Monodromy Of Knizhnik-Zamolodchikov Equations

Florent Schaffhauser

Keio University, Yokohama

Abstract

In this poster, we recall, following [1], two constructions of (families of) representations of Artin's braid group B_n:

$$\rho_n^{KZ} : B_n \longrightarrow \mathrm{Aut}_{\mathbb{C}[[h]]}(W[[h]])$$

and

$$\rho_n^{R_h} : B_n \longrightarrow \mathrm{Aut}_{\mathbb{C}[[h]]}(W[[h]]).$$

The representation ρ_n^{KZ} is obtained analytically: it is the monodromy representation of a certain flat connection called the Knizhnik-Zamolodchikov connection. The representation $\rho_n^{R_h}$ is itself obtained algebraically: it is the braid group representation associated to the universal R-matrix of the quantum enveloping algebra $\mathcal{U}_h(\mathfrak{g})$. Both these representations will be constructed starting from a complex semisimple Lie algebra $\mathfrak{g}$ and objects attached to $\mathfrak{g}$. The purpose of this poster is to give some of the tools needed to understand the statement of the following theorem:

Theorem 1 (The Kohno-Drinfeld theorem) *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let V be a $\mathfrak{g}$-module. The monodromy representation of a certain system of differential equations with values in $V^{\otimes n}$, called the Knizhnik-Zamolodchikov equations, is equivalent to the braid group representation associated to the universal R-matrix of the quantum enveloping algebra $\mathcal{U}_h(\mathfrak{g})$.*

Monodromy representations

Starting from a vector bundle $E \to X$ over a manifold X endowed with a flat connection ∇, the associated *monodromy representation* is a representation of the fundamental group $\pi_1(X)$ of the base space in the automorphism group of the fibre of the bundle. In other words, there is a map

$$\{\text{flat connections on } (E \to X)\} \longrightarrow \{\text{morphisms } \pi_1(X,x) \to \mathrm{Aut}(E_x)\}$$

whose definition we now recall.

A connection ∇ on a vector bundle $(E \to X)$ can be seen as a way of deriving sections, that is, as a map:

$$\nabla : \Omega^0(X,E) = \Gamma(E) \longrightarrow \Omega^1(X,E) = \Gamma(T^*X \otimes E)$$

(satisfying $\nabla(f.\sigma) = df \otimes \sigma + f.\nabla\sigma$) or, equivalently, as parallel transport along paths in X:

$$(\gamma : [0,1] \longrightarrow X) \longmapsto (T_\gamma : E_{\gamma(0)} \xrightarrow{\simeq} E_{\gamma(1)})$$

(linear isomorphism between the fiber at the origin of γ and the fiber at the end of γ).

Lemma 2 *One has: $T_{\gamma\gamma'} = T_\gamma \circ T_{\gamma'}$ for composition of compatible paths (in particular for loops at the same base point).*

By definition, the *holonomy group at* $x \in X$ is the subgroup of $\mathrm{Aut}(E_x)$ generated by the T_γ for γ a *loop* based at x. One says that one is in the presence of *monodromy* when the condition (γ homotopic to γ') implies ($T_\gamma = T_{\gamma'}$). Observing that a connection $\nabla : \Omega^0(X,E) \to \Omega^1(X,E)$ uniquely extends to a covariant derivative

$$\Omega^0(X,E) \xrightarrow{\nabla} \Omega^1(X,E) \xrightarrow{\nabla} \Omega^2(X,E) \xrightarrow{\nabla} \dots$$

satisfying $\nabla(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^{|\omega|}\omega \wedge \nabla\omega'$, one may state a necessary and sufficient condition for monodromy to hold:

Definition 3 (Curvature of a connection) *The map*

$$K^\nabla := \nabla \circ \nabla : \Omega^0(X,E) \to \Omega^2(X,E)$$

is called the curvature *of the connection ∇. The connection ∇ is said to be* flat *if $K^\nabla = 0$.*

Proposition 4

$$\big((\gamma \text{ homotopic to } \gamma') \implies (T_\gamma = T_{\gamma'})\big) \quad \text{iff} \quad (K^\nabla = 0).$$

Therefore, in the presence of a *flat connection*, the following map is well-defined and it is a group morphism by lemma 2:

$$\begin{aligned} \rho : \pi_1(X,x) &\longrightarrow \mathrm{Aut}(E_x) \\ [\gamma] &\longmapsto T_\gamma. \end{aligned}$$

This representation of $\pi_1(X,x)$ is called the *monodromy representation* associated to the flat connection ∇ on $(E \to X)$. We have seen that such a representation arises when holonomy depends only on homotopy.

The KZ system

1. Monodromy representations of Artin's braid group. In order to construct, as in the above section, a monodromy representation of Artin's braid group

$$(*) \quad B_n = <\sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geqslant 2 \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}>$$

the first step is to find a manifold X_n satisfying $\pi_1(X_n) = B_n$. Here, X_n will be the configuration space of n pairwise distinct points in the complex plane, up to permutation:

$$X_n = Y_n/\mathfrak{S}_n \text{ where } Y_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid i \neq j \implies z_i \neq z_j\}.$$

Then, we need to construct a vector bundle on X_n and a flat connection on this bundle. We start with a vector space W and consider the trivial bundle $((Y_n \times W) \to Y_n)$. The Knizhnik-Zamolodchikov system is the following system of differential equations:

$$(\text{KZ}): \quad dw = \sum_{1 \leqslant i < j \leqslant n} \frac{dz_i - dz_j}{z_i - z_j} A_{ij}.w,$$

where $w : Y_n \to W$ is a function and the A_{ij} are endomorphisms of W. Here, the bundle is trivial, so the connection whose horizontal sections ($\nabla\sigma = 0$) correspond to solutions of the KZ system is:

$$\nabla^{KZ} = d - \Gamma, \text{ where } \Gamma = \sum_{i<j} \frac{dz_i - dz_j}{z_i - z_j} A_{ij}.$$

Proposition 5 *The following conditions are sufficient for the KZ connection ∇^{KZ} to be flat:*

$$(**) \begin{cases} [A_{ij}, A_{kl}] = 0 \text{ for all pairwise distinct } i,j,k,l \\ [A_{ij}, A_{ik} + A_{jk}] = 0, \\ [A_{jk}, A_{ij} + A_{ik}] = 0. \end{cases}$$

From now on, we assume that the connection ∇^{KZ} is flat, that is, that conditions $(**)$ are satisfied by the endormorphisms A_{ij} of W. Then, by the considerations of the first section of this poster, we have a monodromy representation $\pi_1(Y_n) \to \mathrm{Aut}(W)$. The fundamental group $\pi_1(Y_n)$ is the pure braid group P_n, satisfying $B_n = P_n/\mathfrak{S}_n$. Recall likewise that $X_n = Y_n/\mathfrak{S}_n$. If we *assume additionally that* $\mathfrak{S}_n$ *acts on* W, then one can consider the (non-trivial) vector bundle ($E_n := (Y_n \times W)/\mathfrak{S}_n \to X_n$), with fibre W above X_n. Here $\mathfrak{S}_n$ acts on $Y_n \times W$ in such a way that $(z_1, \dots, z_n, \sigma.w) = (z_{\sigma(1)}, \dots, z_{\sigma(n)}, w)$ in E_n. Then, if the connection ∇^{KZ} on $((Y_n \times W) \to Y_n)$ is invariant under the action of $\mathfrak{S}_n$ (permutation of indices $i \mapsto \sigma(i)$), it induces a connection ∇^{KZ} on $(E_n \to X_n)$. Since the two bundles have the same fibre W, relations $(**)$ are automatically satisfied and this induced connection is flat. Hence a monodromy representation

$$\rho_n^{KZ} : B_n = \pi_1(X_n) \longrightarrow \mathrm{Aut}(W).$$

2. Construction of a KZ system. For the above construction to make sense, one needs to specify a vector space W and endomorhisms A_{ij} of W satisfying relations $(**)$, as well as an action of $\mathfrak{S}_n$ on W leaving the connection ∇^{KZ} invariant. To construct such a system, the initial data will be:

- a complex semisimple Lie algebra $\mathfrak{g}$.
- a symmetric $\mathfrak{g}$-invariant 2-tensor $t = \sum_r x_r \otimes y_r \in \mathfrak{g} \otimes \mathfrak{g}$ (such an element can always be constructed from the Casimir element $C \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ of $\mathcal{U}(\mathfrak{g})$, see [1]).
- a complex parameter $h \in \mathbb{C}$.
- an integer $n \geqslant 2$.
- a finite-dimensional $\mathfrak{g}$-module V.

First, set $W = V^{\otimes n}$: W is a $\mathfrak{g}$-module and $\mathfrak{S}_n$ acts on W by permutation of coordinates. Second, define:

$$t_{ij} = \sum_r \alpha_r^{(1)} \otimes \dots \otimes \alpha_r^{(n)} \in (\mathcal{U}(\mathfrak{g}))^{\otimes n}$$

where

$$\alpha_r^{(i)} = x_r, \alpha_r^{(j)} = y_r \text{ and } \alpha_r^{(k)} = 1 \text{ if } k \neq i,j.$$

Lemma 6 *Since V is a $\mathfrak{g}$-module (or equivalently a $\mathcal{U}(\mathfrak{g})$-module), the $t_{ij} \in (\mathcal{U}(\mathfrak{g}))^{\otimes n}$ induce endomorphisms of $W = V^{\otimes n}$. These endormorphisms satisfy relations $(**)$.*

The proof of this lemma follows from the construction of the t_{ij} and the $\mathfrak{g}$-invariance of $t \in \mathfrak{g} \otimes \mathfrak{g}$.

Lemma 7 *For all i,j, one has $t_{ij} = t_{ji}$.*

The proof of this lemma follows from the symmetry of the 2-tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$. We can then set $A_{ij} = \frac{h}{2i\pi} t_{ij} \in \mathrm{End}(V^{\otimes n})$ and the KZ system becomes:

$$(\text{KZ}'): \quad dw = \frac{h}{2\sqrt{-1}\pi} \sum_{1 \leqslant i < j \leqslant n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij}.w.$$

By lemmas 6 and 7, the associated KZ connection is flat and $\mathfrak{S}_n$-invariant. Therefore, by the considerations of subsection **1**, one has a monodromy representation

$$\rho_n^{KZ} : B_n = \pi_1(X_n) \longrightarrow \mathrm{Aut}(V^{\otimes n}).$$

Observe that the KZ system (KZ$'$) depends linearly on the parameter h, and therefore its solutions depend *analytically* on h. This dependance may be expressed by a group morphism:

$$\rho_n^{KZ} : B_n = \pi_1(X_n) \longrightarrow \mathrm{Aut}_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]]). \tag{1}$$

Thus, we have a *family* of monodromy representations of the braid group B_n and the objective of the next section is to describe this monodromy in the best possible way.

The quantum group $\mathcal{U}_h(\mathfrak{g})$

We now proceed with a more algebraic construction of braid group representations.

1. Braid group representations from R-matrices. Recall the presentation $(*)$ of the braid group B_n. To construct representations of B_n, one may use Artin's theorem:

Lemma 8 (Universal property of the braid group) *Given a group G and elements $c_1, \dots, c_n \in G$ satisfying $c_ic_j = c_jc_i$ if $|i-j| \geqslant 2$ and $c_ic_{i+1}c_i = c_{i+1}c_ic_{i+1}$, there exists a unique group morphism $B_n \to G$ sending σ_i to c_i.*

Let now W be a vector space and c be a linear automorphism of $W \otimes W$. Set $c_i = \mathrm{Id}_{W^{\otimes(i-1)}} \otimes c \otimes \mathrm{Id}_{W^{\otimes(n-i-1)}}$ for $1 \leqslant i \leqslant n-1$ and observe that by construction $c_ic_j = c_jc_i$ if $|i-j| \geqslant 2$ (note that $c_i \in \mathrm{Aut}(W^{\otimes n})$). Then:

Proposition 9 *One has $c_ic_{i+1}c_i = c_{i+1}c_ic_{i+1}$ iff $c \in \mathrm{Aut}(W \otimes W)$ satisfies the* Yang-Baxter equation*, that is, iff one has, in $\mathrm{Aut}(W \otimes W \otimes W)$, the equality:*

$$(c \otimes \mathrm{Id}_W)(\mathrm{Id}_W \otimes c)(c \otimes \mathrm{Id}_W) = (\mathrm{Id}_W \otimes c)(c \otimes \mathrm{Id}_W)(\mathrm{Id}_W \otimes c).$$

Corollary 10 *If $c \in \mathrm{Aut}(W \otimes W)$ is a solution of the Yang-Baxter equation (such a solution is called an R-matrix), then there exists a unique group morphism $\rho_n^c : B_n \longrightarrow \mathrm{Aut}(W^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for all i.*

Hence, starting from a solution of the Yang-baxter equation, one may construct a representation of Artin's braid group B_n. To produce solutions of the Yang-Baxter equation, one may use the theory of braided bialgebras.

2. Topological braided bialgebras. A topological braided bialgebra is a topological bialgebra H endowed with an element $R \in H \hat{\otimes} H$, invertible in the algebra $H \hat{\otimes} H$, which makes the coproduct Δ of H quasi-cocommutative and which satisfies $(\Delta \hat{\otimes} \mathrm{Id}_H)(R) = R_{13}R_{23}$ and $(\mathrm{Id}_H \hat{\otimes} \Delta)(R) = R_{13}R_{12}$ in $H \hat{\otimes} H \hat{\otimes} H$. Such an element $R \in H \hat{\otimes} H$ is called a *universal* R-matrix because it produces solutions to the Yang-Baxter equation on *every* H-module W:

Proposition 11 *If (H,R) is a topological braided bialgebra then, given an H-module W, there exists a solution of the Yang-Baxter equation c_W^R on W and therefore, by corollary 10, a braid group representation $\rho_n^R : B_n \longrightarrow \mathrm{Aut}(W^{\hat{\otimes} n})$. Explicitly, $c_W^R(w_1 \hat{\otimes} w_2) = \tau_{W,W}(R.w_1 \hat{\otimes} w_2)$, where $\tau_{W,W}(w_1 \hat{\otimes} w_2) = w_2 \hat{\otimes} w_1$.*

3. Quantum enveloping algebras. We now need examples of topological braided bialgebras. Consider the Drinfeld-Jimbo deformation $\mathcal{U}_h(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ (as a vector space, one has $\mathcal{U}_h(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g})[[h]]$):

Theorem 12 *The vector space $\mathcal{U}(\mathfrak{g})[[h]]$ is a topological bialgebra and possesses a universal R-matrix denoted R_h, with respect to which it is a braided quasi-cocommutative topological bialgebra.*

Then, if V is a $\mathfrak{g}$-module (or equivalently a $\mathcal{U}(\mathfrak{g})$-module), $V[[h]]$ is a $\mathcal{U}_h(\mathfrak{g})$-module and if one sets $W = V[[h]]$, one has:

$$W^{\hat{\otimes} n} = (V[[h]])^{\hat{\otimes} n} \simeq (V^{\otimes n})[[h]].$$

The braid group representation associated to the universal R-matrix R_h of $\mathcal{U}_h(\mathfrak{g})$ by proposition 11 is therefore a group morphism

$$\rho_n^{R_h} : B_n \longrightarrow \mathrm{Aut}_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]]). \tag{2}$$

We can now state the Kohno-Drinfeld theorem, which asserts the *equivalence* between the two representations (1) and (2) whose constructions we recalled:

Theorem 13 (The Kohno-Drinfeld theorem) *There exists a $\mathbb{C}[[h]]$-automorphism u of $V^{\otimes n}[[h]]$ such that, for all $b \in B_n$, $\rho_n^{KZ}(b) = u\rho_n^{R_h}(b)u^{-1}$.*

References

[1] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

florent@math.keio.ac.jp