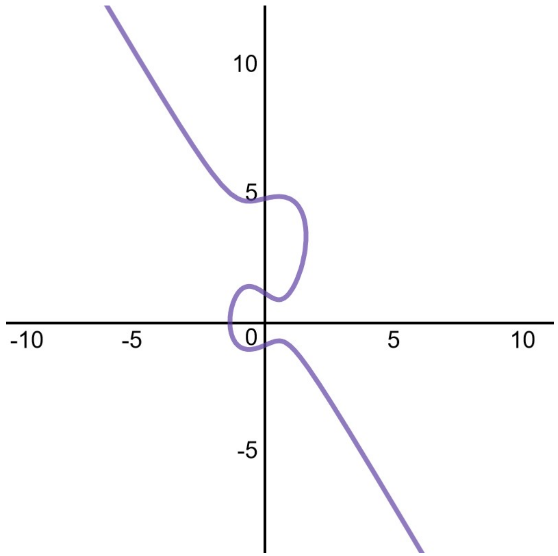
Classification modulo homeomorphisms of $\mathbf{R}^2$

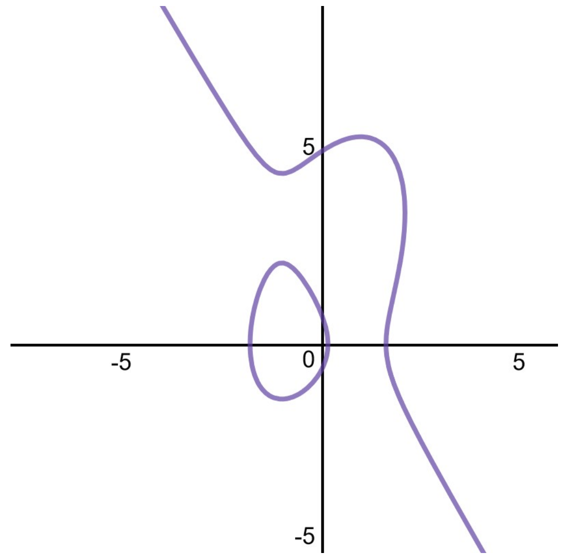
$\rightarrow$ Ambient isotopy

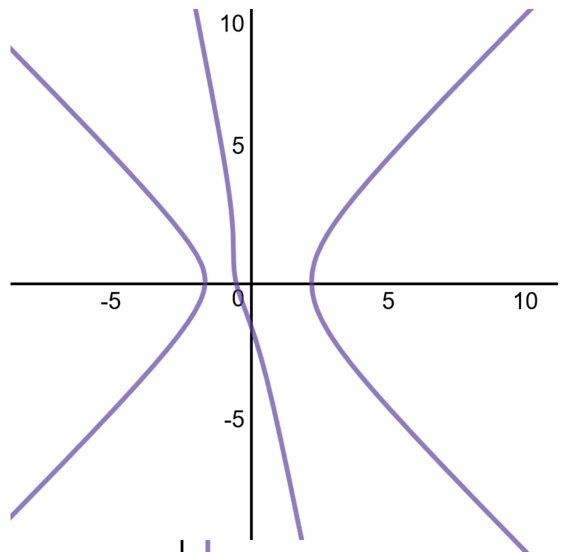
Example

Generic curves of degree 3

$$a_1x^3+a_2x^2y+a_2xy^2+a_4y^3+a_5x^2+a_6xy+a_7y^2+a_8x+a_9y+a_{10}=0$$

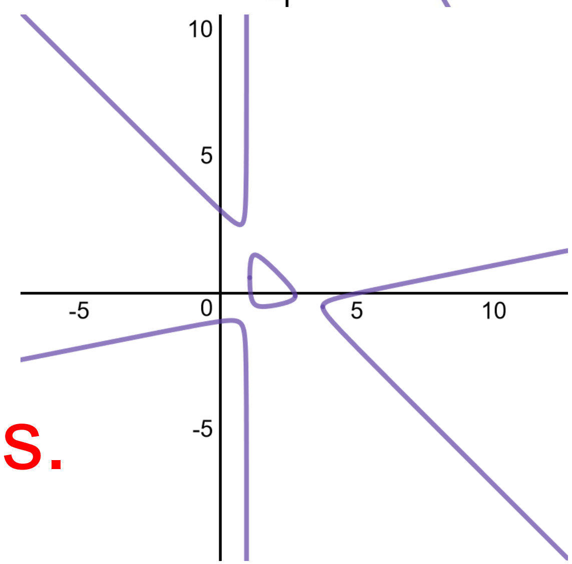


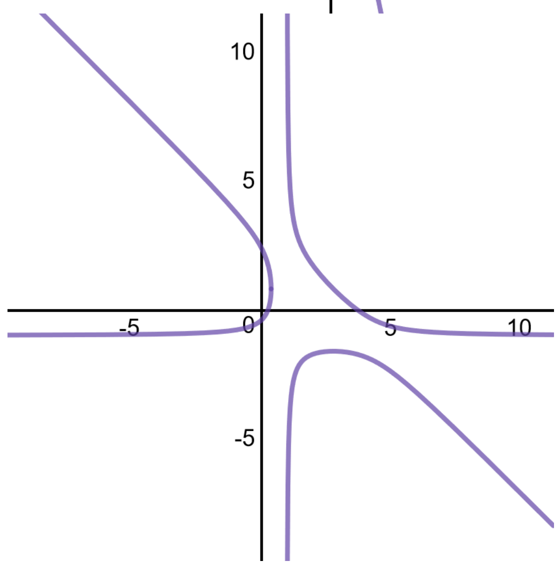




Theorem

Modulo homeo-morphisms of $\mathbf{R}^2$, there are 5 classes.



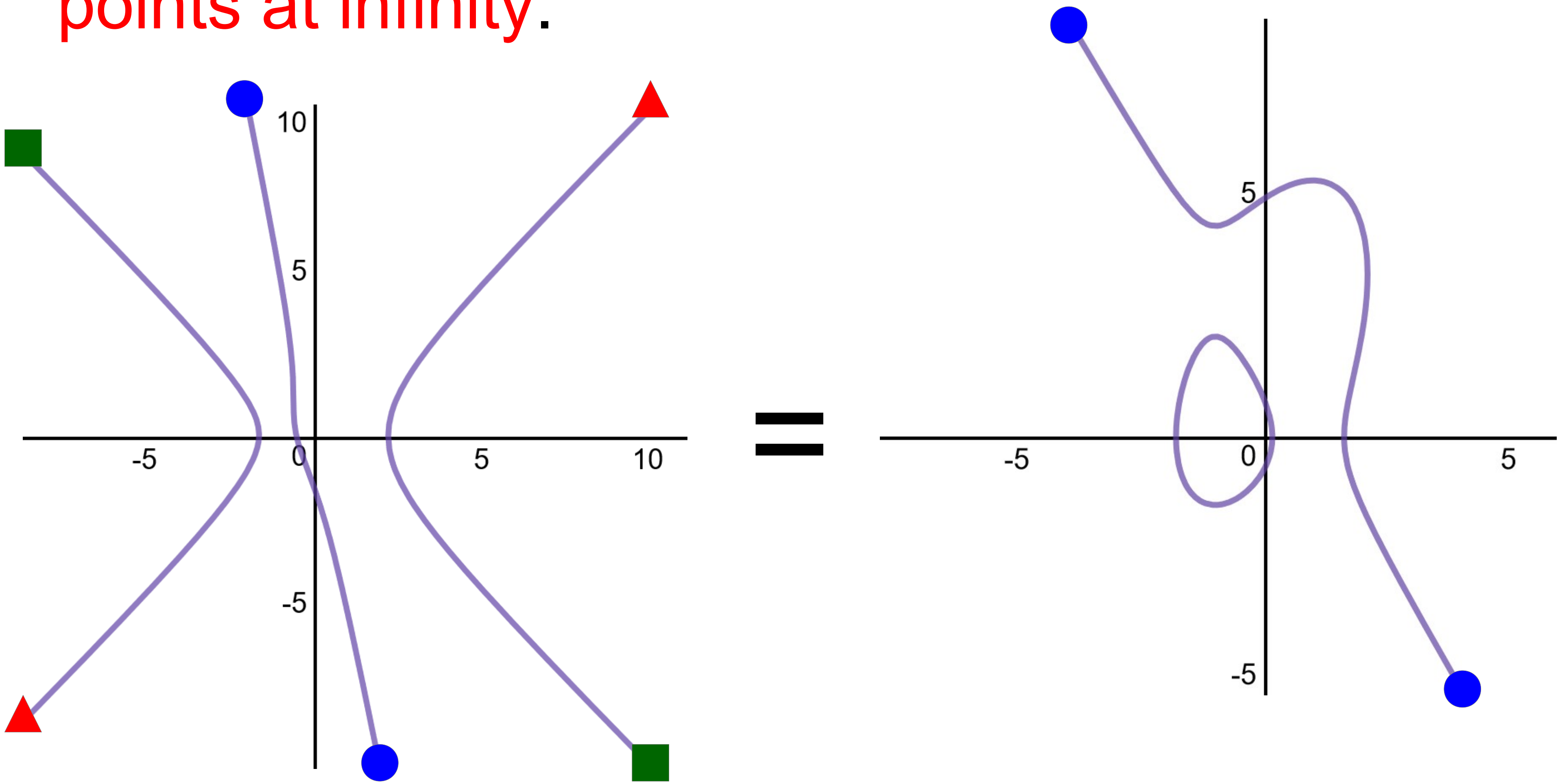


Now: Projective curves

in real projective plane $\mathbf{RP}^2$

decribed by homogenous polynomials of 3 variables

→ Asymptotes get glued/close up via points at infinity.



Hilbert's 16th problem

Hilbert's 23 problems
International Congress of Mathematicians 1900

„The upper bound of closed and separate branches of an algebraic curve of degree n was decided by Harnack.
From this arises the further question as of the relative positions of the branches in the plane.“

Hilbert's 16th problem

„As of the curves of degree 6, I have – admittedly in a rather elaborate way – convinced myself that the 11 branches, that they can have according to Harnack, never all can be separate, rather there must exist one branch, which have another branch running in its interior and nine branches running in its exterior, or opposite. [...]“

Theorem (Harnack)

The number l of connected components of a smooth curve of degree d in real projective plane is bounded by:

$$l \leq \frac{(d-1)(d-2)}{2}$$

For $d = 6$ we get: $l \leq 11$

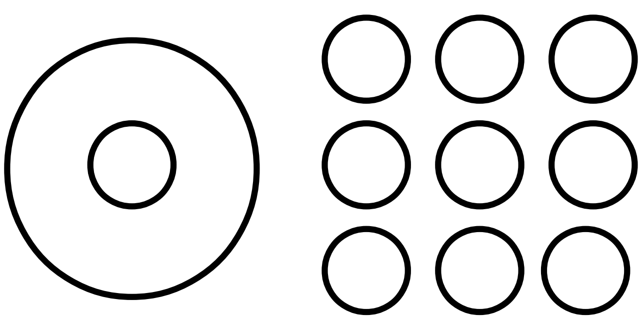
Theorem (Harnack)

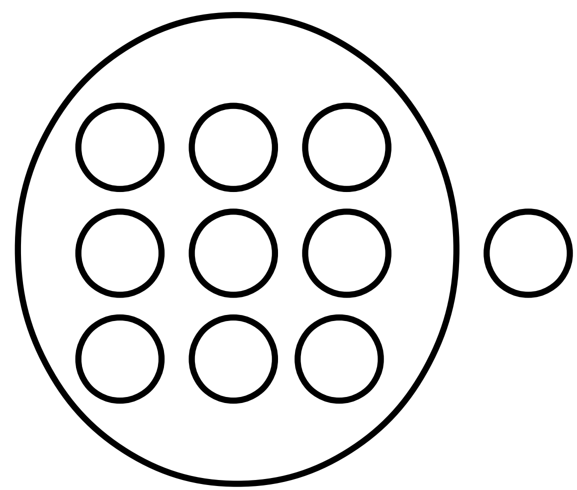
The number l of connected components of a smooth curve of degree d in real projective plane is bounded by:

$$l \leq \frac{(d-1)(d-2)}{2}$$

Conjecture (Hilbert)

Consider a curve of degree 6 with exactly 11 components. Then they must be arranged in one of the two follwing ways:

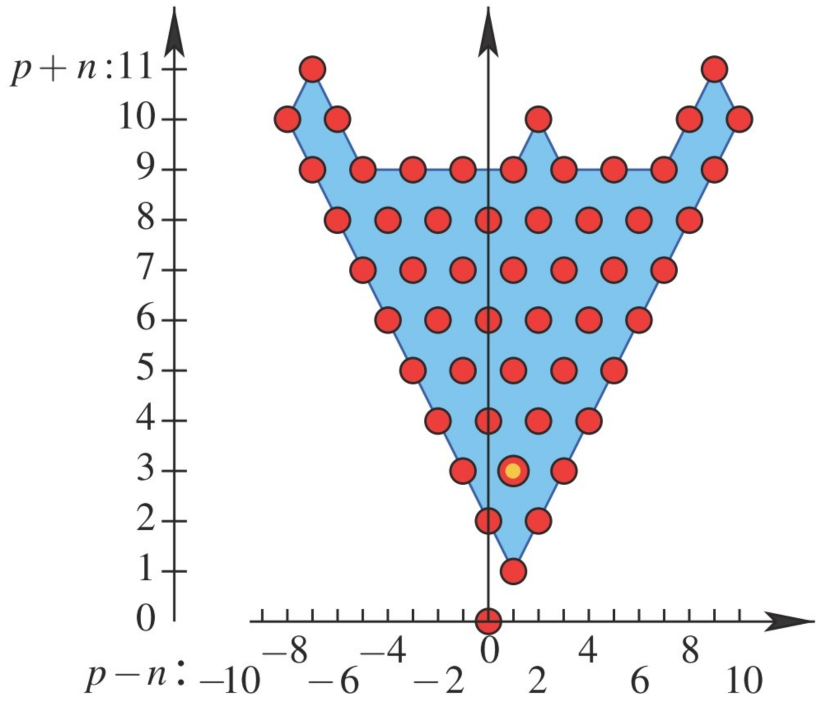




Gudkov

1954 PhD thesis

- Proof of Hilberts conjecture
- Full classification of sextics



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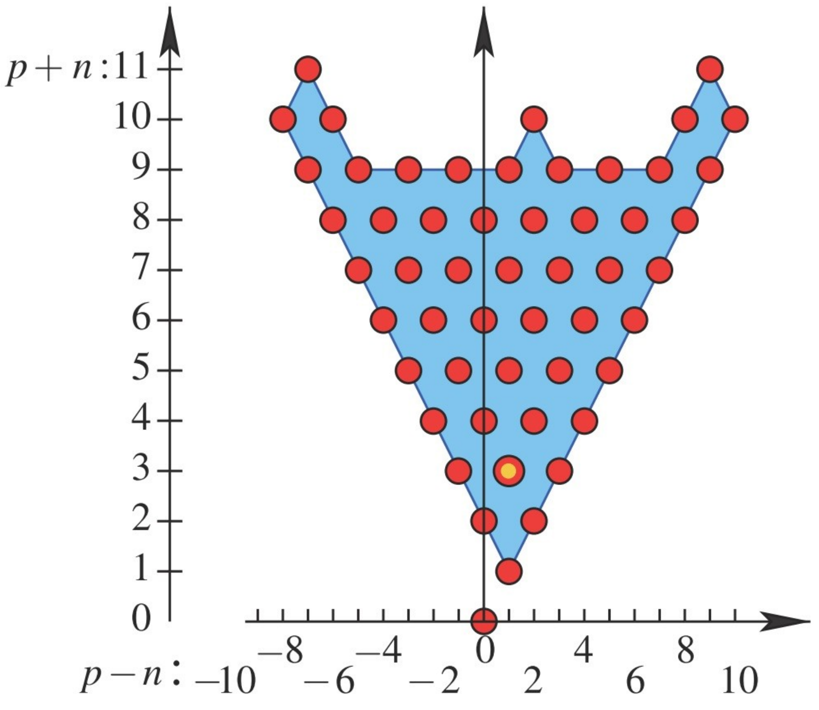

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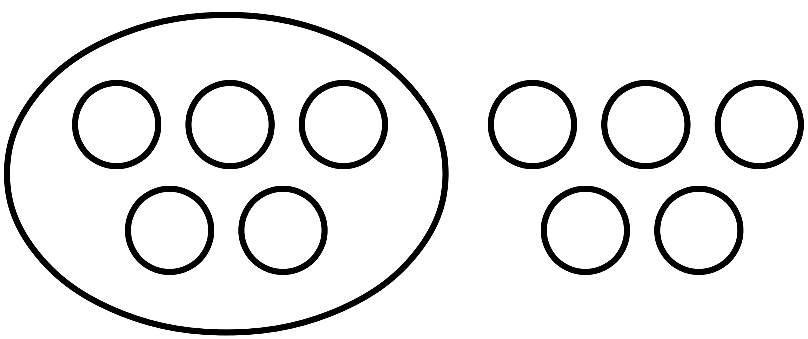
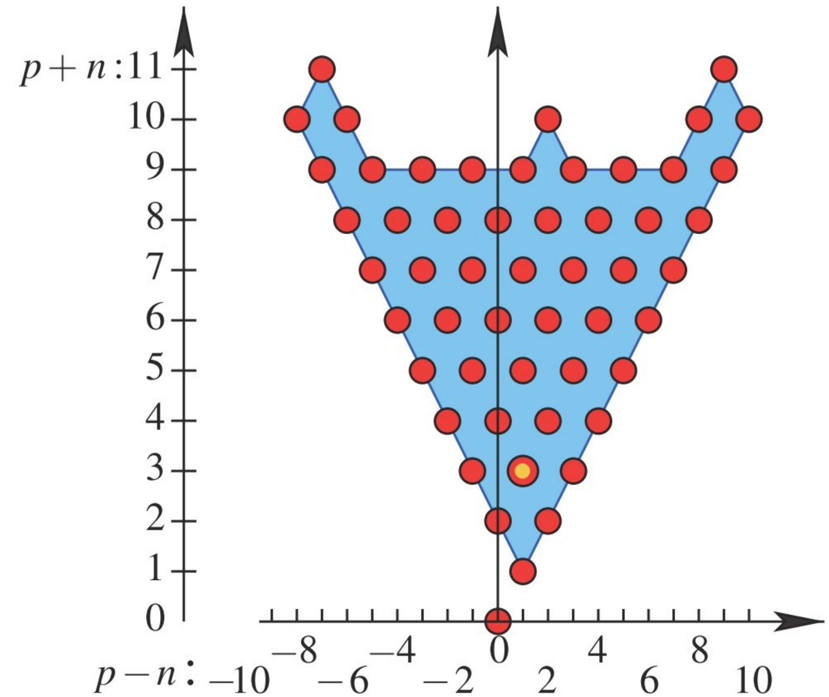
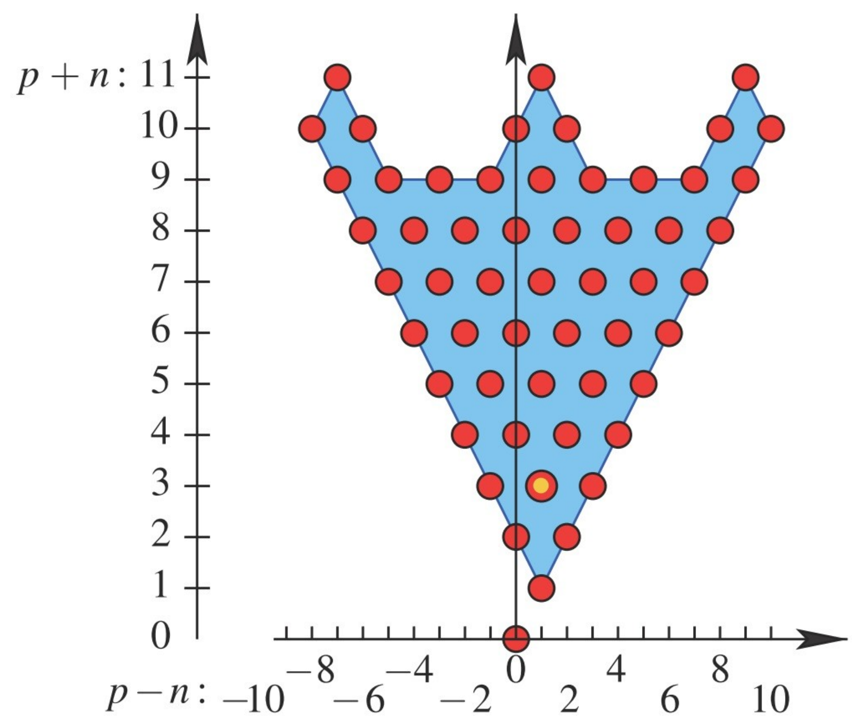
1969 Habilitation thesis

- Conjecture/Proof wrong!
- New, symmetric classification



Gudkov Sextik





Gudkov, Viro, ...

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- Proof of Hilberts conjecture
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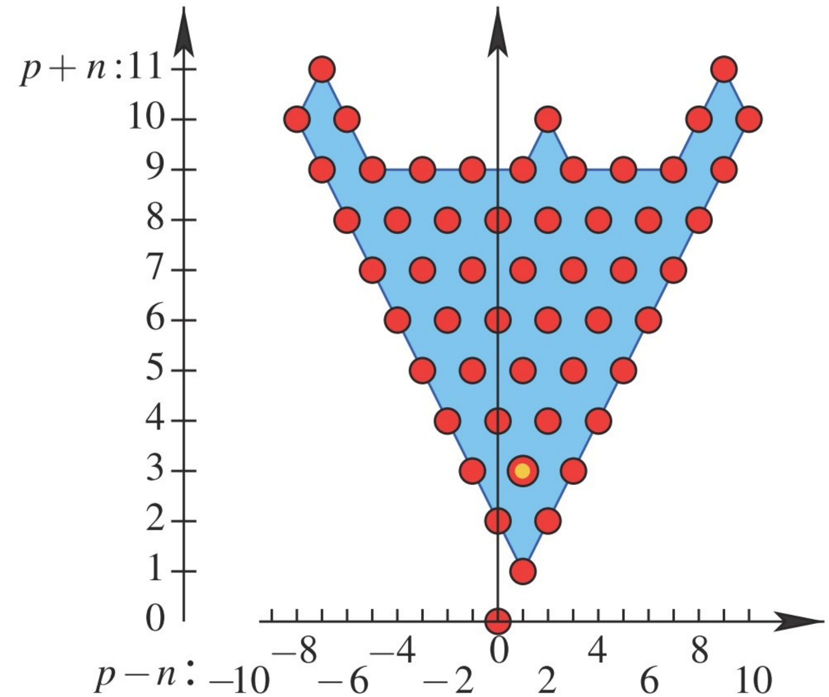
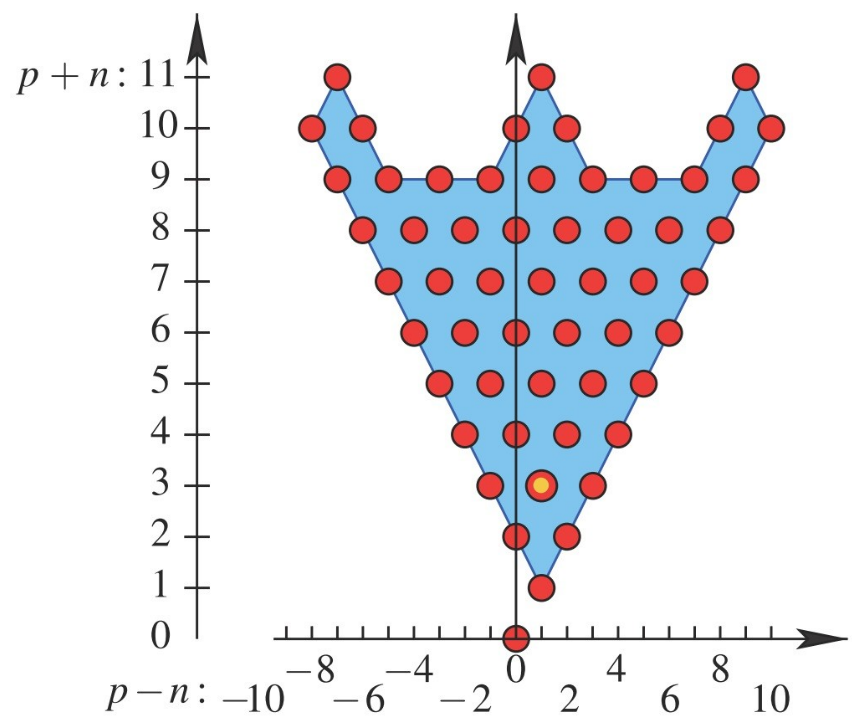
1979

- Viro patchworking
- Full classification of septics

1989? Higher degrees? 8,9,...

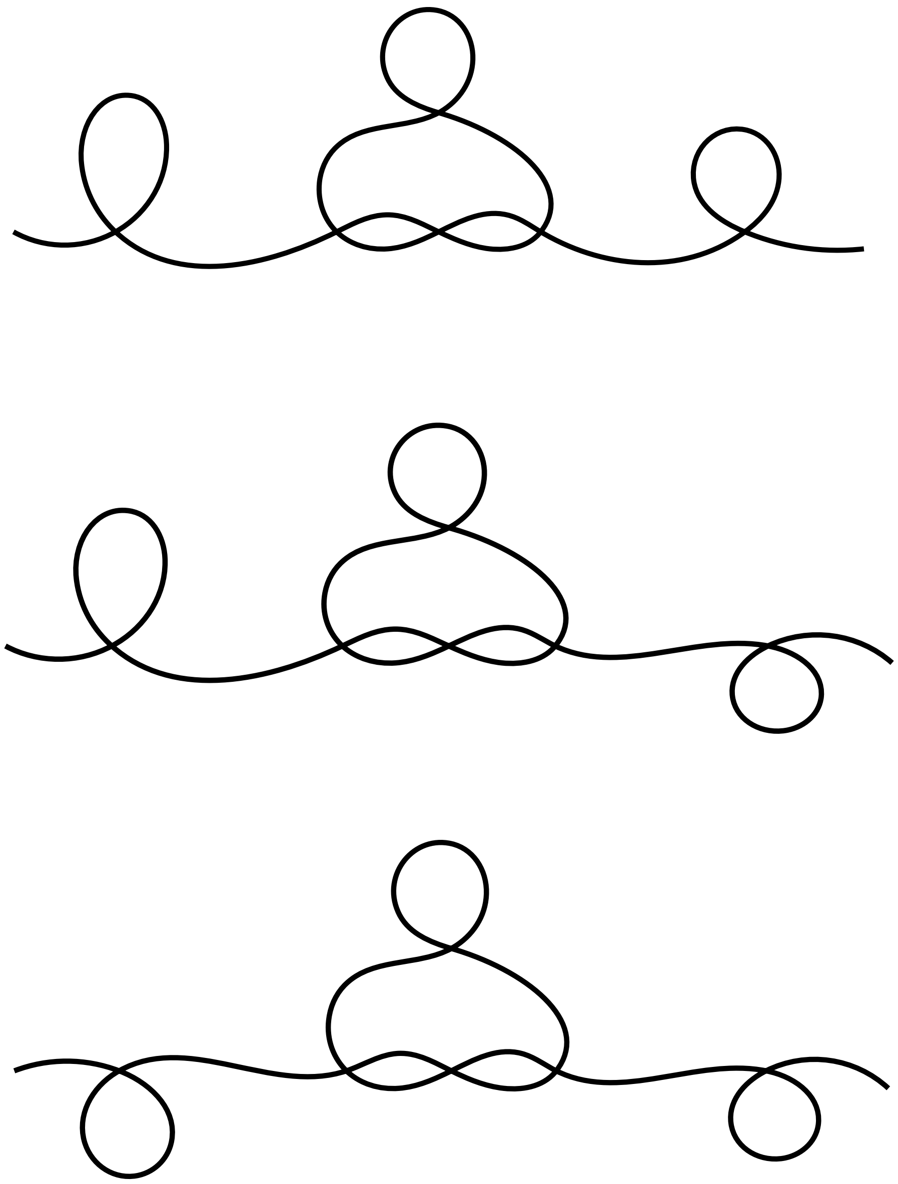
- Still unknown!





Singular curves

Quintics (degree 5) with 6 ordinary double points



Up to isotopy!

Singular curves

Quintics (degree 5) with 6 ordinary double points

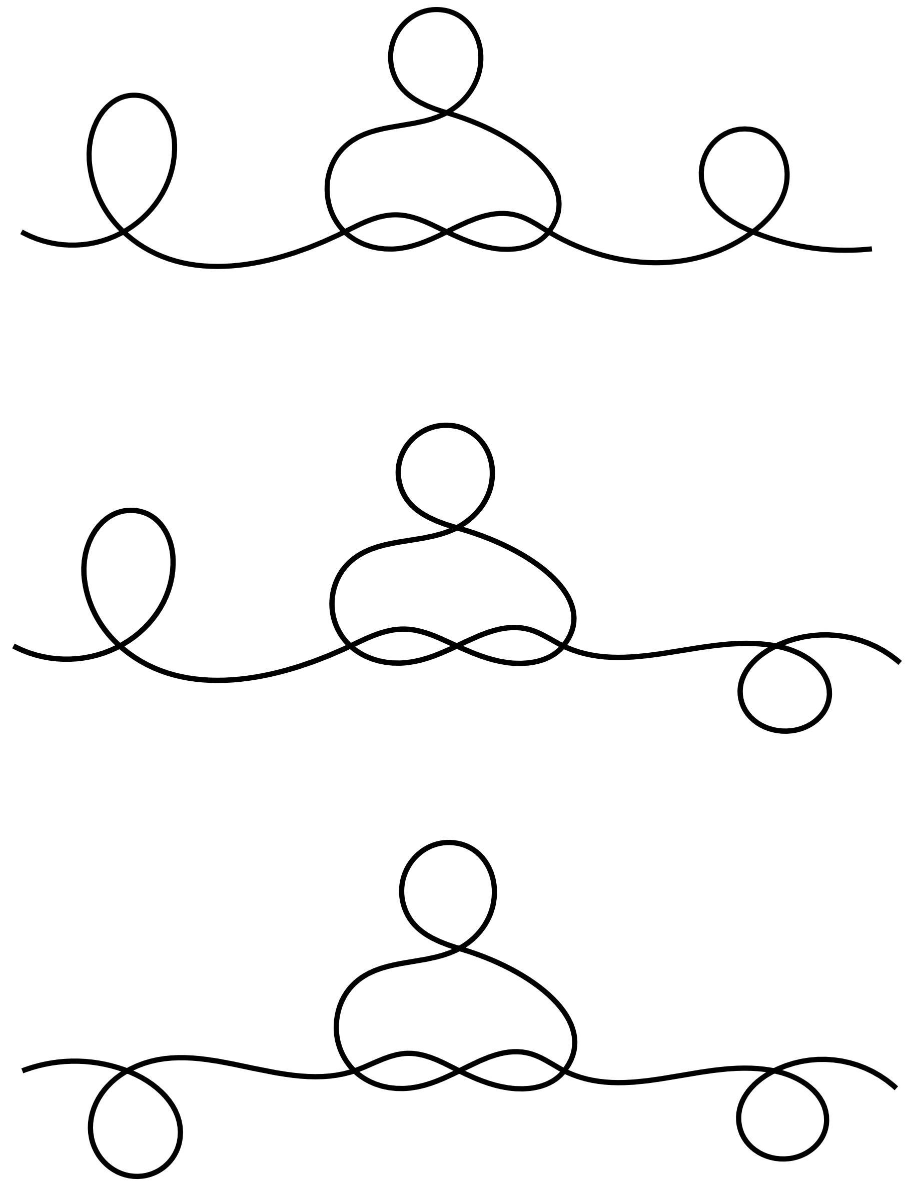
or equivalently

Rational curves

$$\mathbf{R} \to \mathbf{R}^2$$

$$x \mapsto \left(\frac{f(x)}{h(x)}, \frac{g(x)}{h(x)} \right)$$

f, g, h polynomials in $\mathbf{R}[x]$ of degree 5



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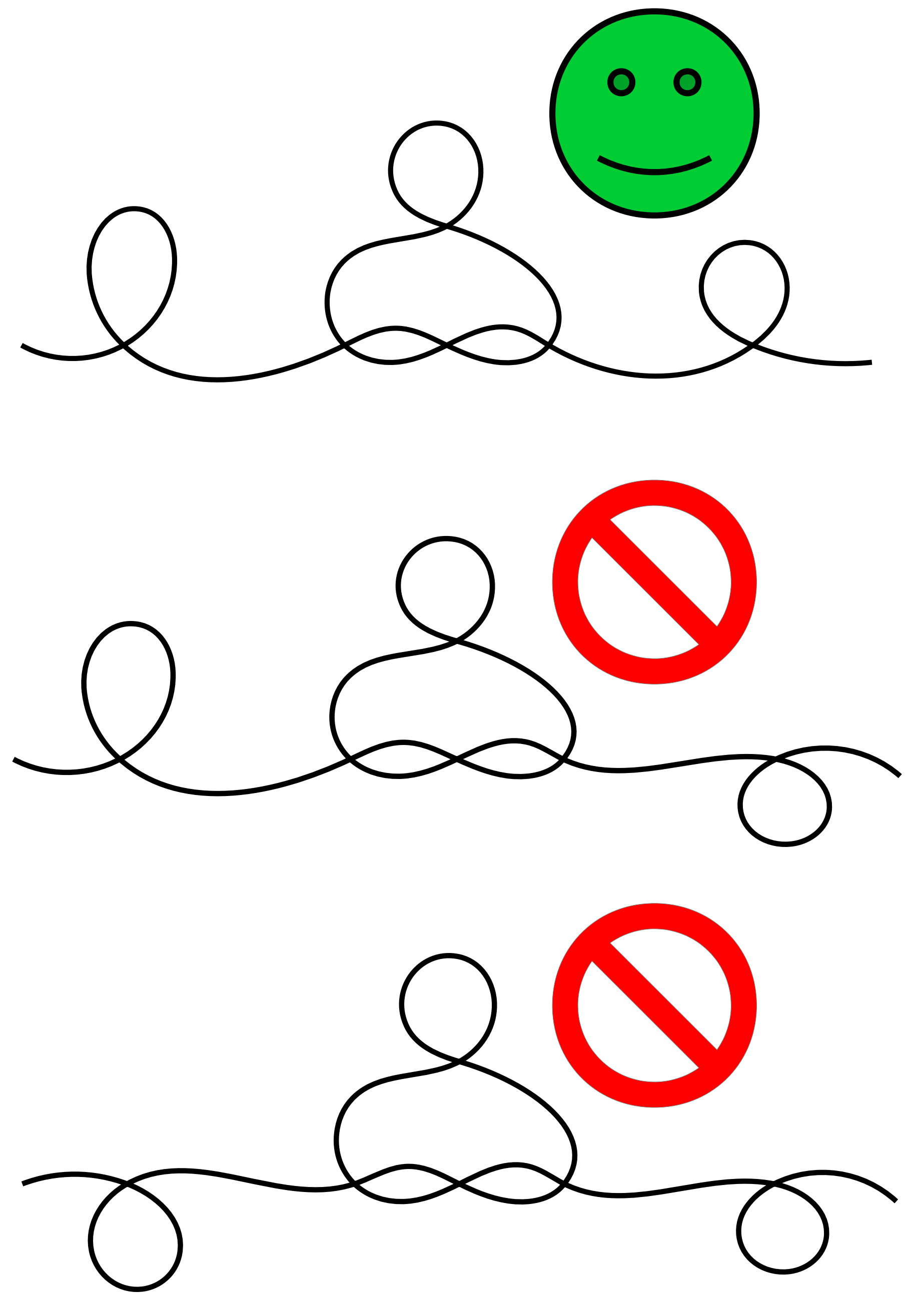


Up to isotopy!

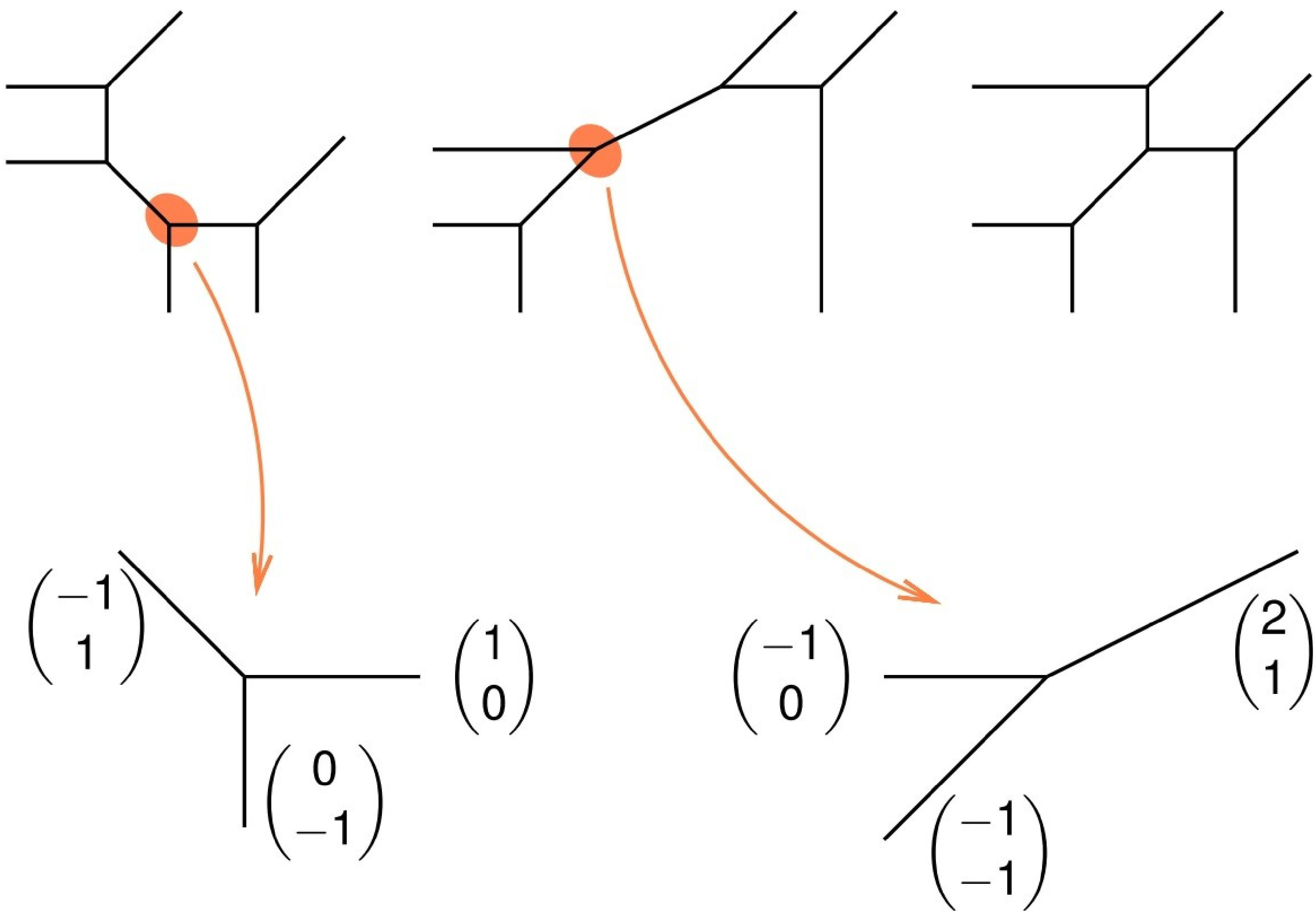
Theorem (Itenberg, Mikhalkin, R., 2016)
There are 121 classes of rational nodal curves of degree 5 in $\mathbf{RP}^2$ up to ambient isotopy (complete classification).

46 classes with only hyperbolic double points

2. Section

Tropical patchworking

Tropical curves



Tropical arithmetics

$$a +_{tr} b = max\{a, b\}$$

$$a \cdot_{tr} b = a + b$$

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Tropical polynomials

$$f = \sum_{tr} a_{ij} \cdot_{tr} x^i \cdot_{tr} y^j$$

$$f = max\{a_{ij} + ix + jy\}$$

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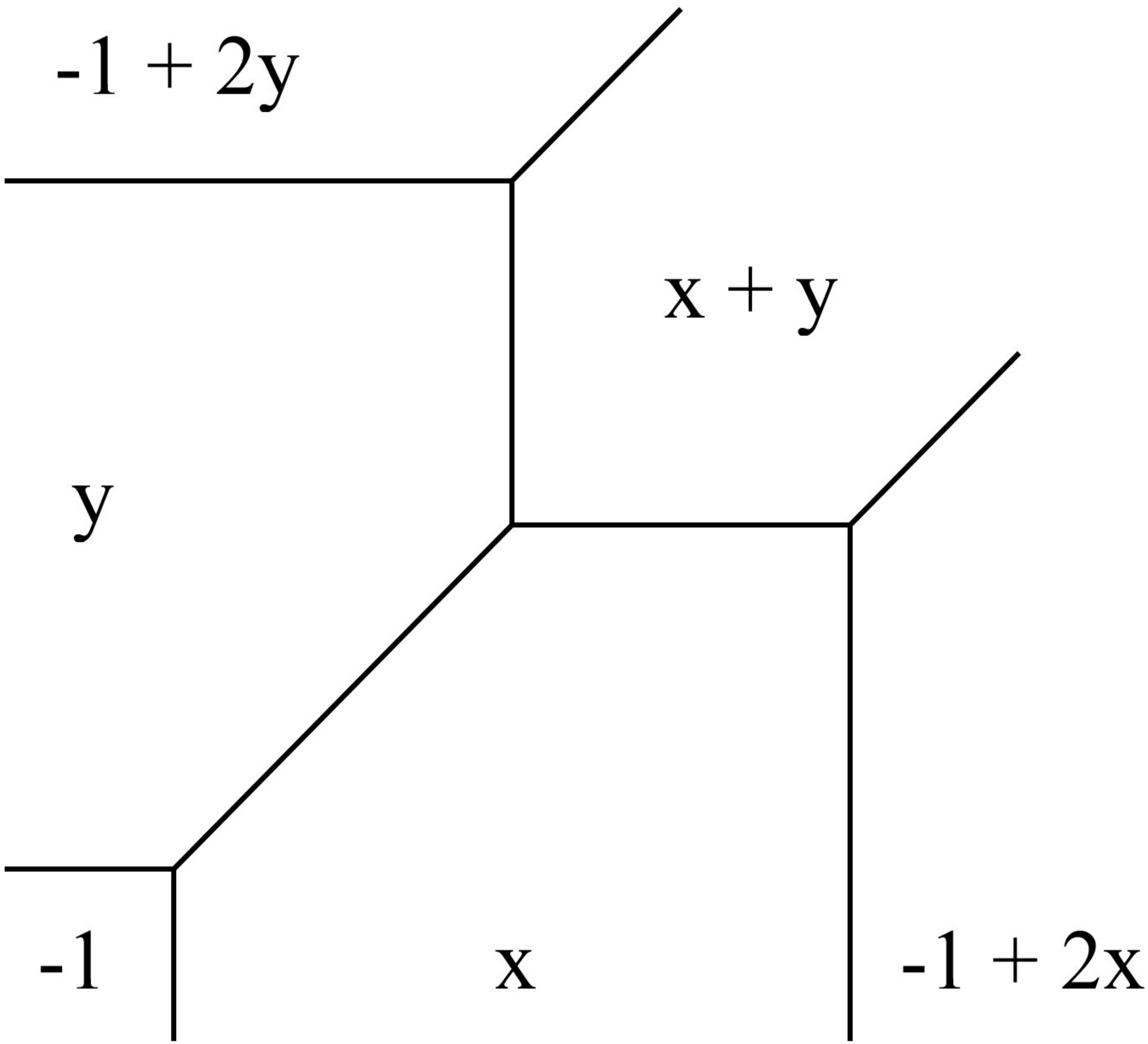
Tropical hypersurface

Definition

Given a tropical polynomial f, the tropical hypersurface $V(f)$ is the set of points x in $\mathbf{R}^2$ where the maximum in $f(x)$ is attained by at least two terms.

$$f = -1 + x + y + xy + (-1)x^2 + (-1)y^2$$
$$= max\{-1, x, y, x + y, -1 + 2\mathrm{x}, -1 + 2\mathrm{y}\}$$



$$Log_t : (\mathbf{C}^\times)^2 \to \mathbf{R}^2, (z, w) \to (\log_t |z|, \log_t |w|).$$

Theorem

Let

$$F_t = \sum A_{ij}(t) z^i w^j$$

be a family of complex polynomials such that

$$A_{ij}(t) \sim c_{ij} t^{a_{ij}} \qquad t \to \infty.$$

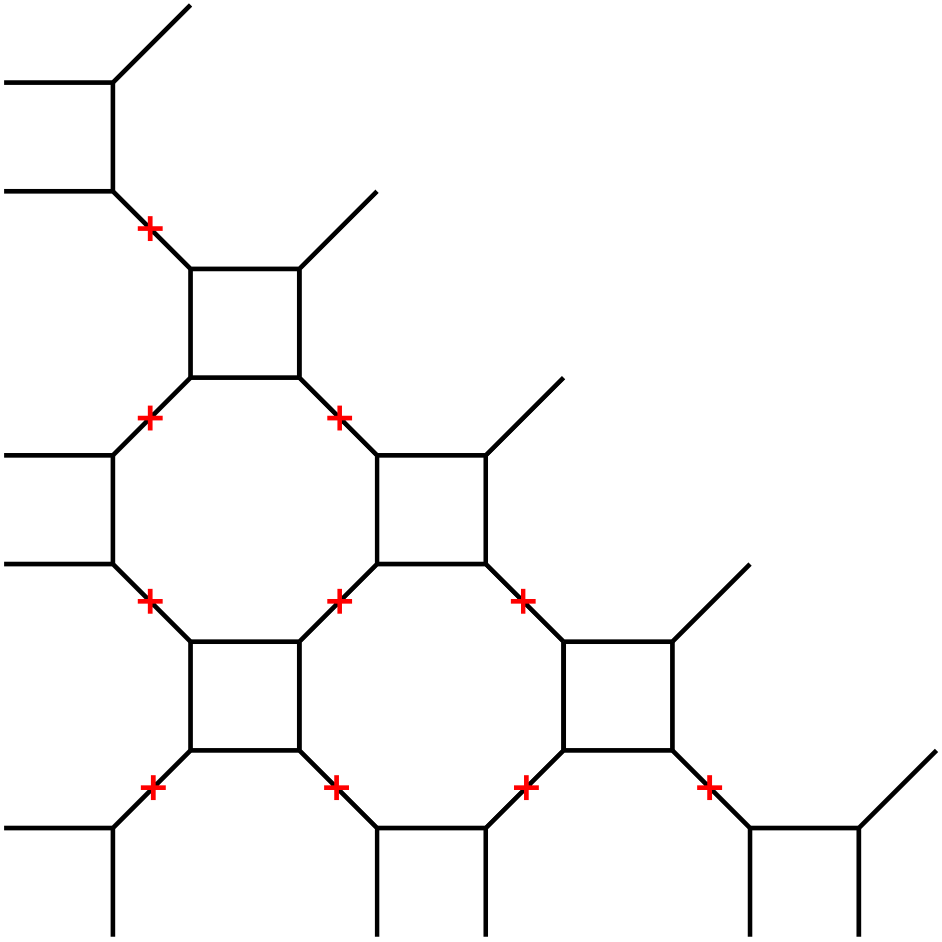
Consider the tropical polynomial

$$f = \sum\nolimits_{tr} a_{ij} \cdot_{tr} x^i \cdot_{tr} y^j.$$

Then

$$\lim_{t \to \infty} Log_t (V(F_t) \cap (\mathbf{C}^\times)^2) = V(f).$$

Real version



Input data: (X,T)

- X smooth tropical curve
- T subset of bounded edges (twists)

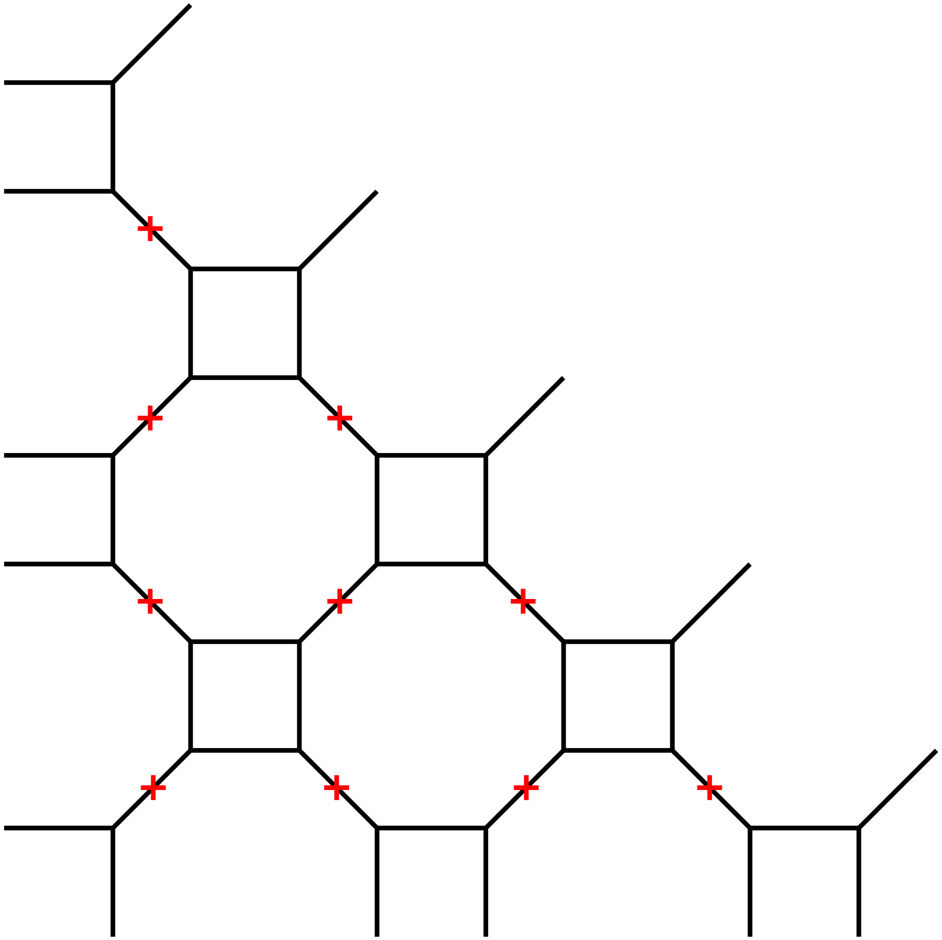
such that for any bounded region B in $\mathbf{R}^2$ we have:

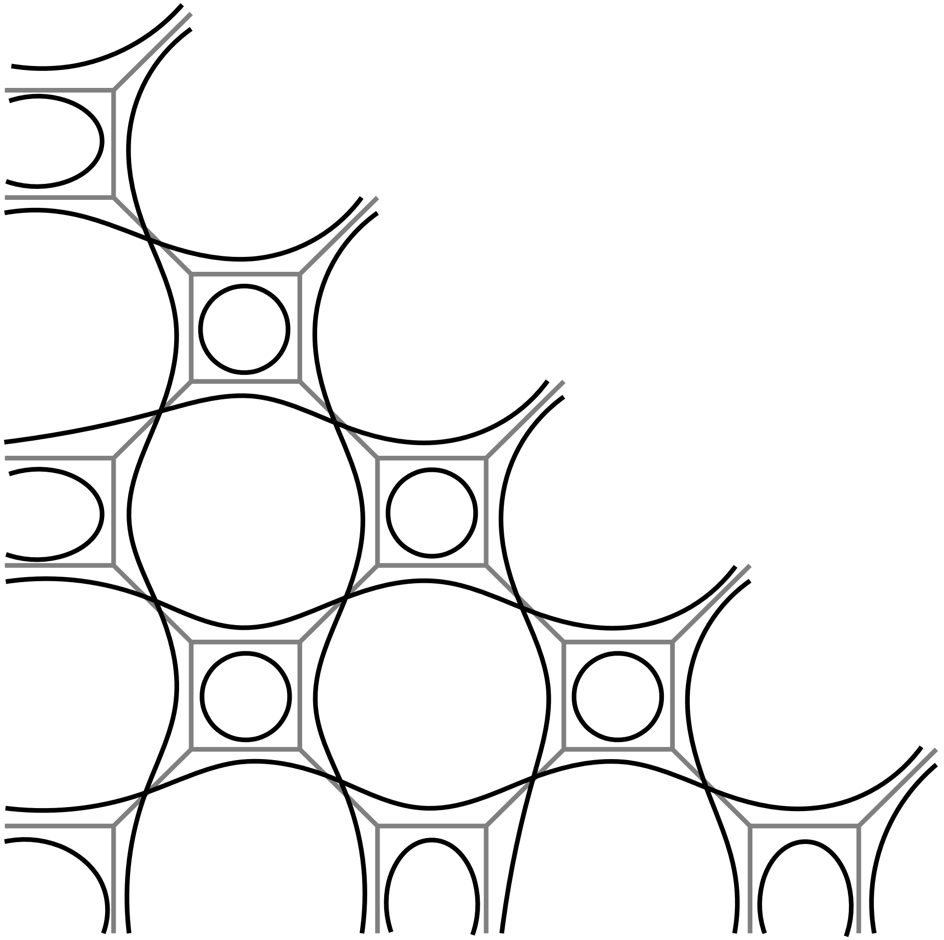
$$\sum_{e \in T \cap \partial B} v_e \in (2\mathbf{Z})^2$$

Real version

First step: $R(X,T)$

- Ribbon graph associated to X with twists T.

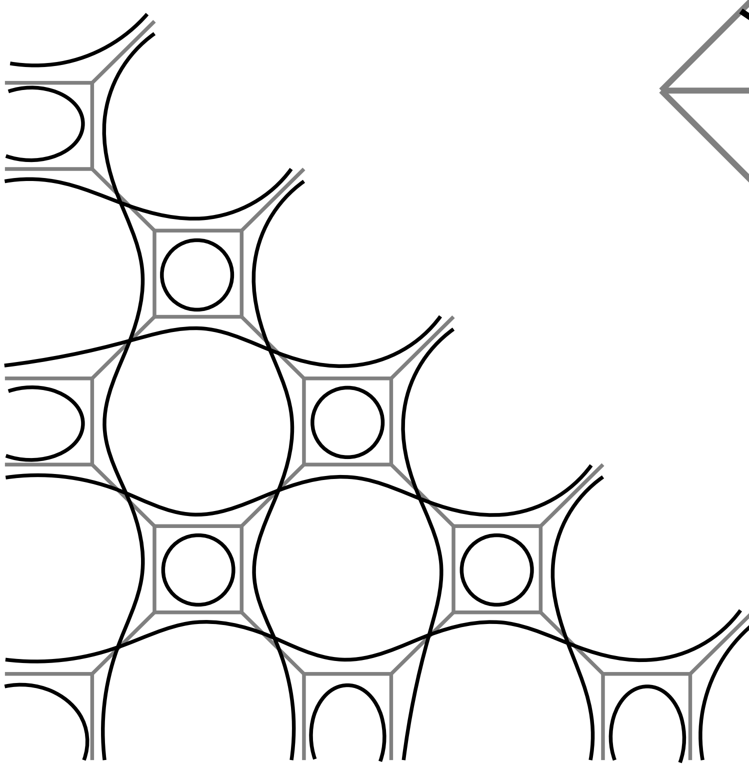


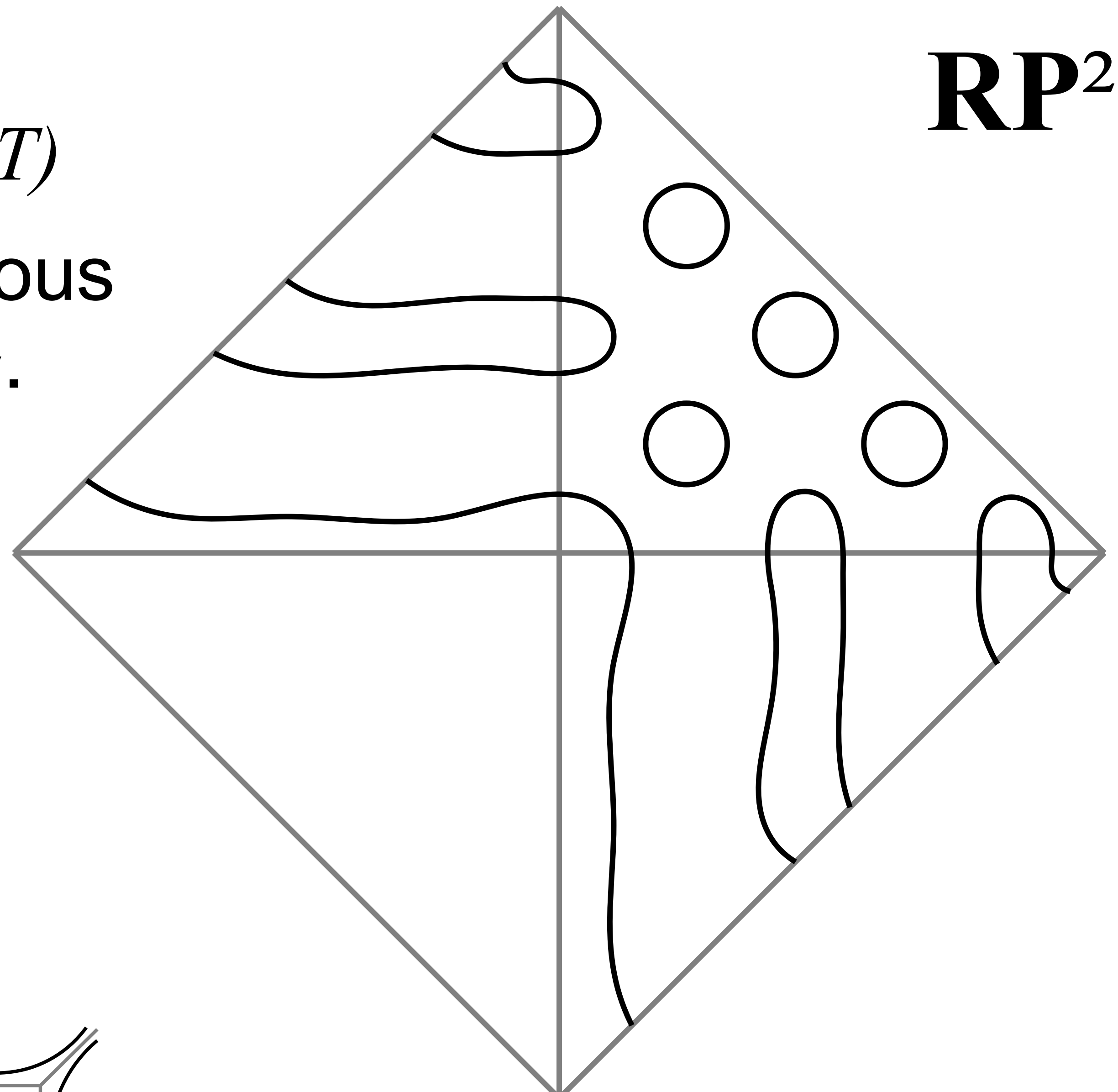


Real version

Second step: $L(X,T)$

- Lift $R(X,T)$ to various quadrants of $\mathbf{RP}^2$.

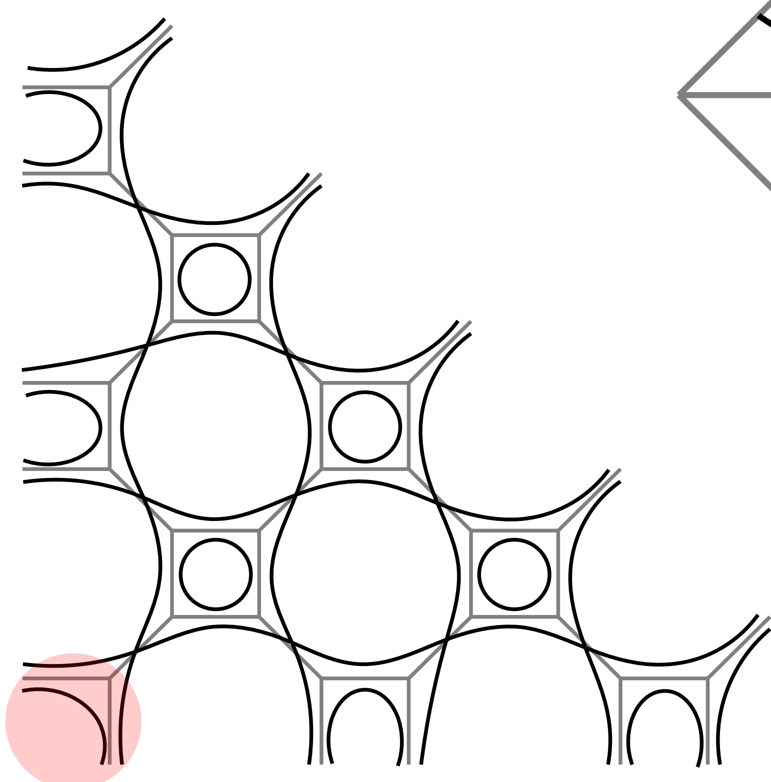


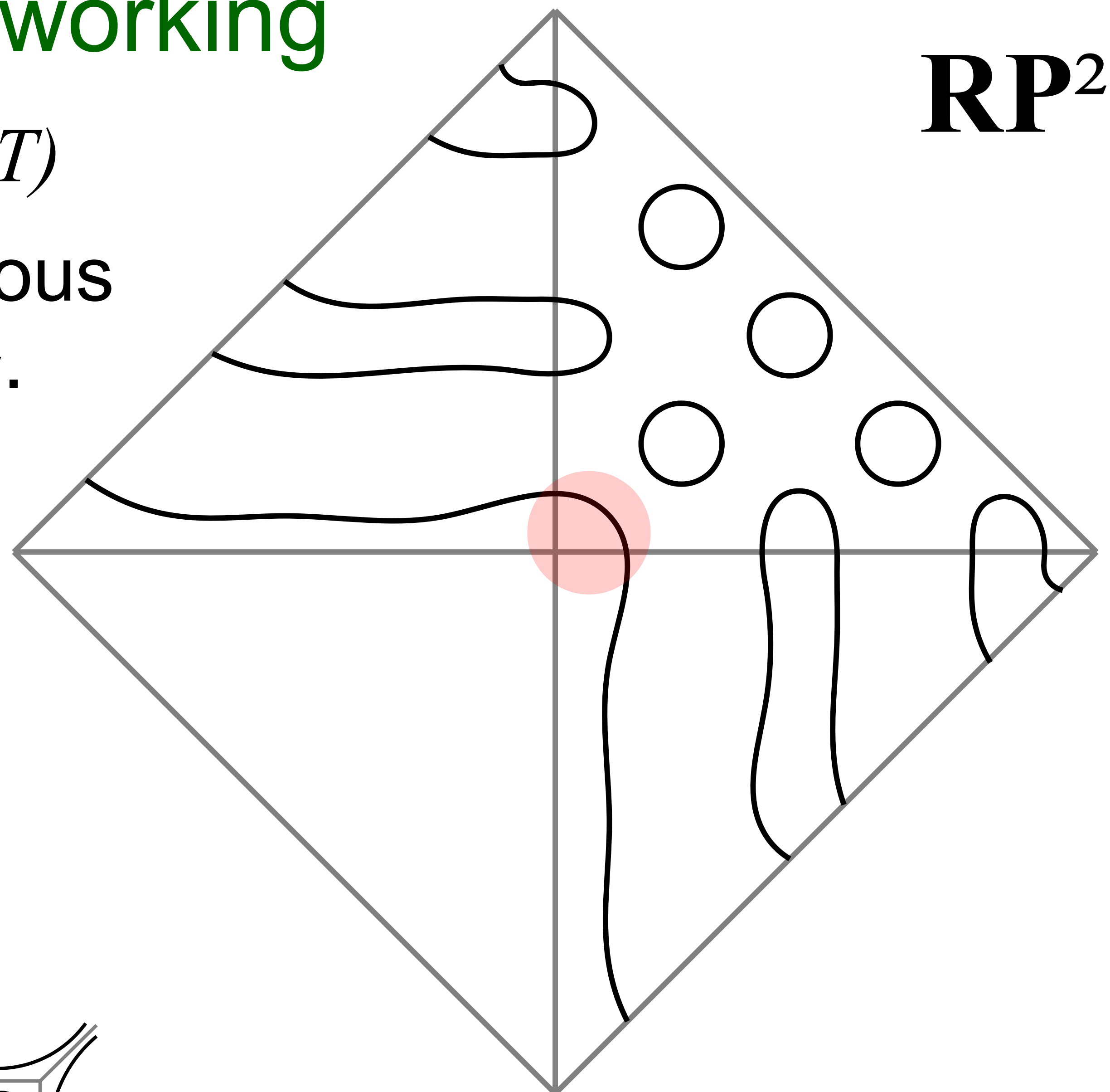


Tropical Patchworking

Second step: $L(X,T)$

- Lift $R(X,T)$ to various quadrants of $\mathbf{RP}^2$.





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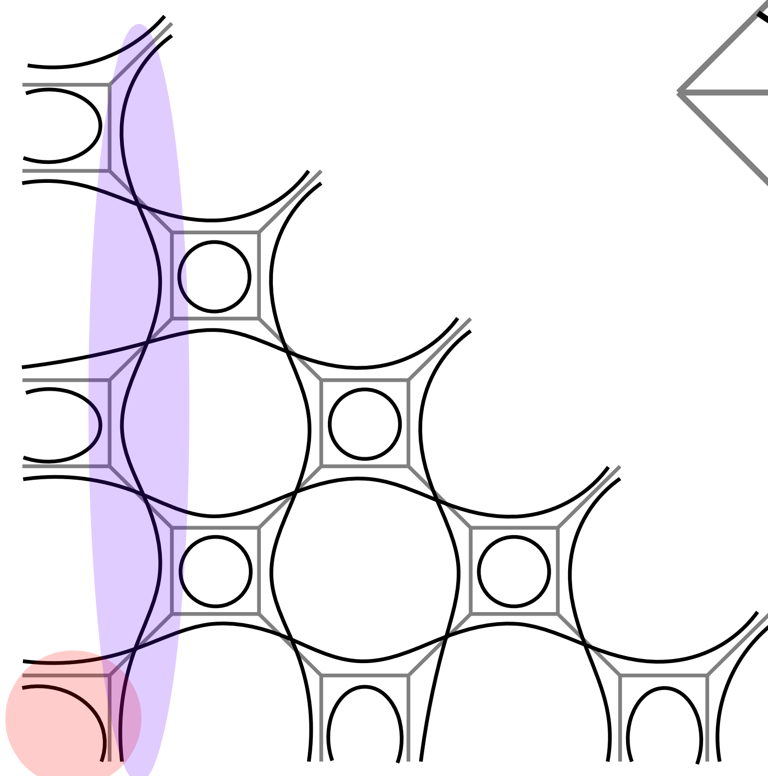
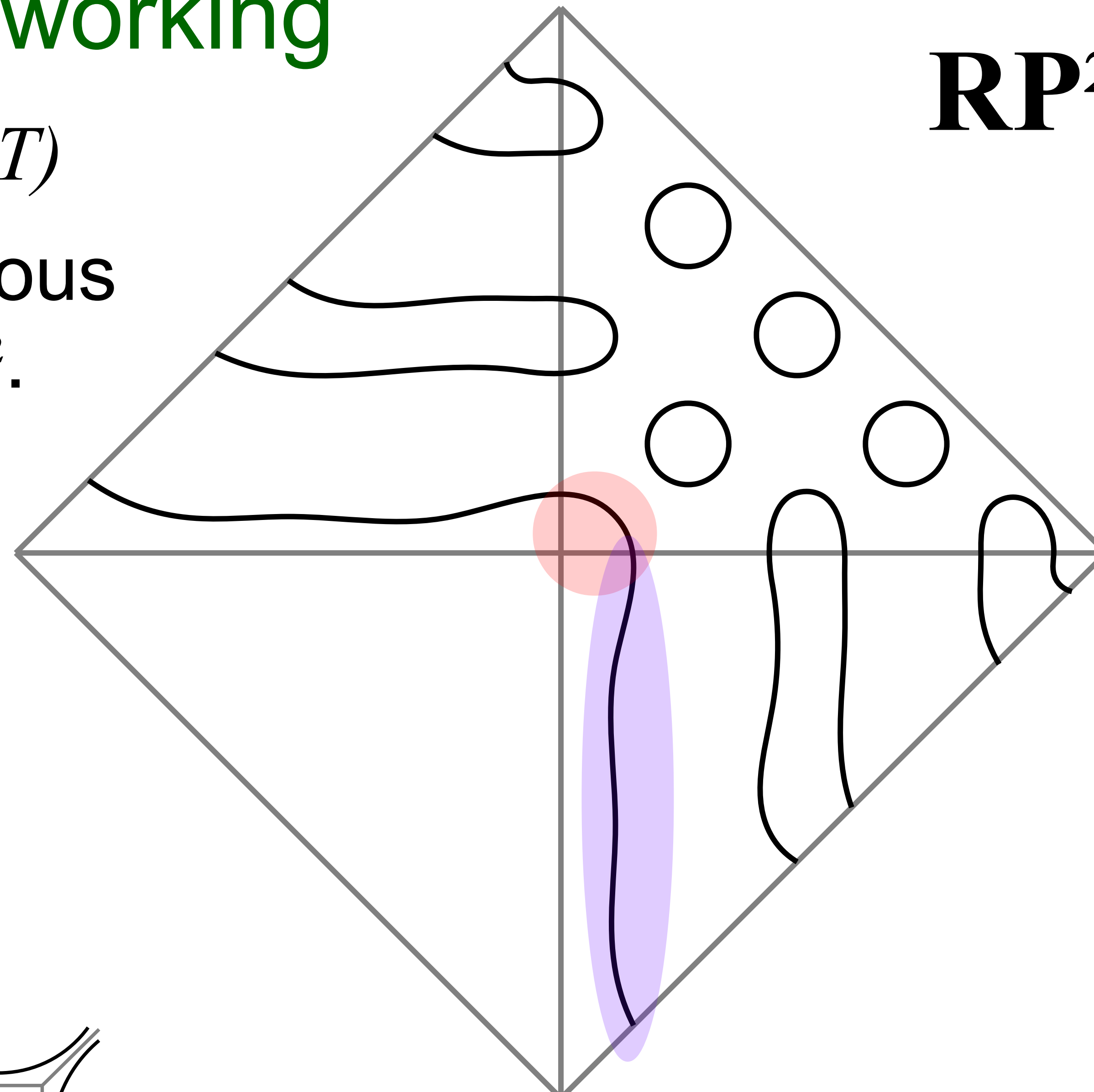




$\mathbf{RP}^2$

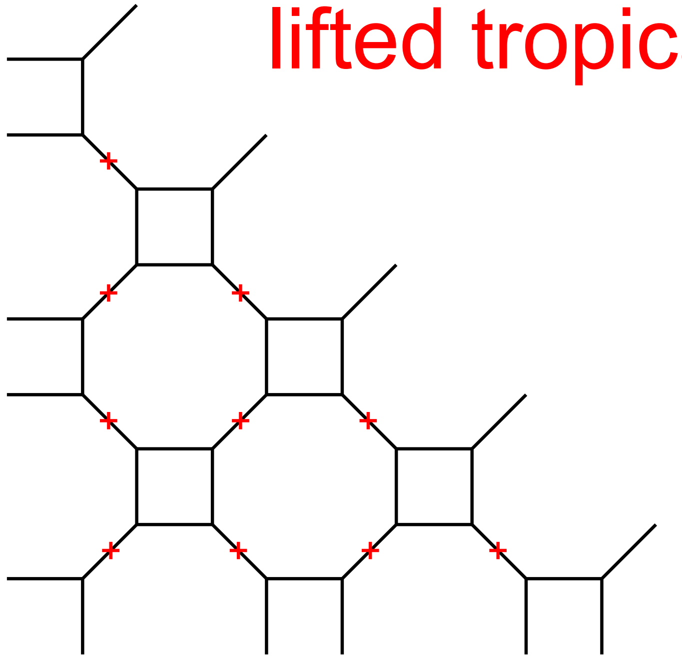
Definition

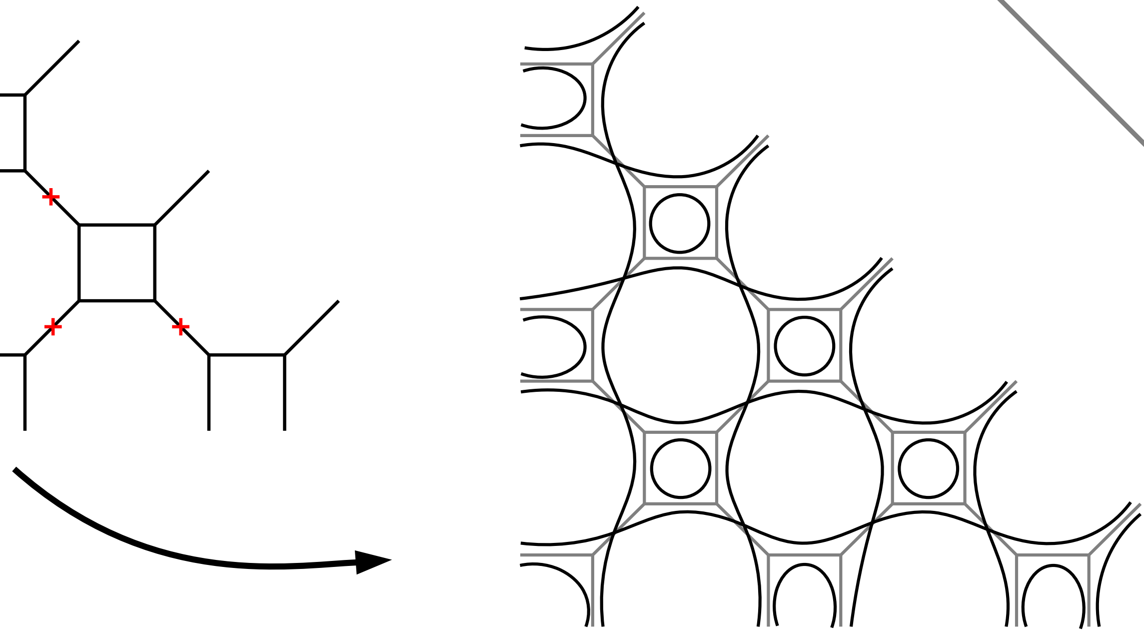
The subset on the right

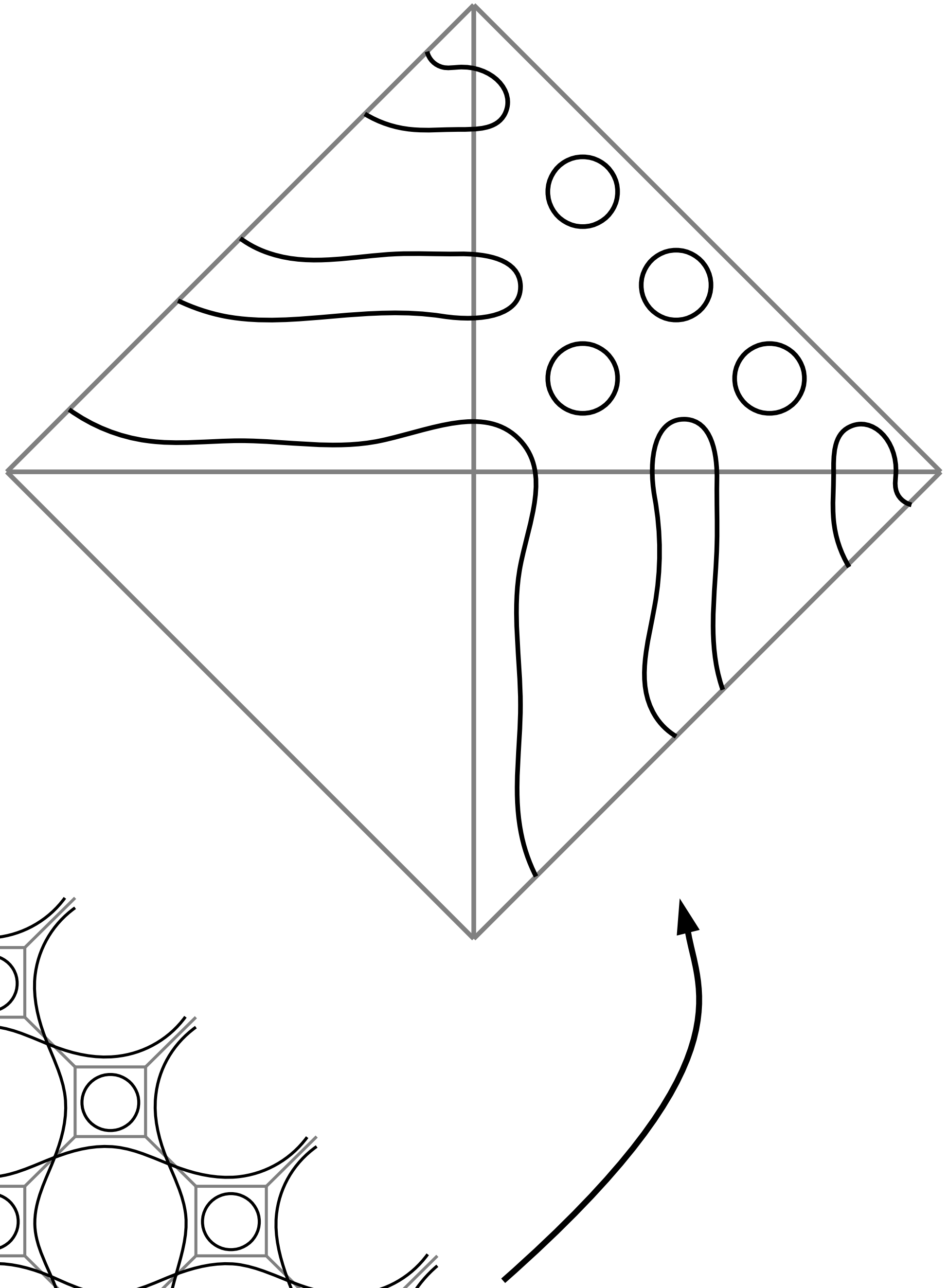
$$L(X,T) \subset \mathbf{RP}^2$$

is the

lifted tropical curve.







Theorem (Viro, 70s, „tropical“ reformulation)

Assume that X is given by the tropical polynomial

$$f = \sum\nolimits_{tr} a_{ij} \cdot_{tr} x^i \cdot_{tr} y^j .$$

Then there exists a choice of signs for the family of real polynomials

$$F_t = \sum \pm t^{a_{ij}} z^i w^j$$

such that for large values of t

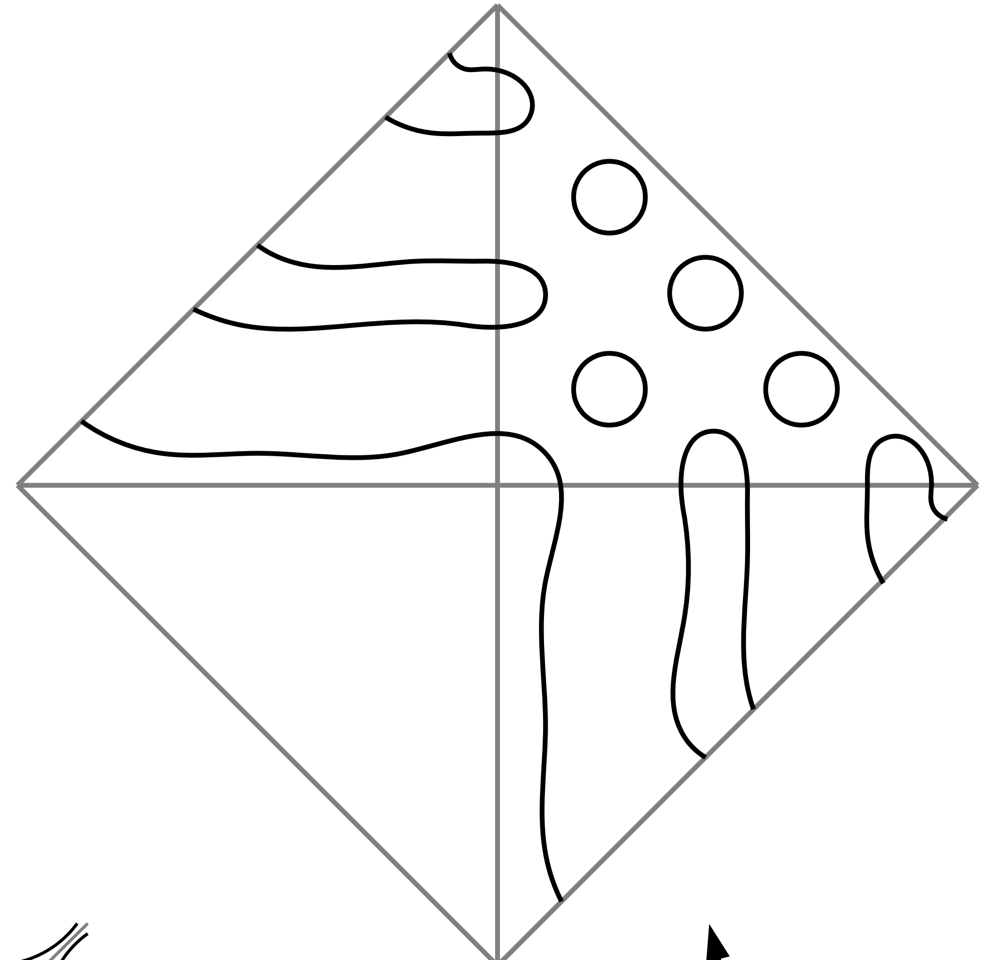
$$V(F_t) \subset \mathbf{RP}^2$$

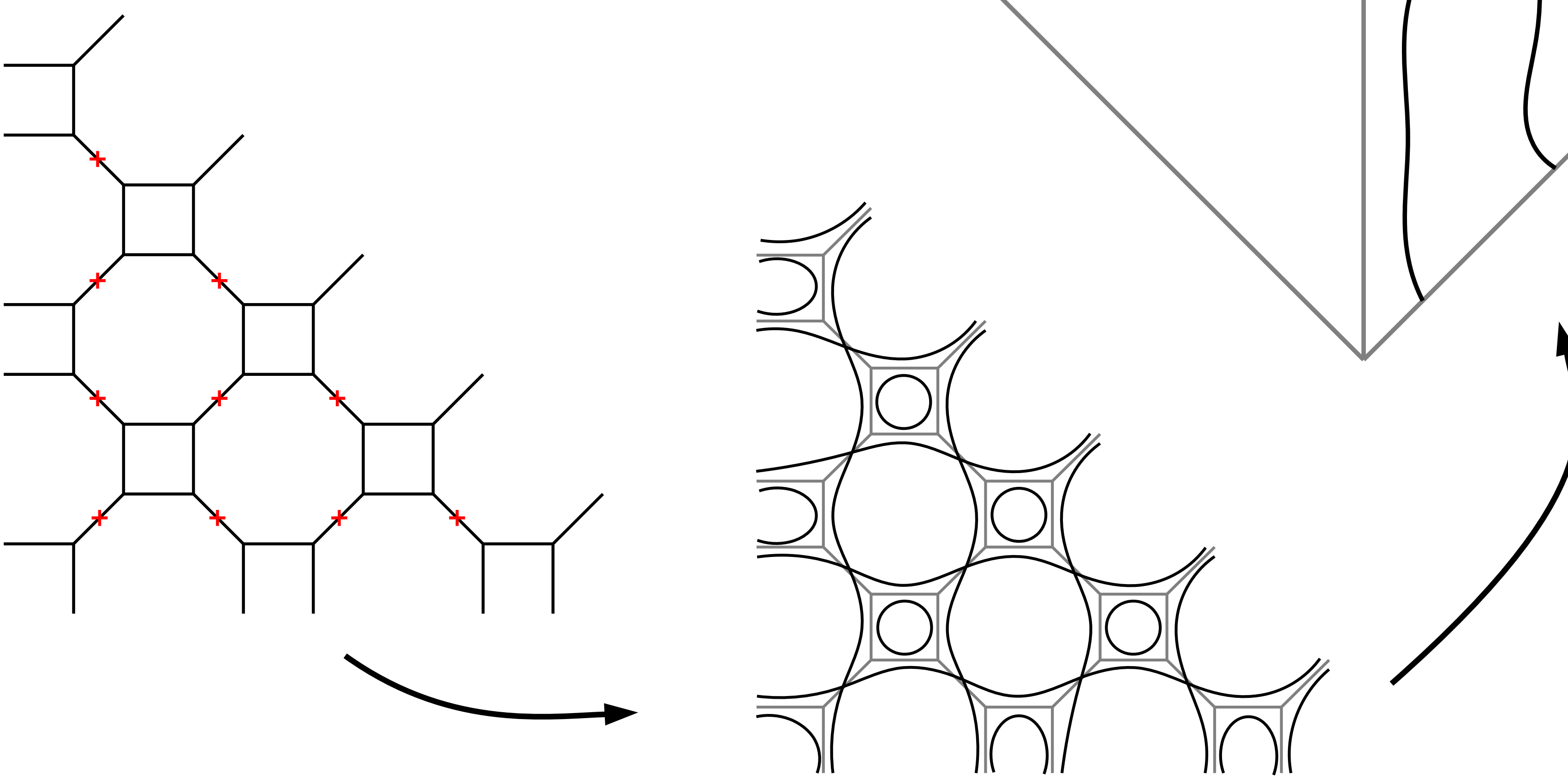
is homeomorphic to the lifted tropical curve

$$L(X, T) \subset \mathbf{RP}^2 .$$

Punchline

L(X,T) is the topological type of a smooth real algebraic curve of the same degree.

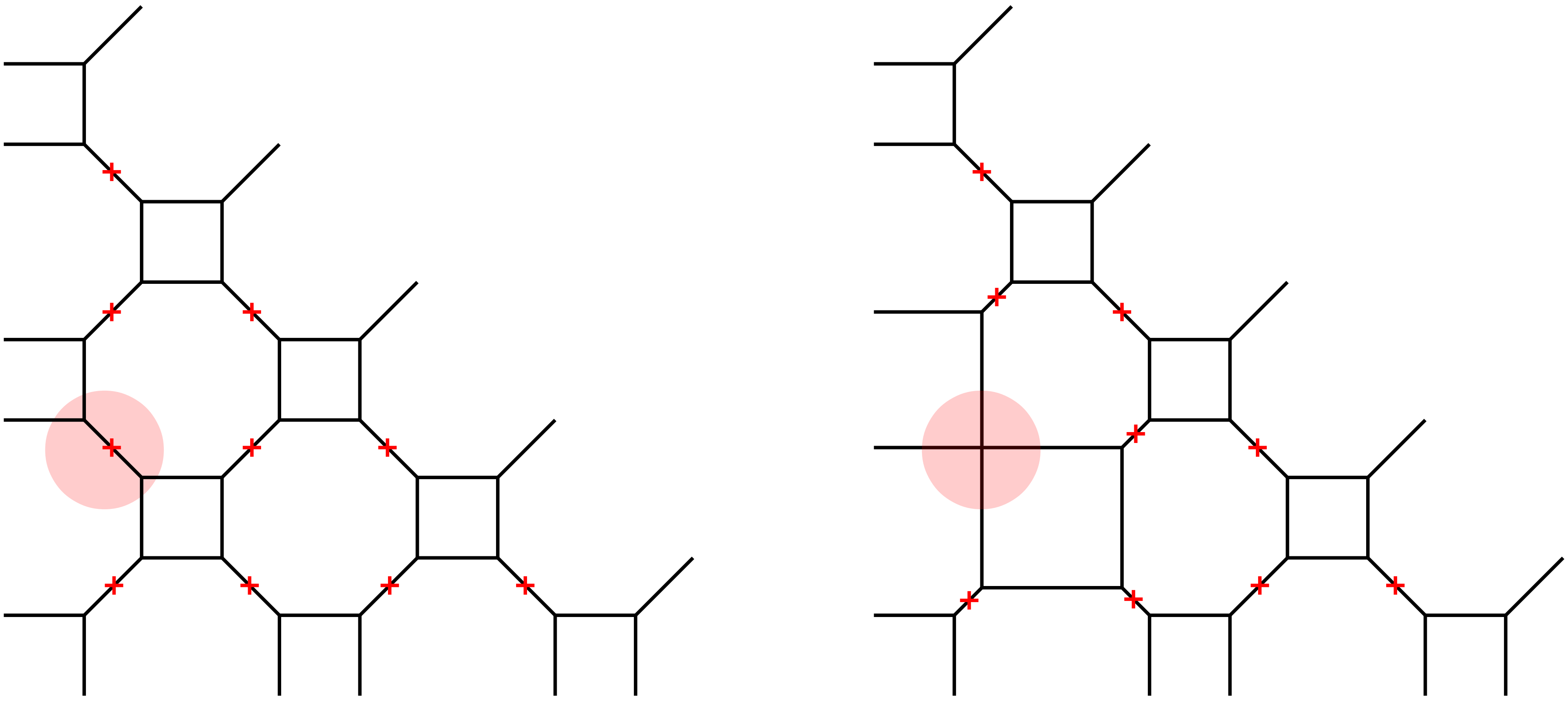




Double points

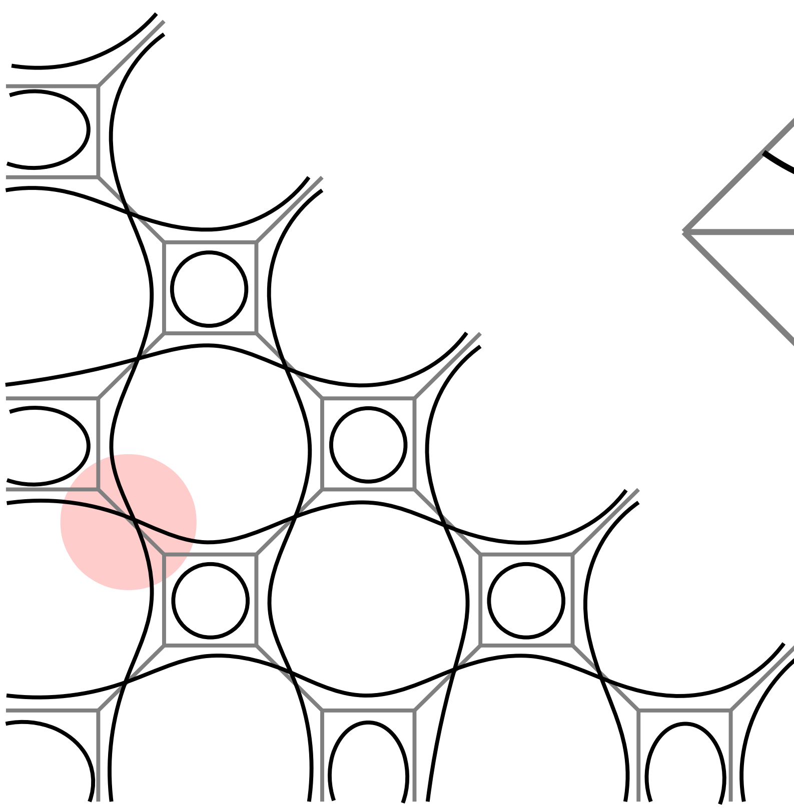
Take e in T a twisted edge such that

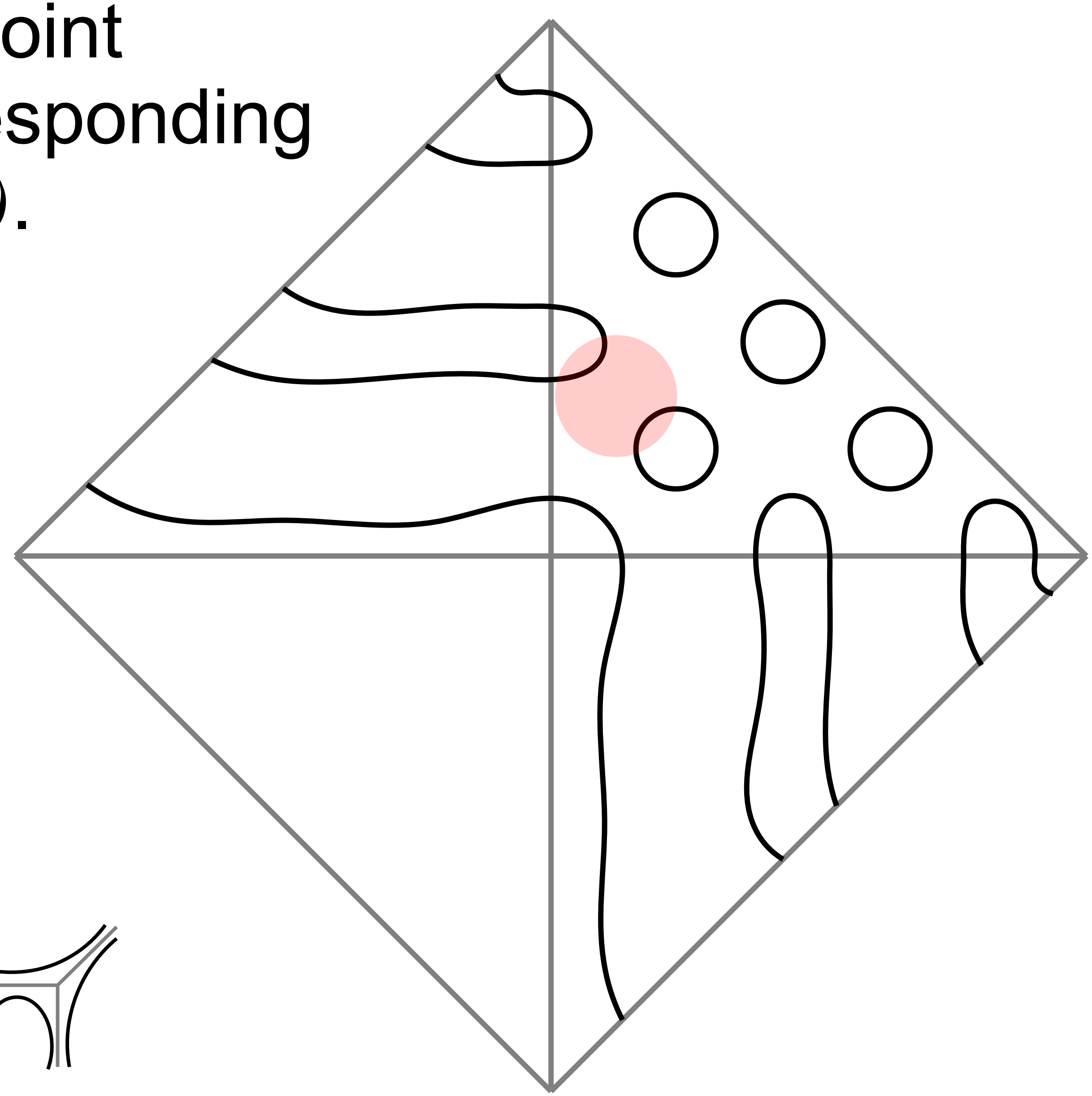
- the edge e can be contracted,
- the new vertex is a „crossing“.



Double points

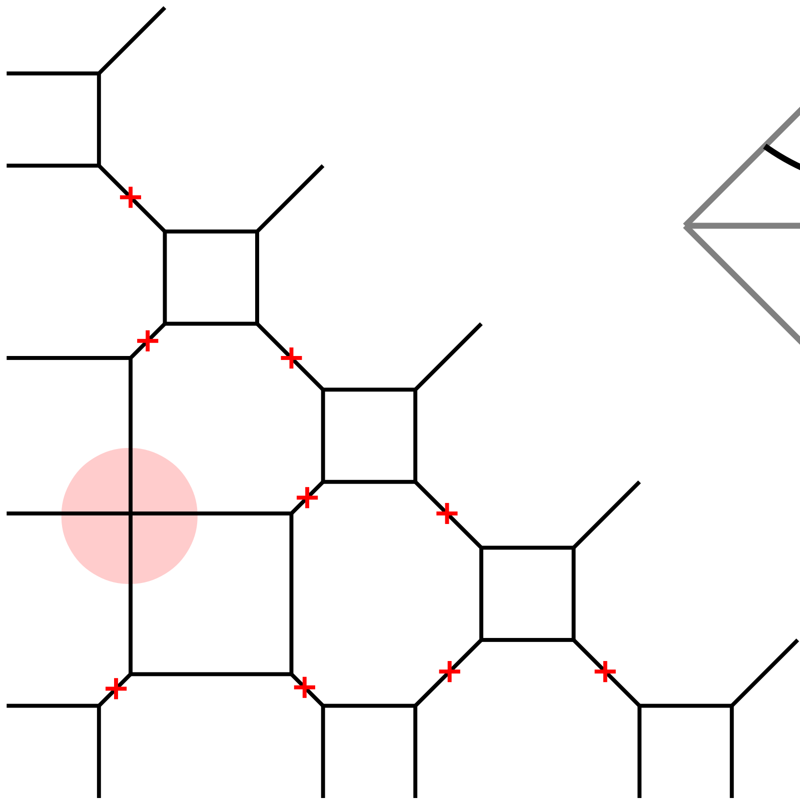
Create a double point between the corresponding branches in $L(X,T)$.

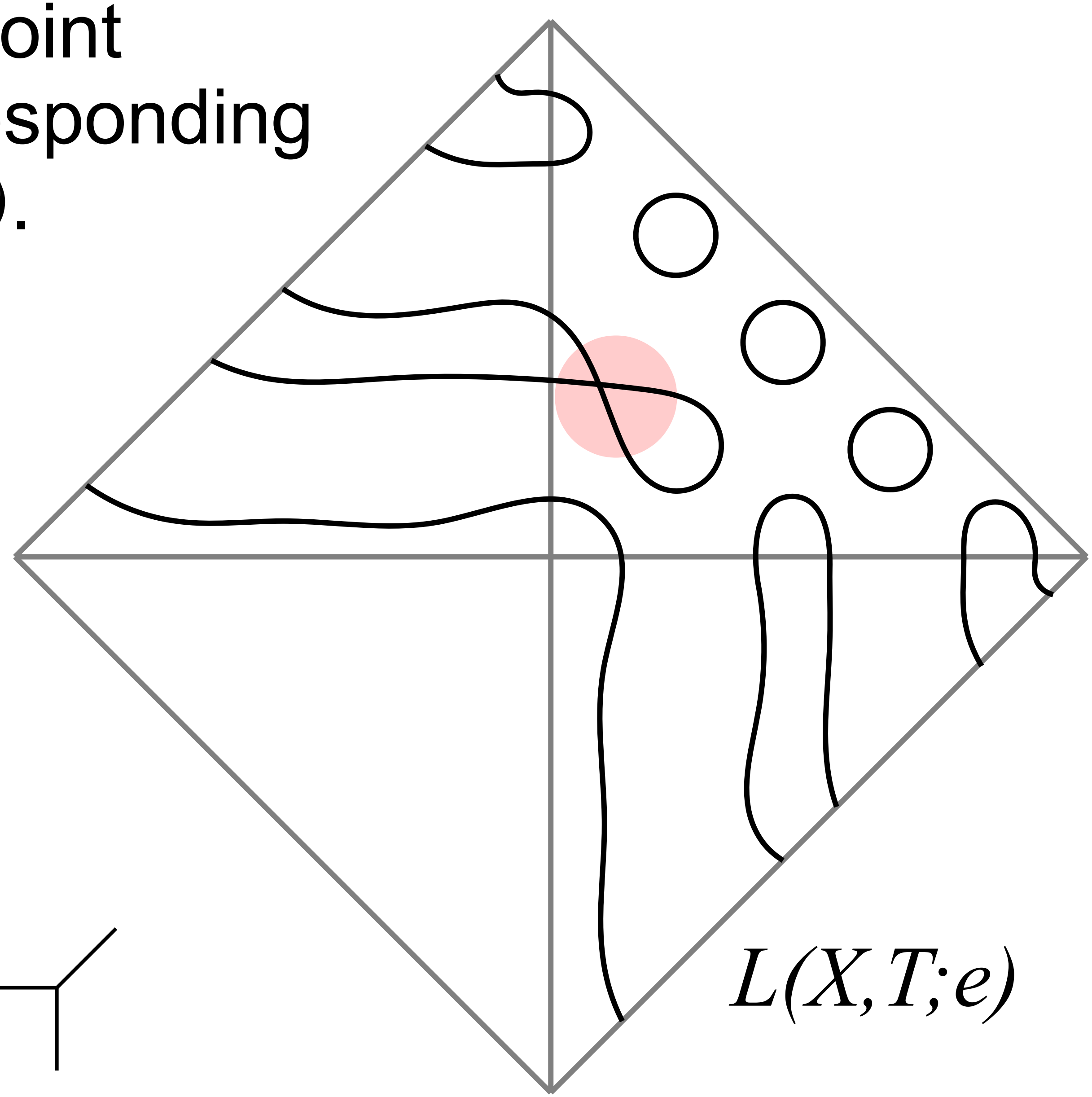




Double points

Create a double point between the corresponding branches in $L(X,T)$.

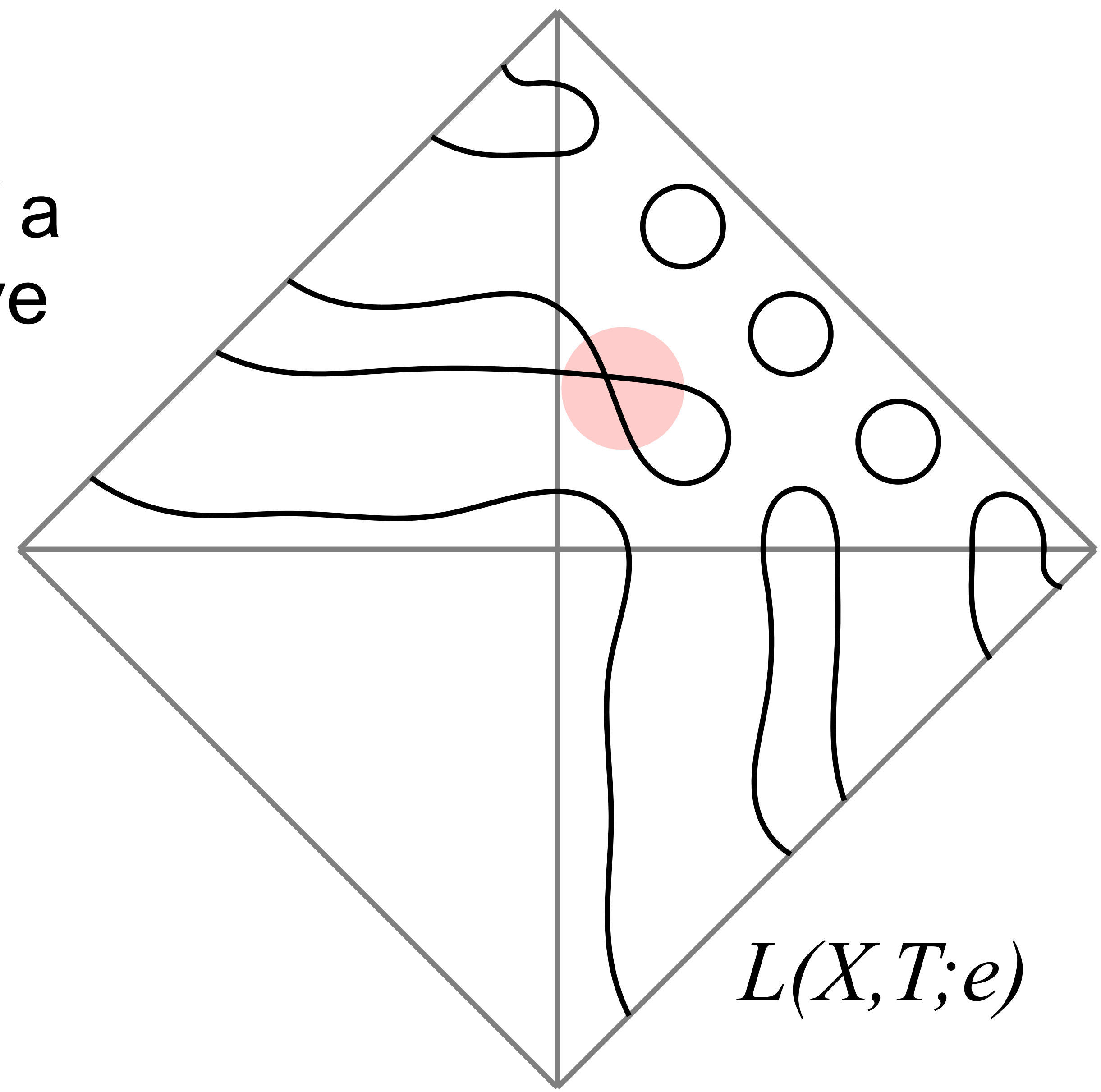




Theorem

(Itenberg-Mikhalkin-R., based on Shustin)

$L(X,T;e)$ is the topological type of a real algebraic curve with one ordinary double point.

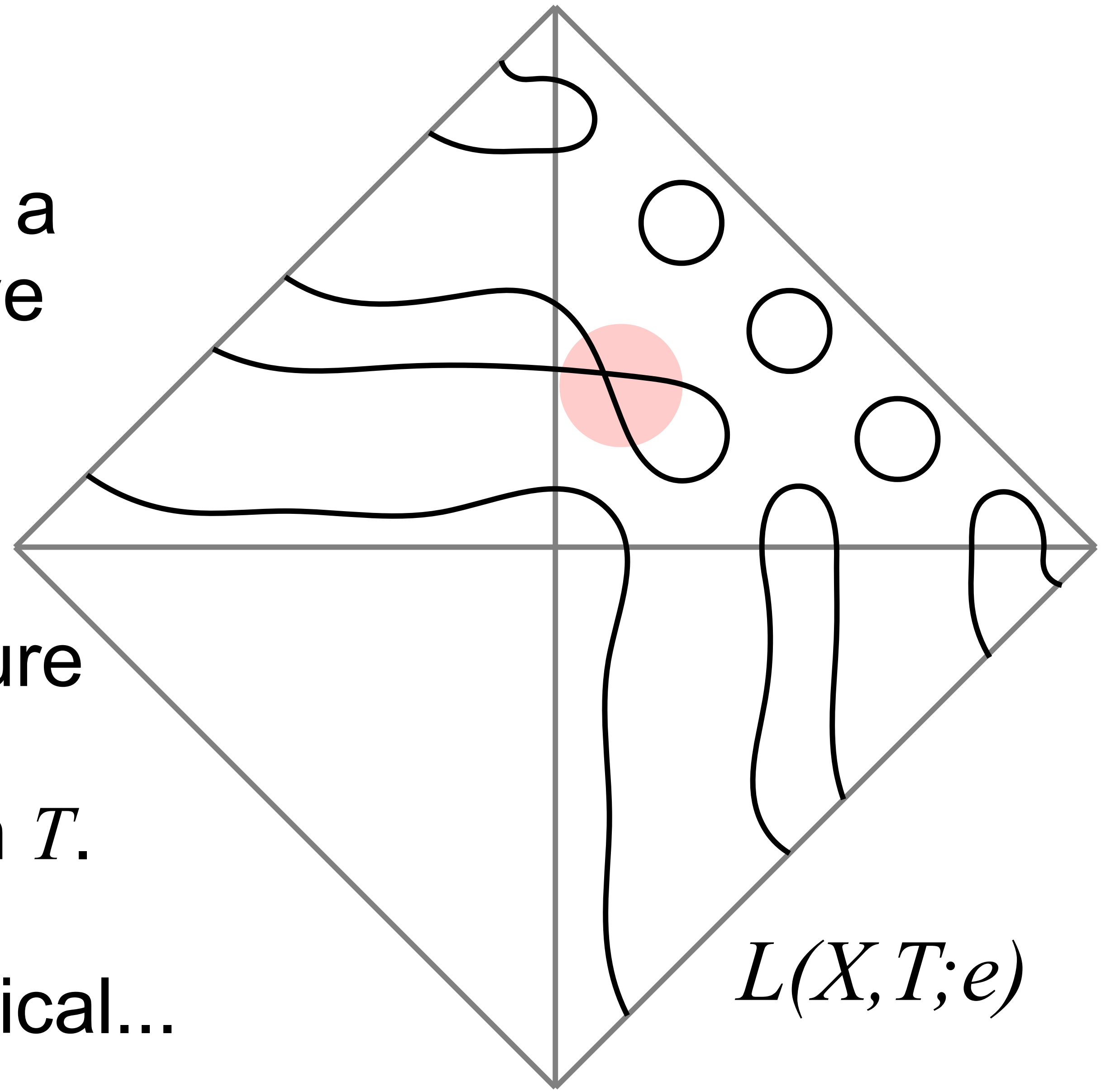


Theorem
(Itenberg-Mikhalkin-R., based on Shustin)

$L(X,T;e)$ is the topological type of a real algebraic curve with one ordinary double point.

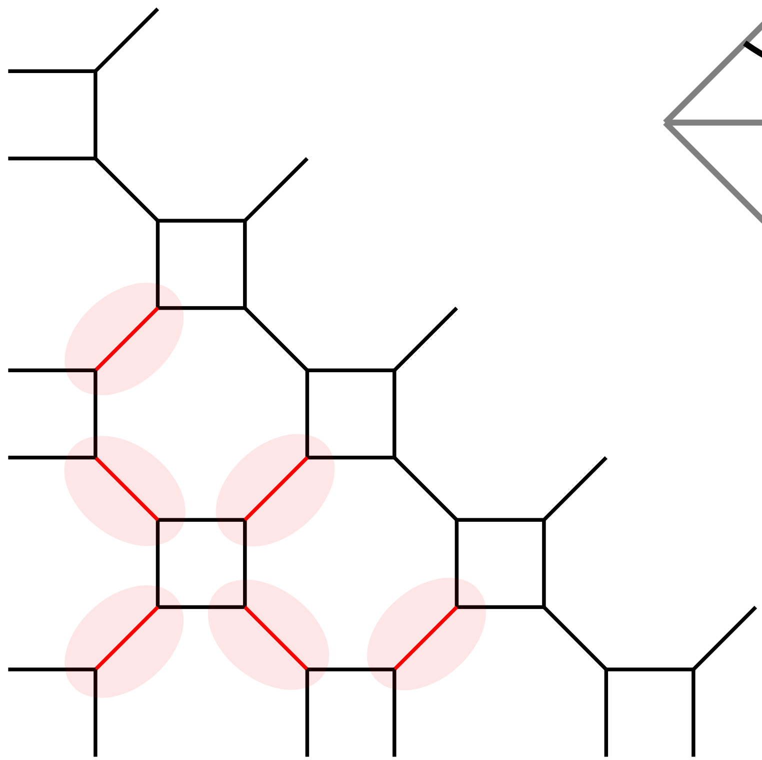
Moreover, procedure can be iterated to several edges C in T.

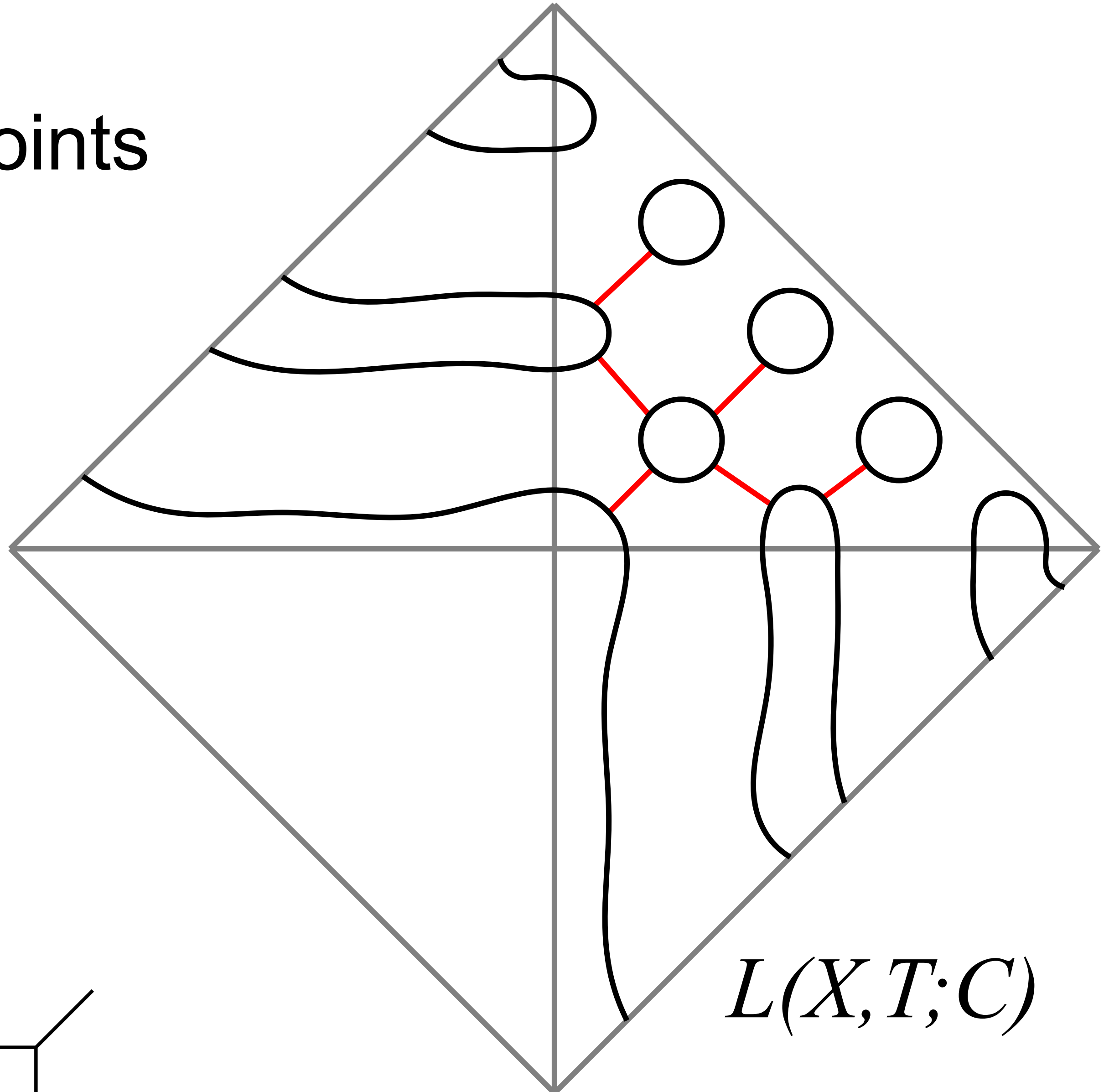
$L(X,T;C)$ is topological...



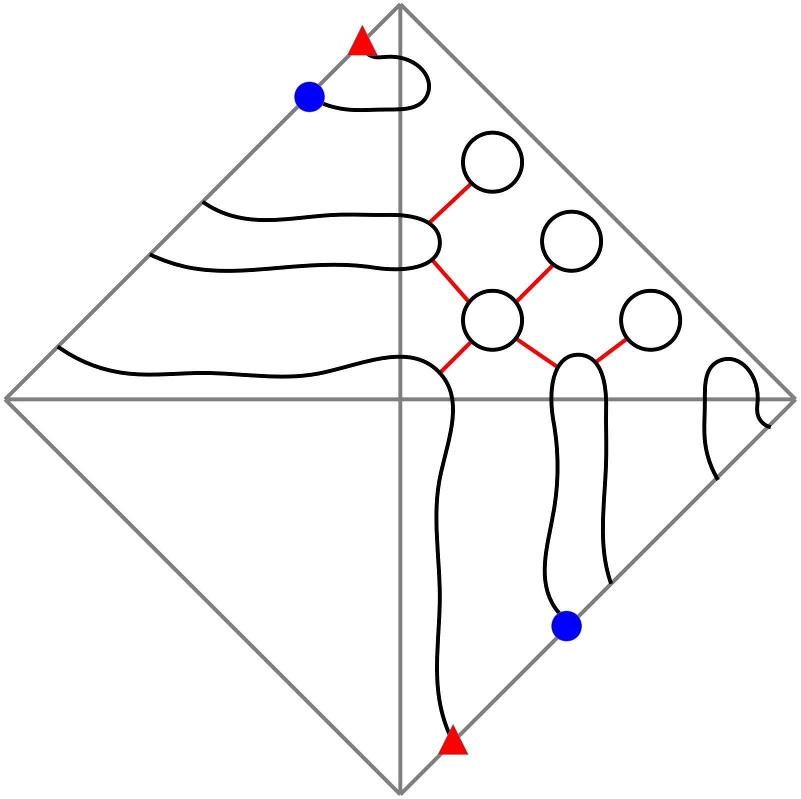
Example

6 edges/double points





Simplify!



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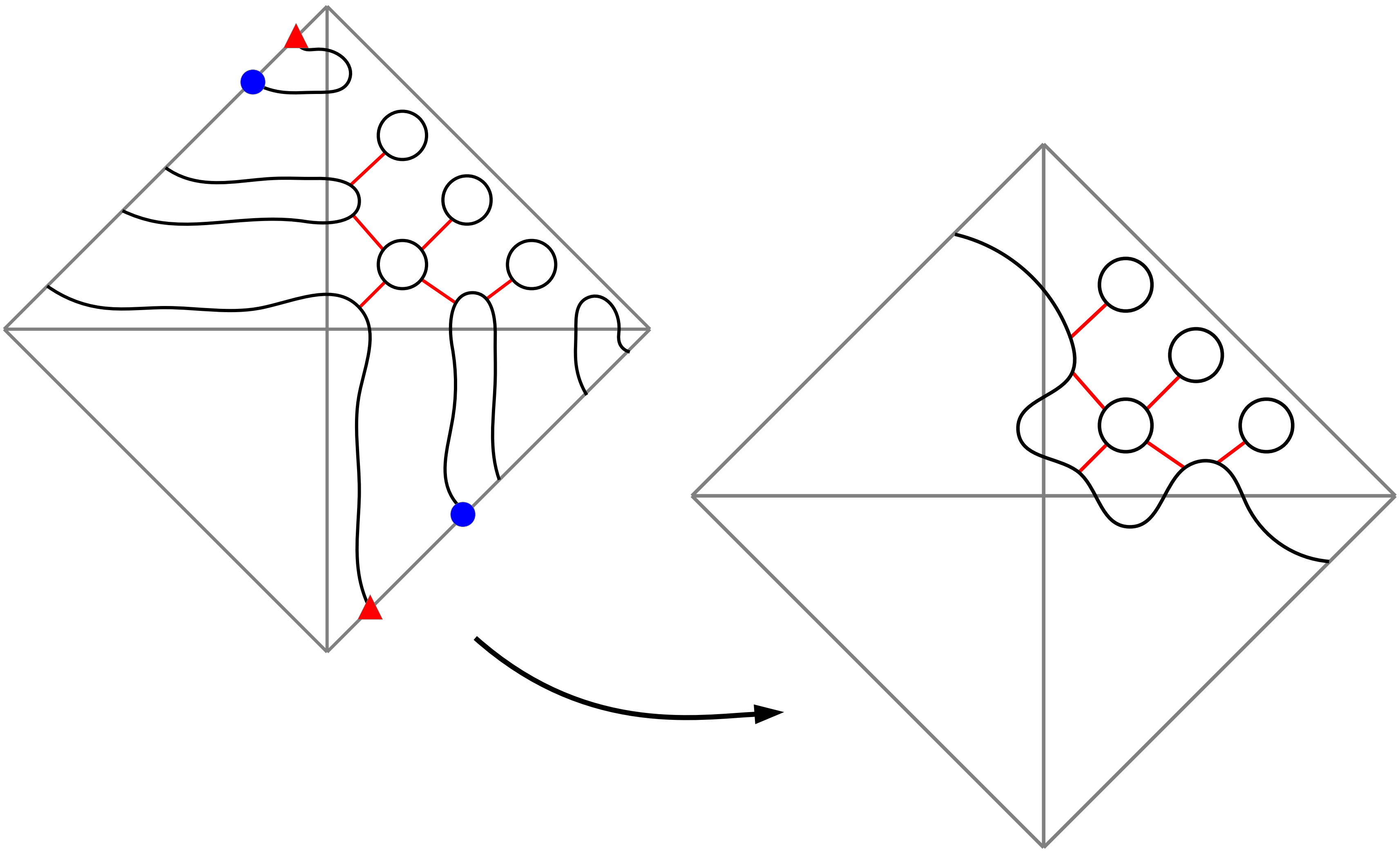

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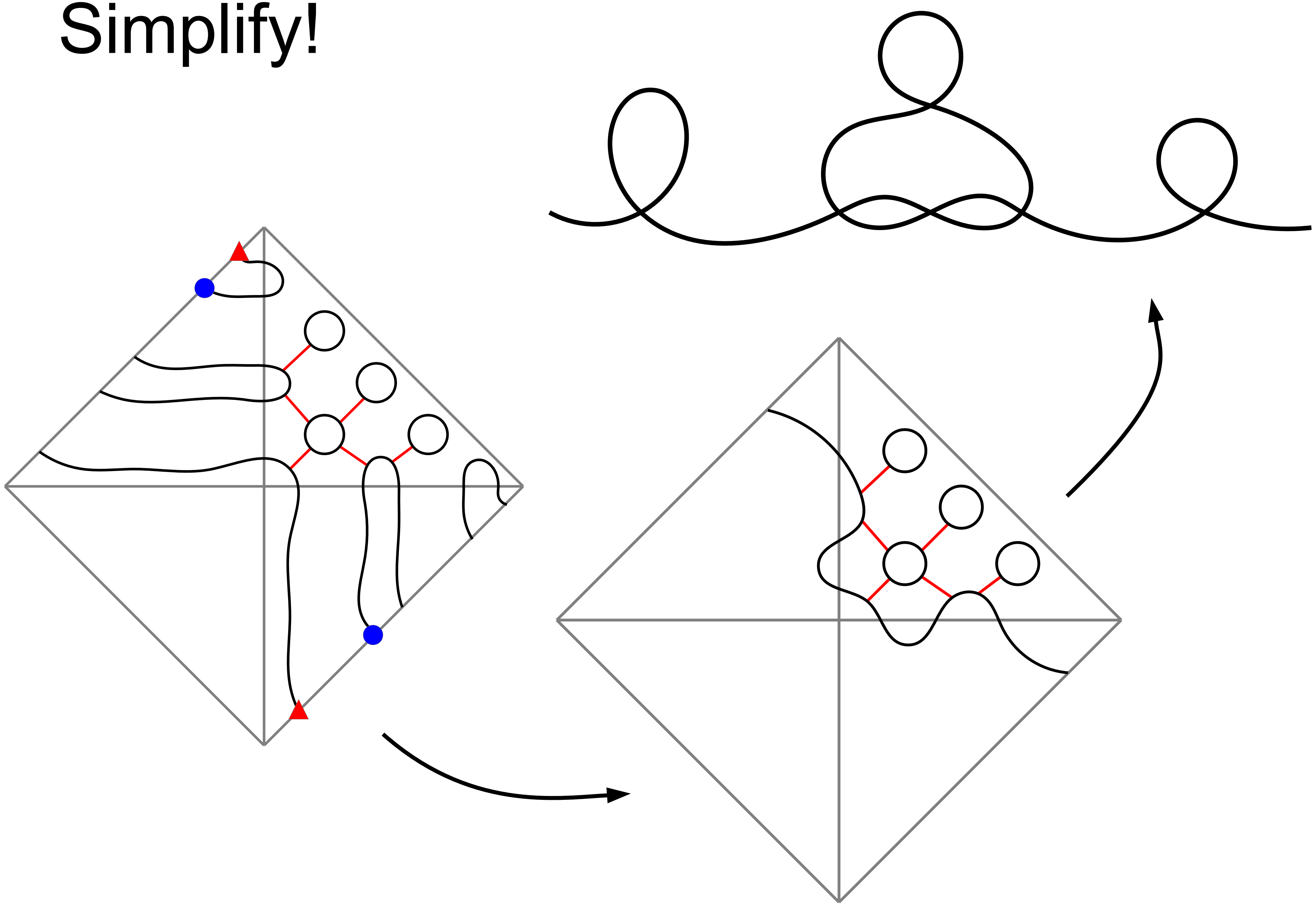

Thank you!