The Dimension of an Amoeba

joint with

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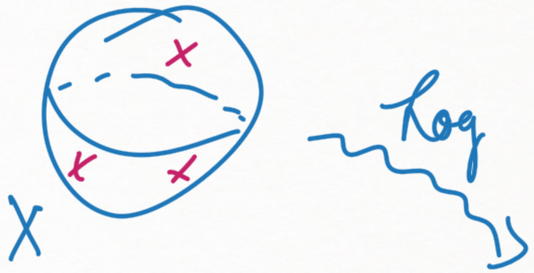
$X \subset (\mathbb{C}^*)^n$

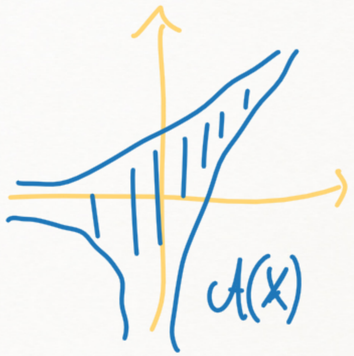
closed irred. variety

$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$

$(z_1, \dots, z_n) \longmapsto (\log|z_1|, \dots, \log|z_n|)$

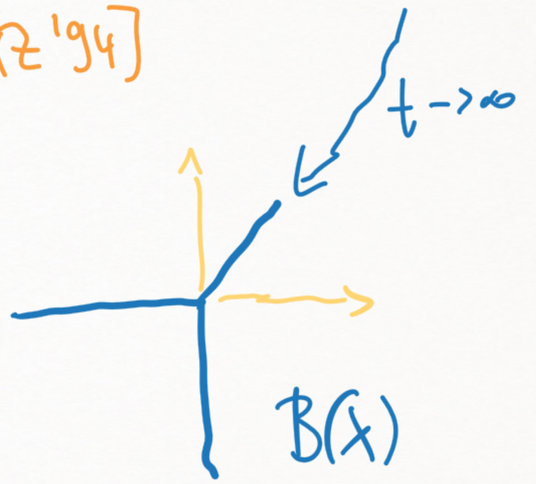
Ex: $z + w + 1 = 0$





DEF: $\mathcal{A}(X) = \mathrm{Log}(X)$ — amoeba [GKZ '94]

$\mathcal{B}(X) = \lim_{t \to \infty} \frac{1}{t} \mathcal{A}(X)$ — Bergman fan [B '11]



Hypersurfaces

$X = V(f)$ $\quad \mathcal{A}(f) = \mathcal{A}(V(f))$

THM 1 [GKZ '94]

Connected components $\mathcal{C}$ of $\mathbb{R}^n \setminus \mathcal{A}(f)$

$\updownarrow$ 1:1

Laurent series expansions of $\overline{F}$

Domain of convergence $\mathrm{Log}^{-1}(\mathcal{C})$ $\Rightarrow \mathcal{C}$ convex

THM 2 [FPT '00]

$\pi_0(\mathbb{R}^n \setminus \mathcal{A}(f)) \to NP(f) \cap \mathbb{Z}^n$

$\mathcal{C} \longmapsto \mathrm{ind}(\mathcal{C})$

(a) injective

(b) $\mathrm{Vert}(NP(f)) \subseteq \mathrm{Im}(\mathrm{ind})$

(c) $rc(\mathcal{C}) = nc(NP(f), \mathrm{ind}(\mathcal{C}))$

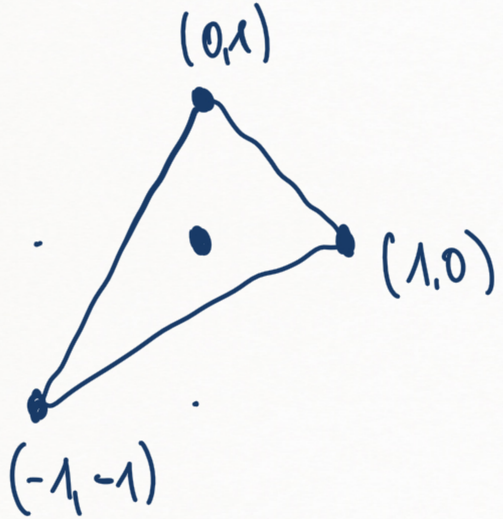
rc: recession cone

nc: normal cone

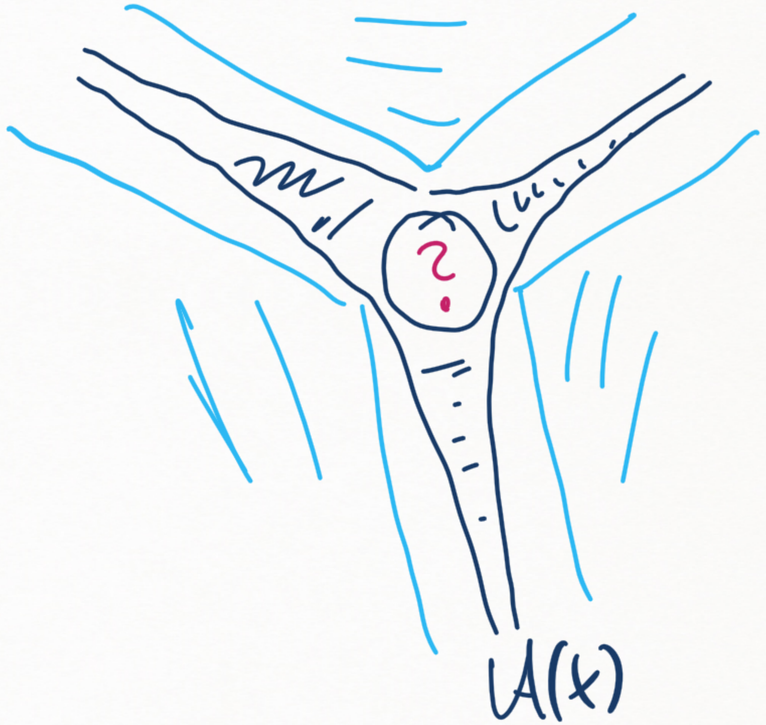


Hypersurfaces

Example $f = a + z + w + \frac{1}{zw}$, $a \in \mathbb{C}$



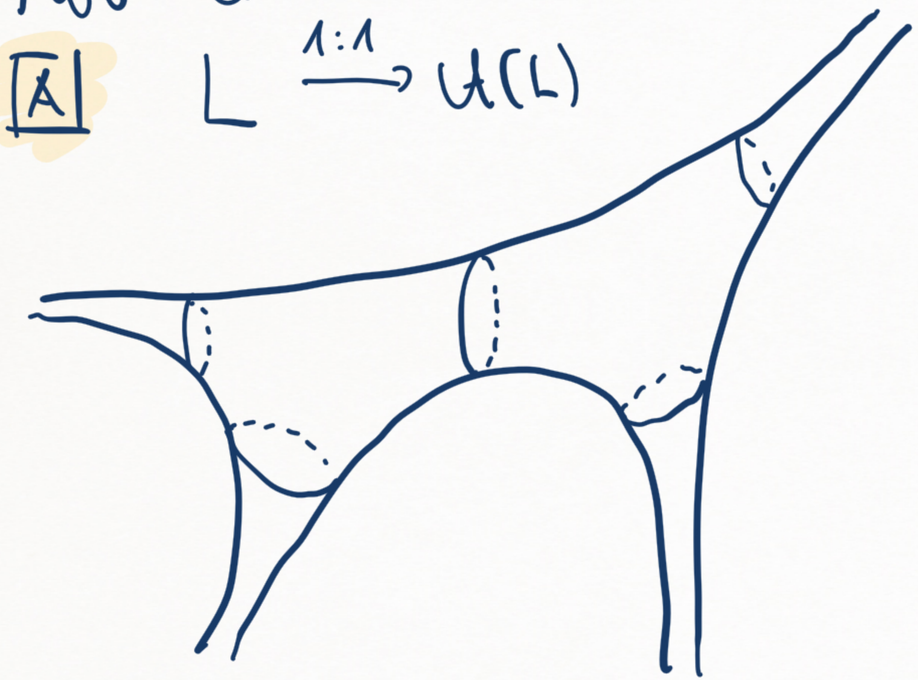




Higher Codimensions

Two cases

A $L \xrightarrow{1:1} \mathcal{A}(L)$

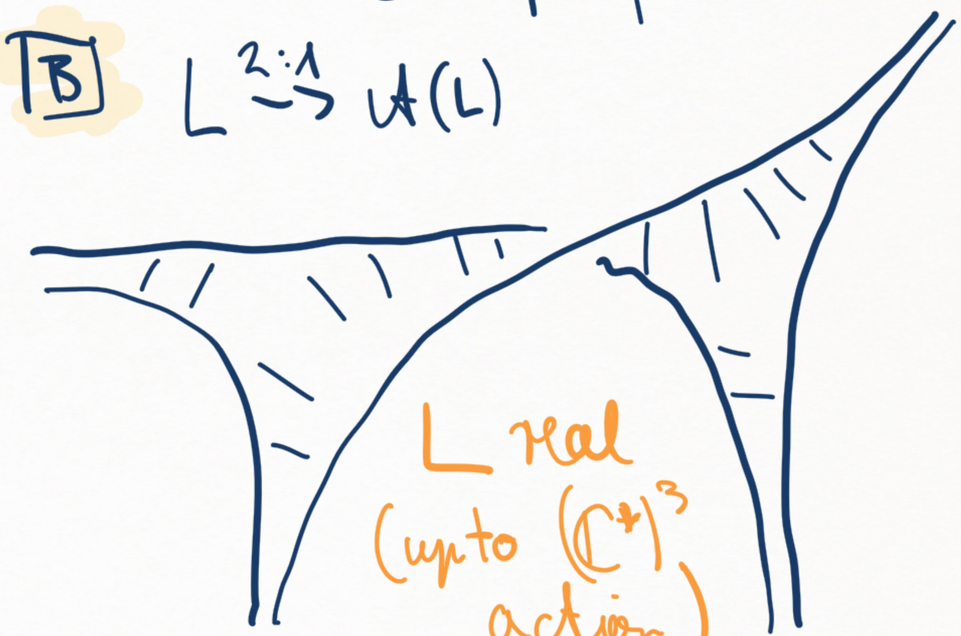


Example $L \subset (\mathbb{C}^*)^3$ line

$\| $

$\mathbb{CP}^1 \setminus 4\,\text{pts}$

B $L \overset{2:1}{\rightarrow} \mathcal{A}(L)$



Higher Codimensions

THM [Hilbert 1890]

$$X = V(f_1) \cap \dots \cap V(f_r)$$

THM [Sturmfels-Speyer '03]

$$B(X) = B(f_1) \cap \dots \cap B(f_r)$$

THM [Bushhoo '08]

$$\mathcal{A}(X) = \bigcap_{f \in I(X)} \mathcal{A}(f)$$

Question: Finitely many?

THM [Nisse-Sottile '18]

X non-degenerate $\Rightarrow$ NO!

$2 \dim_{\mathbb{C}} X < n$

$\dim \mathcal{A}(X) = ??$

Ex: $L \subset (\mathbb{C}^*)^3$ line

$\dim \mathcal{A}(X) = ??$

TRIVIAL BOUNDS

$$\dim_{\mathbb{R}} \mathcal{A}(X) \leq 2 \dim_{\mathbb{C}} X, n$$

SUBTORI

$T \subseteq (\mathbb{C}^*)^n$ subtorus,

$\mathcal{A}(T) = V \qquad V \subset \mathbb{R}^n$ rat'l subspace

$\dim_{\mathbb{R}} \mathcal{A}(T) = \dim_{\mathbb{C}} T$

BASIC EXAMPLES (for $\subsetneq$)

A $X \subseteq \mathfrak{a}T$ contained in subtorus

$\dim \mathcal{A}(X) \leq \dim T$

B $T \cdot X = X$ subtorus acting on X

$\dim \mathcal{A}(X) \leq 2 \dim X - \dim T$

$\dim \mathcal{A}(X/T) + \dim \mathcal{A}(T) \leq 2 \dim X/T + \dim T$

$\dim T$

Our Formula

first version

THM (Draisma - R - Yuen)

$$\dim \mathcal{A}(X) = \min_{\substack{T \subset (\mathbb{C}^*)^n \\ \text{subtorus}}} \left\{ 2 \dim TX - \dim T \right\}$$

TRIVIAL BOUNDS

$T = \{1\}$ and $T = (\mathbb{C}^*)^n$

BASE EXAMPLES

[A] $X \subset aT$

[B] $TX = X$

COROLLARY (also NS'18)

$\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{C}} X \iff X = aT$

since $2 \dim TX \geq \dim T + \dim X$ ...

Proof

Induction of $\dim X$ using

LEMMA

If $\dim \mathcal{A}(X) < 2\dim X$,

then here exists $v \in \mathbb{Q}^n$ s.th.

$$v \in T_p \mathcal{A}(X)$$

for all (smooth) $p \in \mathcal{A}(X)$.

IDEA [1] $\mathrm{Log}_{\mathbb{C}} : (\mathbb{C}^*)^n \to \mathbb{R}^n \times \mathbb{R}^n/\mathbb{Z}^n$

$\circlearrowleft \cdot i$ on $(\mathbb{C}^*)^n$; $\cdot i$ on $\mathbb{R}^n \times \mathbb{R}^n/\mathbb{Z}^n$

[2] X is complex subvariety

$\Rightarrow$ $T_q X \circlearrowleft \cdot i$ invariant

[3] Suffices to find $v \in T_q F_p$

with $F_p = \mathrm{Log}_X^{-1}(p) \subset \mathbb{R}^n/\mathbb{Z}^n$

F_p fiber

[4] "$<$" implies $\dim F_p > 0$ ....

Our formula

second version

THM (—) Let $F = B(X)$

$$\dim \mathcal{A}(X) = \min_{\substack{V \subset \mathbb{R}^n \\ \text{rat'l subsp.}}} \left\{ 2\dim(V+F) - \dim V \right\}$$

$$= \{v + f | \ldots\}$$

The dimension $\dim \mathcal{A}(X)$ can be computed in terms of $B(X)$.

PROOF

$\dim_{\mathbb{C}} X = \dim_{\mathbb{R}} F$

(Bergman, Biesi-Graves)

Linear spaces / matroids

$X \subset (\mathbb{C}^*)^n$ linear $\Rightarrow$ $F = B(X) = B(M)$ matroidal fan

↳ matroid of X

DEF: $a(M) = \min\{\dots\} = \dim \mathrm{Gr}(X)$

CONJECTURE

Minimum is attained at

$$V = \bigcap H_{ij}$$

with

$$H_{ij} = \{x_i = x_j\} \subset \mathbb{R}^n$$

COROLLARY

$$a(M) = \min_{P_1 \sqcup \dots \sqcup P_\ell = \{0,\dots,n\}} 2\left(\mathrm{rk}\, M|_{P_1} + \dots + \mathrm{rk}\, M|_{P_\ell}\right) - \ell - 1$$

leading term in characteristic polynomial of M ??

THANK YOU!