The Yang-Mills equations over Klein surfaces

Chiu-Chu Melissa Liu & Florent Schaffhauser

Columbia University (New York City) & Universidad de Los Andes (Bogotá)

Seoul ICM 2014

Outline

Klein surfaces

- X a compact connected Riemann surface of genus $g \geq 2$.
- $\sigma : X \longrightarrow X$ an anti-holomorphic involution.
- Complete topological invariants of the pair (X, σ): the numbers (g, n, a), where
  - g is the genus of X,
  - n is the number of connected components of X^σ $(n \leq g + 1)$.
  - a is 2 minus the number of connected components of $X \setminus X^\sigma$.
- The pair (X, σ) will be referred to as a *Klein surface*.
- We fix a Kähler metric of unit volume on X.

Real vector bundles (Atiyah)

- $E \longrightarrow X$ a smooth Hermitian vector bundle of rank r and degree d.
- $\tau : E \longrightarrow E$ an isometry such that:
  - the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X \end{array}$$

    is commutative.
  - $\forall v \in E, \forall \lambda \in \mathbb{C},\ \tau(\lambda v) = \overline{\lambda}\tau(v)$.
  - $\tau^2 = \mathrm{Id}_E$.

Quaternionic vector bundles

- (E, τ) as earlier, except that now $\tau^2 = -\mathrm{Id}_E$.
- Explanation: both real and quaternionic vector bundles are fixed points of the involution

$$E \longmapsto E^\sigma := \overline{\sigma^* E}.$$

If $\mathrm{Aut}(E) \simeq \mathbb{C}^*$ and E is a fixed point, then E is either real or quaternionic (and cannot be both).

- Equivalently, (E, τ) is real (resp. quaternionic) if and only if there is an isomorphism

$$\varphi_\sigma : E \xrightarrow{\simeq} E^\sigma$$

satisfying $\varphi_\sigma^\sigma = \varphi_\sigma^{-1}$ (resp. $\varphi_\sigma^\sigma = -\varphi_\sigma^{-1}$).

Holomorphic structures

- $\mathcal{A}_E := \{\text{unitary connections on E}\}$
  $\simeq \{\text{holomorphic structures on E}\}$, since $\dim_{\mathbb{C}} X = 1$.
- $\mathcal{G}_E := \mathbf{U}(E)$, $\mathcal{G}_{\mathbb{C}} := \mathbf{GL}(E) \simeq \mathcal{G}_E^{\mathbb{C}}$.
- $\varphi_\sigma : E \longrightarrow E^\sigma$ such that $\varphi_\sigma^\sigma = \pm \varphi_\sigma^{-1}$.

Proposition ([Sch12])

$A \in \mathcal{A}_E$ makes τ anti-holomorphic if and only if

$$\varphi_\sigma^* A^\sigma = A.$$

Note that $\beta : A \longmapsto \varphi_\sigma^* A^\sigma$ is an involutive transformation, even if $\varphi_\sigma^\sigma = -\varphi_\sigma^{-1}$. It is anti-symplectic with respect to the Atiyah-Bott symplectic form $\omega_A(a, b) = \int_X -\mathrm{tr}(a \wedge b)$.

Moduli spaces

- $\mathcal{A}_E^\tau := \mathrm{Fix}(\beta) = \{\tau-\text{compatible hol. structures on } E\}$.
- $\mathcal{G}_E^\tau \subset \mathcal{G}_{\mathbb{C}}^\tau$, automorphisms of E that commute with τ. These groups act on $\mathcal{A}_E^\tau$.
- $F : \mathcal{A}_E \longrightarrow \Omega^2(X; \mathfrak{u}(E)) \simeq \Omega^0(X; \mathfrak{u}(E))$, curvature map (momentum map of the $\mathcal{G}_E$-action on $\mathcal{A}_E$).
- The following result is an analogue of Donaldson's formulation of the Narasimhan-Seshadri correspondence:

Theorem ([Sch12])

The set $\mathcal{M}^{\mathrm{ss}}(E,\tau)$ of S-equivalence classes of semi-stable τ-compatible holomorphic structures on E is in bijection with the set $F^{-1}(\{i2\pi\frac{d}{r}\mathrm{Id}_E\})^\tau/\mathcal{G}_E^\tau$ of Galois-invariant, projectively flat unitary connections on E, modulo the action of the real part of the unitary gauge group.

The Yang-Mills equations over a Klein surface

- Given (E, φ_σ) real or quaternionic over (X, σ), the associated Yang-Mills equations are:

$$\left\{ \begin{array}{ccc} F_A & = & i2\pi \frac{d}{r} \mathrm{Id}_E \\ \varphi_\sigma^* A^\sigma & = & A \end{array} \right.$$

- The space of gauge orbits of solutions is a Lagrangian quotient

$$\big(F^{-1}(\{i2\pi \frac{d}{r} \mathrm{Id}_E\}) \cap \mathcal{A}_E^\tau\big) / \mathcal{G}_E^\tau \simeq \mathcal{M}^{\mathrm{ss}}(E, \tau)$$

  which interprets nicely as a moduli space of semistable τ-compatible holomorphic structures on E.

- Note that $\mathcal{M}^{\mathrm{ss}}(E, \tau)$ inherits in this way a topology.

Topology of the moduli space

- We want to understand the topology of $\mathcal{M}^{\mathrm{ss}}(E,\tau)$.
  - We know that it is a compact connected space ([Sch12]).
  - We know its (equivariant) mod 2 Betti numbers ([LS13]).
- Our main tool for the computation of the equivariant mod 2 Betti numbers will be the *real Kirwan map*, whose definition follows from the fact that $\mathcal{M}^{\mathrm{ss}}(E,\tau)$ is a Lagrangian quotient:

$$\mathcal{M}^{\mathrm{ss}}(E,\tau) \simeq \left(F^{-1}(\{i2\pi\frac{d}{r}\})\right)^{\tau}/\mathcal{G}_E^{\tau}.$$

Strategy, *à la* Atiyah-Bott

- The inclusion map $(F^{-1}(\{i2\pi\frac{d}{r}\}))^\tau \hookrightarrow \mathcal{A}_E^\tau$ is $\mathcal{G}_E^\tau$-equivariant, so it induces a map (called the *real Kirwan map*)

$$H^*_{\mathcal{G}_E^\tau}(\mathcal{A}_E^\tau; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*_{\mathcal{G}_E^\tau}\big((F^{-1}(\{i2\pi\frac{d}{r}\}))^\tau; \mathbb{Z}/2\mathbb{Z}\big).$$

  Moreover, when r and d are coprime, we prove in [LS13] that

$$H^*_{\mathcal{G}_E^\tau}\big((F^{-1}(\{i2\pi\frac{d}{r}\}))^\tau\big) \simeq H^*(B\mathbf{O}(1)) \otimes_{\mathbb{Z}/2\mathbb{Z}} H^*(\mathcal{M}^{\mathrm{ss}}(E,\tau)).$$

- The space $\mathcal{A}_E^\tau$ is contractible (it is an affine space), so

$$H^*_{\mathcal{G}_E^\tau}(\mathcal{A}_E^\tau; \mathbb{Z}/2\mathbb{Z}) = H^*(B(\mathcal{G}_E^\tau); \mathbb{Z}/2\mathbb{Z}).$$

Surjectivity of the real Kirwan map

- The following is an analogue of a theorem of Atiyah and Bott:

Theorem ([LS13])

The real Kirwan map is surjective.

- Ingredients of the proof:
  - The space $\mathcal{A}_E$ is stratified by the Harder-Narasimhan type of holomorphic vector bundes: $\mathcal{A}_E = \bigcup_\mu \mathcal{A}_\mu$.
  - The strata are invariant under the involution β.
  - The invariant part of the stratification forms a $\mathcal{G}_E^\tau$-equivariantly perfect stratification of $\mathcal{A}_E^\tau$ (over mod 2 integers), meaning essentially that the equivariant (mod 2) Euler class of the normal bundle to any given stratum is not a zero divisor in the cohomology ring of that stratum (note that the normal bundles to the strata are not orientable in general, hence the necessity of mod 2 coefficients).

A recursive formula

- The equivariant perfection of the stratification by Harder-Narasimhan type, combined with the fact that strata of positive codimension are cohomologically equivalent to products of semi-stable strata for bundles of lower rank, gives a recursive formula for the equivariant Poincaré polynomial of the semi-stable stratum:

$$P^{\tau}_{(g,n,a)}(r,d) = P_t(B\mathcal{G}^{\tau}_E) \; - \sum_{\mu \neq \mu_{ss},\, \mathcal{A}^{\tau}_{\mu} \neq \emptyset} t^{d_\mu} \prod_{i=1}^{l} P^{\tau}_{(g,n,a)}(r_i, d_i)$$

where we denote by
$\mu = (\underbrace{\frac{d_1}{r_1}, \dots, \frac{d_1}{r_1}}_{r_1 \text{ times}}, \dots, \underbrace{\frac{d_l}{r_l}, \dots, \frac{d_l}{r_l}}_{r_l \text{ times}}), \frac{d_1}{r_1} > \dots > \frac{d_l}{r_l}$ the Harder-Narasimhan type of a holomorphic vector bundle and $d_\mu = \mathrm{codim}_{\mathcal{A}^{\tau}_E} \mathcal{A}^{\tau}_{\mu} = \sum_{1 \leq i < j \leq l} \big(r_j d_i - r_i d_j + r_i r_j (g-1)\big)$.

Comments

- As the notation suggests, the result only depends on the topology of (X, σ) and (E, τ), not on the holomorphic structure of X.
- The recursion can be solved by applying a theorem of Zagier, thus giving a closed formula for the equivariant Poincaré polynomial of the semi-stable stratum.
- When $r = 1$, the formula indeed gives $(1+t)^g$ for the Poincaré polynomial of moduli spaces of real bundles (which in this case are connected components of the real part of the Jacobian of X, the latter being a $2g$-dimensional torus).
- The most involved step in our computation is the determination of the Poincaré series of the classifying space of the unitary gauge group $\mathcal{G}_E^\tau$ of a real or quaternionic vector bundle. The answer depends on whether the bundle is real or quaternionic, as well as on the invariants (g, n, a) and r.

The holonomy map (complex case)

- Fix a point $x \in X$. The based gauge group $\mathcal{G}_E(x)$ acts freely on the contractible space $\mathcal{A}_E$. So $B(\mathcal{G}_E(x)) \sim \mathcal{A}_E/\mathcal{G}_E(x)$.
- Fixing a frame at x, one has an evaluation map

$$1 \longrightarrow \mathcal{G}_E(x) \longrightarrow \mathcal{G}_E \xrightarrow{\mathrm{ev_x}} \mathbf{U}(r) \longrightarrow 1$$

and the usual cellular decomposition of a genus g Riemann surface gives a fibration called the holonomy map

$$\Omega\mathbf{SU}(r) \sim B(\Omega^2\mathbf{U}(r)) \longrightarrow B(\mathcal{G}_E(x)) \xrightarrow{\mathrm{Hol}} \mathbf{U}(r)^{2g}.$$

- The group $\mathbf{U}(r)$ acts diagonally by conjugation on $\mathbf{U}(r)^{2g}$, hence a fibration

$$\mathbf{U}(r)^{2g} \longrightarrow \mathbf{U}(r)^{2g} \times_{\mathbf{U}(r)} E\mathbf{U}(r) \longrightarrow B\mathbf{U}(r).$$

The Poincaré series of $B(\mathcal{G}_E)$

- The previous considerations lead to the following commutative diagramme, whose rows and columns are fibrations:

$$
\begin{array}{ccccc}
\Omega\mathbf{SU}(r) & \longrightarrow & B\big(\mathcal{G}_E(x)\big) & \longrightarrow & \mathbf{U}(r)^{2g} \\
\| & & \downarrow & & \downarrow \\
\Omega\mathbf{SU}(r) & \longrightarrow & B\big(\mathcal{G}_E\big) & \longrightarrow & \mathbf{U}(r)^{2g} \times_{\mathbf{U}(r)} E\mathbf{U}(r) \\
\downarrow & & \downarrow & & \downarrow \\
\{\mathrm{pt}\} & \longrightarrow & B\mathbf{U}(r) & = & B\mathbf{U}(r)
\end{array}
$$

- Atiyah and Bott proved that the middle column and the top row are cohomologically trivial fibrations (over any coefficient field):

$$
\begin{aligned}
P_t(B(\mathcal{G}_E)) &= P_t(B\mathbf{U}(r))P_t(\mathcal{G}_E(x)) \\
&= P_t(B\mathbf{U}(r))P_t(\mathbf{U}(r)^{2g})P_t(\Omega\mathbf{SU}(r)).
\end{aligned}
$$

The holonomy map (real and quaternionic cases)

- Again we proceed similarly to Atiyah and Bott. But:
  - We work with the middle row and right column (would also work in the complex case but Atiyah and Bott did not need it).
  - The target space of the holonomy map will depend on the topogical type of the Klein surface (X, σ) as well as on the topological type of the real or quaternionic bundle (E, τ).
  - We need an appropriate cellular decomposition for each topological type of Klein surface (5 cases to distinguish).
  - The cases $X^\sigma \neq \emptyset$ and $X^\sigma = \emptyset$ are rather different.
  - For the most part of the computation, we need to work with mod 2 coefficients.
- Example (real case, $X^\sigma \neq \emptyset$):

$$P_t(B(\mathcal{G}_E^{\mathcal{T}_\mathbb{R}}); \mathbb{Z}/2\mathbb{Z}) = P_t(B\mathbf{O}(r))P_t(Z_{(g,n,a)}^{\mathcal{T}_\mathbb{R}})P_t(\Omega\mathbf{SU}(r))$$

where

$$P_t(Z_{(g,n,a)}^{\mathcal{T}_\mathbb{R}}) = P_t(\mathbf{U}(r))^{g-n+1}P_t(\mathbf{U}(r)/\mathbf{O}(r))^{n-1}P_t(\mathbf{SO}(r))^n.$$

Maximal real algebraic varieties

Let Y be a real algebraic variety defined over $\mathbb{R}$, smooth, projective, of dimension n. Then, by Milnor-Thom-Smith,

$$\sum_{i=0}^{n} b_i(Y(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) \leq \sum_{i=0}^{2n} b_i(Y(\mathbb{C}); \mathbb{Z}/2\mathbb{Z}).$$

Y is said to be maximal if

$$\sum_{i=0}^{n} b_i(Y(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) = \sum_{i=0}^{2n} b_i(Y(\mathbb{C}); \mathbb{Z}/2\mathbb{Z}).$$

Theorem ([LS13])

The moduli variety $\mathcal{M}^{2,1}_{(X,\sigma)}$ of semi-stable vector bundles of rank 2 and degree 1 over (X,σ) is a maximal real algebraic variety if and only if the Klein surface (X,σ) is itself maximal.

References

M. F. Atiyah and R. Bott.
The Yang-Mills equations over Riemann surfaces.
Philos. Trans. Roy. Soc. London Ser. A, 308(1505):523–615, 1983.

C. C. M. Liu and F. Schaffhauser.
The Yang-Mills equations over Klein surfaces.
J. Topol., 6(3):569–643, 2013.

F. Schaffhauser.
Real points of coarse moduli schemes of vector bundles on a real algebraic curve.
J. Symplectic Geom., 10(4):503–534, 2012.