GIT quotients and symplectic reduction: the Kempf-Ness theorem

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Introduction

The Kempf-Ness theorem is a fundamental result at the intersection of the complex algebraic geometry and symplectic geometry. It induces a symplectic structure on quotient algebraic varieties, and then in moduli spaces that arise there. This point of view has been fundamental in the works of Atiyah, Bott and Donaldson about the Yang-Mills equations on Riemann surfaces.

The objective of this thesis was to study this relation in the finite dimensional case: this is the original statement of Kempf and Ness.

The first chapter introduces the main strategy to solve the problem of finding quotients in algebraic geometry, by means of geometric invariant theory (GIT) in characteristic 0. In this chapter we expose the link between GIT and (linear) reductive groups and geometric quotients, which turns out to be fundamental to prove a result of Nagata and Mumford that implies the existence of quotients of affine algebraic varieties by reductive groups. We give two examples that illustrate how GIT works in the practice. Some remarks about the link between reductive algebraic groups and Lie groups are given, and these are important to understand the last chapter of the thesis.

The second chapter introduces the symplectic reduction, which is the natural notion of quotient in symplectic geometry. We treat the same examples that appear in the first chapter, but from a symplectic point of view. The chapter concludes with the proof of the Marsden-Winstein-Meyer theorem.

The third chapter states, proves and illustrates (with the examples given in the first two chapters) the symplectic quotient in the case of affine algebraic varieties. The proof is slightly different from the one generally given in the original articles.

I want to thank my advisor Florent Schaffhauser, who patiently introduced me to the subject.

Chapter 1

Geometric Invariant Theory

1.1 Affine Quotient

An algebraic group is a group with a structure of affine algebraic variety, such that the inverse and multiplication maps are morphisms of varieties (i.e. the component functions of this maps are polynomials).

We will denote with k an algebraically closed field of characteristic 0 and V a finite-dimensional vector space over k.

Example 1. *The main examples that we will work with are:*

- $(k^*)^n$ *is called an algebraic n-torus, and it is an algebraic group.*
- *The linear groups $GL(V)$, $SL(V)$, $Sp(V, \omega)$, where V is complex vector space and ω is a symplectic structure on V, are examples of complex algebraic groups.*

In this thesis we will be interested in actions of algebraic groups on affine and projective algebraic varieties defined over the complex numbers. However, not all the actions are considered. We will consider two cases. If the variety X is affine and is contained in $\mathbb{C}^n$, then we are interested in actions of G through a homomorphism $\rho : G \to GL(\mathbb{C}^n)$ and hence we will focus on linear representations of the group G on a complex vector space V. On the other hand, if X is a projective variety contained in $\mathbb{C}P^n = P(\mathbb{C}^{n+1})$, then we are interested in the action of G through a linear representation $\rho : G \to GL(\mathbb{C}^{n+1})$. This defines an action on $\mathbb{C}P^n$ because $\rho(g)(\lambda v) = \lambda\rho(G)(v)$ and then $\rho(g)[v] = [\rho(g)v]$.

The main problem, in terms of affine varieties first, is:

Problem 1. *Let X be an affine algebraic variety invariant under the linear action of G. Does the orbit space X/G have a structure of affine algebraic variety?*

This problem is not easy to solve in general. The most common counterexample for such a question is the following:

Example 2. *Consider the action of k^* on the space k^2 given by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$. This space has as orbits: $\{(0,0)\}$, the x-axis without $(0,0)$, the y-axis without $(0,0)$ and the sets $\mathcal{O}_\lambda := \{(x,y) : xy = \lambda\}$ with $\lambda \neq 0$. Now suppose that the set of orbits is a variety. Then, every point (orbit) must be closed. Then if we take all k^*-invariant polynomials in two variables, there should be one that attains different values on any couple of points we pick. In particular the origin and the x-axis without the origin should be separated by a k^*-invariant polynomial. But any polynomial is continuous in the Zariski topology, and even more in the complex topology. Then, if such a polynomial has value a in the x-axis then it must have value a in the origin. Then there is no polynomial that distinguishes the two orbits we picked, and hence the orbit space cannot be a variety.*

The strategy to solve this problem goes in two different directions.

- First, we should consider groups for which it is possible to determine if the quotient will exist.
- Second, we should change our notion of quotient, by a much more suitable one for the category of algebraic varieties.

Both directions could be translated if one knows a little bit of basic algebraic geometry. Recall that an affine algebraic set is the zero set of a set of polynomials in $k[x_1, \ldots, x_n]$. The ideal generated by such a set of polynomials gives rise to the same zero set. On the other hand, such an ideal is finitely generated by Hilbert's basis theorem. Then we can restrict ourselves to the zero sets of a finite number of polynomials. Moreover, $k[x_1, \ldots, x_n]$ is Noetherian implies that the ideal generated by the polynomials mentioned above is Noetherian as well. Hence, if $I(X)$ is the set of polynomials that vanish in X and $V(f_1, \cdots, f_m)$ is the set of common zeros of the polynomials $f_1, \cdots, f_m$, then the quotient $A(V(f_1, \ldots, f_m)) = k[x_1, \ldots, x_n]/I(V(f_1, \ldots, f_m))$ is Noetherian as well, and it is called the *affine coordinate ring* of $V = V(f_1, \ldots, f_m)$. This means that $A(V(f_1, \ldots, f_m))$ is a finitely generated k-algebra.

By the Hilbert's Nullstellensatz the coordinate k-algebra is reduced, whenever k is algebraically closed. Recall on the other hand that if one has a finitely generated and reduced k-algebra A over an algebraically closed field k, then there is an algebraic set having A as coordinate algebra. This set is given by the maximal spectrum of A, $Spm(A)$.

On the other hand, if one has an algebraic set $X \subseteq k^n$, and the action of any group G on X, this induces an action of G on the coordinate algebra of X

$$g \cdot f(x) = f(g^{-1} \cdot x).$$

Then the coordinate algebra of a possible quotient variety should be given by $A(X)^G$: the subalgebra of G-invariant elements in $A(X)$. Obviously, a subalgebra of a reduced algebra is reduced as well. Thus, to solve our problem we must ask if in general $A(X)^G$ is finitely generated, in order to have a corresponding variety.

In general, $A(X)^G$ is not finitely generated. However, there is a certain class of groups for which this holds. This is explained in section 1.1.1.

Finally the second direction to solve the problem stated above is quite subtle, and depends as well on the description given above. This will be explained in sections 1.1.2 and 1.1.3.

1.1.1 A few words about reductive groups

A class of groups G that make G-invariant subalgebras inherit finite generatedness is the class of reductive groups. To give a definition we have to recall some concepts of representation theory.

In our context, any group G acts linearly on a complex vector space V. This means that our actions are actually representations of the group G. Then, for us, a (*linear*) *representation* of G (or a *G-module*) is a pair (V, ρ) such that V is a complex vector space, and $\rho : G \to GL(V)$ is a group homomorphism. If ρ is clear, we will only mention V, and vice versa.

A *G-submodule* is a subspace $W \leq V$ invariant under ρ: this makes sense, since a ρ-invariant subspace gives rise to a representation $\rho' : G \to GL(W)$ by restricting to W the evaluation maps. There are obvious G-submodules: the *trivial* one $\{0\}$ and V itself. A morphism of G-modules is a G-equivariant linear map $\psi : V \to W$.

A G-module is *irreducible* if it has no proper, non-trivial G-submodules. The direct sum of two G-modules V, W is given by the representation $\rho_V \oplus \rho_W : G \to GL(V \oplus W)$ defined by

$$(\rho_V \oplus \rho_W)(g)(v \oplus w) = \rho_V(v) \oplus \rho_W(w).$$

A G-module is *completely reducible* if it can be decomposed as the direct sum of irreducible G-submodules. A group G is called (linearly) reductive if any G-module is completely reducible.

Remark 1: Let V be a G-module. We denote V^G the G-submodule of invariant elements of V. Moreover, any morphism of G-modules induces a morphism $\psi^G : V^G \to W^G$.

The definition of reductive group we gave above is not the easiest to work with. This is why we have the following

Proposition 1. *Let G be any group. Then the following are equivalent:*

1. *G is reductive,*
2. *for any G-module V, if V is a G-submodule then it has a direct complement which is a G-module,*
3. *any surjective morphism of G-modules $\psi : V \to W$ induces a surjective map $\psi^G : V^G \to W^G$.*

Many of the groups we already know are reductive. This makes the notion of reductive group natural and not too restrictive.

Example 3 (Finite groups). *This is a theorem due to Maschke. For the proof we use (2) in Proposition 1. Let $\rho : G \to GL(V)$ be a representation and W a G-submodule. There exist a direct complement U and hence a projection $\pi : V \to W$. Define $\tilde{\pi} : V \to W$ as $\tilde{\pi}(v) := \sum_{g\in G}(g \cdot \pi(g^{-1} \cdot v))$. $\tilde{\pi}$ is:*

- *linear: it is a finite sum of linear operators.*
- *G-invariant: the action of an element $g \in G$ only permutates the sum.*
- *surjective: for any $v \in W$, $\tilde{\pi}(v) = |G|v$. Since k has 0 characteristic, then $|G| \neq 0$*

By the isomorphism theorem, V splits as $W \oplus \ker \tilde{\pi}$. Moreover $\ker \tilde{\pi}$ is ρ-invariant. Thus G is reductive.

Example 4 (Compact real Lie groups). *A compact real Lie group K is reductive for the following reasons:*

- *(Haar theorem): If K is a compact Lie group then there exists only one K-invariant integral (complex-valued linear functional in $\mathcal{C}(K, \mathbb{C})$) $\int_K dg$ such that $\int_K dg = 1$.*
- *Given a Hermitian product $\langle , \rangle$, one can always define a new K-invariant Hermitian product $\langle , \rangle_K$ as follows:*

$$\langle v, w\rangle_K := \int_K \langle g \cdot v, g \cdot w\rangle \, dg$$

Now we proceed to find a direct complement of any K-invariant subspace W by means of the K-invariant Hermitian product. Hence K is reductive.

Example 5 (Algebraic tori). *Any group isomorphic to $\mathbb{C}^n$ is called an algebraic n-torus. These groups are all reductive. For a proof see [13].*

The relation between reductive groups and finitely generated algebras is a consequence of the following theorem of Hilbert:

Theorem 1. *Let k be an algebraically closed field of characteristic 0. If G is a (linearly) reductive group then $k[x_1, \dots, x_n]^G$ is finitely generated.*

Proof. It is clear that $A = k[x_1, \dots, x_n] = \bigoplus_{i\geq 0} A_i$ (where $A_i := k[x_1, \dots, x_n]_i^G$ are the degree i elements of $k[x_1, \dots, x_n]$) and then

$$k[x_1, \dots, x_n]^G = \bigoplus_{i\geq 0}(A_i \bigcap A^G).$$

Now let $A_+^G := \bigoplus_{i>0}(A_i \bigcap A^G)$ and J be the ideal generated by A_+^G in A. Since J is an ideal of a Noetherian ring, J is finitely generated in A by a finite subset of A_+^G $\{f_1, \dots, f_m\}$ (hence all of them of positive degree). Moreover, it is clear that J and A are k-algebras endowed with a linear action of G. This simply says that they are G-modules. This means that $\phi : A \oplus \cdots \oplus A \to J$ defined by $\phi(h_1, \ldots, h_m) = \sum_{i=1}^m h_i f_i$ is a G-modules homomorphism. Note that it is surjective. Since G is reductive, the induced map $\phi^G : A^G \oplus, \ldots, \oplus A^G \to J^G$ is surjective.

We want to prove that $A^G = k[f_1, \ldots, f_m]$. ($\supseteq$) is clear. Let us prove $\subseteq$: let $h \in A^G$. We proceed by induction on the degree of h. If $deg(h) = 0$ then it is a constant polynomial and then $h \in k[f_1, \ldots, f_m]$. If $deg(h) > 0$ then $h \in J$. Since h is G-invariant, $h \in J^G$. By surjectivity of ϕ^G, there are G-invariant elements $h_i \in A^G$ such that $h = \sum_{i=0}^m h_i f_i$. But all the elements f_i have positive degree, then $deg(h_i) < deg(h)$. By the induction hypothesis, $h_i \in k[f_1, \ldots, f_m]$, and then $h \in k[f_1, \ldots, f_m]$. $\square$

Remark: If $k = \mathbb{C}$ there are other ways to build up invariant h_i by using the Haar integral defined in example 4.

Corollary 1. *Let A be a finitely generated k-algebra and G a reductive group. Then A^G is finitely generated.*

Proof. Since A is finitely generated, one can build up the following surjective homomorphism of k-algebras $\phi : k[x_1, \ldots, x_n] \to A$ such that $x_i \mapsto a_i$, and $\{a_i : 1 \leq i \leq n\}$ are generators of A. This induces an action of G on $k[x_1, \ldots, x_n]$, and by the theorem above $k[x_1, \ldots, x_n]^G$ is finitely generated. We have then an induced morphism of G-modules $k[x_1, \ldots, x_n]^G \to A^G$, which is surjective by Proposition 1. Hence the image of generator of $k[x_1, \ldots, x_n]^G$ is a set of generators of A^G. $\square$

Remark 2: For the sake of completeness, a few words should be mentioned about fields of non-zero characteristic. The problem of determining whether the set of invariants of a k-algebra under the linear action of a group is or not finitely generated was formulated by Hilbert. He himself found a class of groups for which finite generatedness holds in the case of fields of 0 characteristic, and is precisely the same that we described above: linear reductivity. In general, if the field has arbitrary characteristic, linear reductivity does not lead to finitely generated k-algebras. That is why there are two more general notions of reductivity, all of which concide in 0 characteristic: reductivity and geometric reductivity. Reductivity implies geometric reductivity (this is called Mumford's conjecture, and was proved by Haboush), and there is a generalization of Hilbert's theorem in this context (due to Nagata) which asserts that the algebra of invariants of a k-algebra under the action of a geometrically reductive group is finitely generated.

Remark 3: In chapter 3 it will be fundamental to understand some relations between algebraic groups and Lie groups. It is clear that any complex algebraic group is always a complex Lie group. On the other hand, any compact real Lie group K is reductive (example 4), and has a *complexification* $K_{\mathbb{C}}$: this means that K is a maximal compact subgroup of $K_{\mathbb{C}}$, such that $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k} \oplus i\mathfrak{k}$. When building up $K_{\mathbb{C}}$, it becomes clear that it is a complex algebraic group [2]. This group inherits (linear) reductivity from K: this is a consequence of a general version of the so called Weyl's unitary trick [4]. But this turns out to be the general situation. Then we will say that G is reductive if it is the complexification of a compact real Lie group K.

1.1.2 The affine quotient

By the theorem just mentioned, and using the dictionary between algebraic sets and finitely generated reduced k-algebras, our intuition says that the corresponding quotient for the action of G on $A(X)$ is given by $Spm(A(X)^G)$. But we already gave an example of the algebraic torus $\mathbb{C}^*$, and the orbit space was not a variety. However, our analysis over the category of k-algebras seems correct. What is the problem?

The problem is the quotient. The right notion of quotient in the category of algebraic varieties is not the orbit space. To fix this, we can try to formulate our problem in a more categorical context. Recall that in the category of sets, orbit spaces have the following property: let X be a set and G any group acting on it and let $\pi : X \to X/G$ be the projection map. It is clear that π is G-invariant. Now let Y be any other set and $\eta : X \to Y$ a G-invariant. Then there is a unique map $\tilde{\eta} : X/G \to Y$ such that $\eta = \tilde{\eta} \circ \pi$. But this does not only happens in the category of sets. For instance, the topological quotient (orbit space with the quotient topology) has the same property in the category of topological spaces with continuous maps.

This suggests the following definition:

Definition 1. *Let X be an affine algebraic variety and G an algebraic group acting on it. We say that an affine algebraic variety Z with a G-invariant morphism $\psi : X \to Z$ is a* ***categorical quotient*** *if for every affine algebraic variety Y and every G-invariant morphism $\phi : X \to Y$, there is a morphism $\overline{\phi} : Z \to Y$ such that $\overline{\phi} \circ \psi = \psi$.*

Now note that if $p(x_1, \dots, x_n)$ is a polynomial in $X \subseteq k^n$, the action of G in k^n induces an action of G on p:

$$g \cdot p(x_1, \dots, x_n) = p(g^{-1}(x_1, \dots, x_n)).$$

This means that if a quotient Z exists, then every $A(Z)$ should be isomorphic to $A(X)^G$. Then if $Spm(A(X)^G)$ is a variety (i.e. $A(X)^G$ is reduced and finitely generated) then it is a good candidate to be the categorical quotient. This will not be completely clear until the end of this section.

The contruction we present here allows one to define the categorical quotient with an inmersion in an affine space. This is a key point to completely understand the projective quotient.

Since every variety can be regarded as $X = Spm(A)$, where A is its coordinate k-algebra, there is a finite number of generators in A, say $\{f_1, \dots, f_m\}$. But then, as one usually does, there is a surjective k-algebra homomorphism $\phi : k[x_1, \dots, x_n] \to A$ such that $x_i \mapsto f_i$. This induces a morphism of algebraic sets $\tilde{\phi} : Spm(A) \to Spm(k[x_1, \dots, x_n]) = k^n$, defined by

$$\mathfrak{m} \mapsto (f_1(\mathfrak{m}), \dots, f_m(\mathfrak{m})).$$

Here, we are regarding $f_i \in A$ as function by means of the composite map $A \to A/\mathfrak{m} \cong k$ (the last k-algebra isomorphism is the only k-linear function that sends the identity to the identity, taking $A/\mathfrak{m}$ as a k-vector space). What we did, is to embed explicitly $Spm(A)$ in k^n.

If one slightly modifies the last statement, one has a similar formulation for $Spm(A^G)$ in terms of $Spm(A)$. Let G be reductive. Then A^G is finitely generated by $\{f_1, \dots, f_m\}$, which are G-invariant elements of A. This induces a G-invariant map

$$\begin{aligned} \psi : & X \to k^m \\ & x \mapsto (f_1(x), \dots, f_m(x)) \end{aligned}$$

Lemma 1. *$A(\psi(X)) \cong A(X)^G$.*

Proof. $A(\psi(X)) := k[x_1, \dots, x_m]/I(\psi(X))$ and $A(X)^G = k[f_1, \dots, f_m]$. Define

$$\alpha : k[x_1, \dots, x_n] \to k[f_1, \dots, f_n].$$

This map is surjective and has kernel

$$ker(\alpha) = \{p \in k[x_1, \dots, x_m] : p(f_1, \dots, f_m) = 0\} = I(\psi(X)).$$

□

By G-invariance, ψ is constant on the orbits of X under G. However, not necessarily two orbits have different values under ψ in k^m.

Example 6 (ex.2 revisited)**.** *It is clear that $A(\mathbb{C}^2) = \mathbb{C}[x, y]$. Let us calculate $\mathbb{C}[x, y]^{\mathbb{C}^*}$. Recall that $\lambda \cdot f(x, y) = f(\lambda^{-1}x, \lambda y)$. Then, if $f(x, y) = \sum_{\alpha \le n, \beta \le m} a_{\alpha\beta} x^\alpha y^\beta$,*

$$\begin{aligned} \lambda \cdot f(x, y) &= \sum_{\alpha \le n, \beta \le m} a_{\alpha\beta} (\lambda^{-1}x)^\alpha (\lambda y)^\beta \\ &= \sum_{\alpha \le n, \beta \le m} a_{\alpha\beta} \lambda^{\beta - \alpha} x^\alpha y^\beta \\ &= \sum_{\alpha \le n, \beta \le m} a_{\alpha\beta} x^\alpha y^\beta \end{aligned}$$

by G-invariance. Then $\lambda^{\beta-\alpha} = 1$, and hence $\beta = \alpha$. Thus $f(x, y) \in \mathbb{C}[xy]$. Obviously a generator of $\mathbb{C}[xy]$ is xy, and then $\psi(x, y) = xy$. This means that precisely ψ assumes the value 0 in the orbits $(0, 0)$, x-axis without $(0, 0)$ and y-axis without $(0, 0)$.

Obviously ψ is continuous in the Zariski topology (because it is a regular map), and then $\psi^{-1}(0)$ must be closed. But the x-axis without $(0, 0)$ is not closed. Moreover, the closure of this orbit (the whole x-axis) intersects the closure of the other two orbits with image 0 by ψ. This means that intuitively, the quotient variety takes the orbit space and identifies orbits whose closures have non-empty intersection. Then this allows one define a relation between orbits as follows:

Definition 2. *$G \cdot x \sim G \cdot y$ if $\overline{G \cdot x} \bigcap \overline{G \cdot y} \neq \varnothing$*

It is clear that $\sim$ is symmetric and transitive. However, it is not clear at all that transitivity holds, and then this relation is not necessarily an equivalence relation.

This leads to the following definition:

Definition 3. *Two orbits $G \cdot p, G \cdot q$ are **closure-equivalent** if there are $a_1, \ldots, a_k \in X$ such that $a_1 = p, a_k = q$ and $\overline{G \cdot a_i} \bigcap \overline{G \cdot a_{i+1}} \neq \varnothing$ for every i.*

Closure-equivalence is obviously an equivalence relation. In the case of reductive groups, it is equivalent to $\sim$, and hence $\sim$ is an equivalence relation.

Theorem 2 (Mumford, Nagata)**.** *If G is reductive acting linearly on X, then $G \cdot p, G \cdot q$ are closure equivalent if and only if $\overline{G \cdot p} \bigcap \overline{G \cdot q} = \varnothing$. Equivalently to these statements, in algebraic terms, no $f \in A(X)^G$ has different values on $G \cdot p$ and $G \cdot q$.*

Proof. If the closures of two orbits intersect, then both orbits are trivially closure-equivalent, and by continuity, if $f \in A(X)^G$ then $f(G \cdot p) = f(G \cdot a_1) = \cdots = f(G \cdot a_{n-1}) = f(G \cdot q)$. Hence, we only need to prove that if $f \in A(X)^G$ does not separate $G \cdot p$ and $G \cdot q$ then their closures have non-empty intersection. Instead, let us prove that if $\overline{G \cdot p} \bigcap \overline{G \cdot q} = \varnothing$ then there is $f \in A(X)^G$ such that $f(G \cdot p) \neq f(G \cdot q)$.

Let $I_1 := I(G \cdot p), I_2 := I(G \cdot q)$. It is clear that $I(A) = I(V(I(A)))$ for any set A and that $\overline{A} = V(I(A))$. This implies that $I_1 = I(\overline{G \cdot p}), I_2 = I(\overline{G \cdot q})$. Then since $\overline{G \cdot p} \bigcap \overline{G \cdot q} = \varnothing$, $V(I_1 + I_2) = V(I_1) \bigcap V(I_2) = \overline{G \cdot p} \bigcap \overline{G \cdot q} = \varnothing$. Recall that the Nullstellensatz says that a proper ideal always has a nonempty zeros set. Then $I_1 + I_2 = A(X)$.

On the other hand $I_1 = I(\overline{G \cdot p}), I_2 = I(\overline{G \cdot q})$ implies that I_1, I_2 are invariant subspaces. Moreover $A(X) = I_1 + I_2 \cong I_1 \oplus I_2 = A(X)$, and then the map $I_1 \oplus I_2 \to A(X)$ defined by $(f, g) \mapsto f + g$ is surjective by the reasoning above. Since G is reductive, $I_1 \bigcap A(X)^G \oplus I_2 \bigcap A(X)^G \mapsto A(X)^G$ is surjective. Hence there are $f \in I_1 \bigcap A(X)^G, g \in I_2 \bigcap A(X)^G$ such that its image is 1, i.e. $f + g = 1$. Then $f(\overline{G \cdot p}) = 0$ and $f(\overline{G \cdot q}) = 1$. ☐

The next theorem establishes the main result of this chapter. A few comments are useful to understand the proof below. In order to have that $\psi(X)$ is a categorical quotient, $\psi(X)$ *should be an affine algebraic variety* (or equivalently a Zariski closed), and there should be a morphism $\overline{\phi}$ for every morphism $\phi : X \to Y$ ($\overline{\phi}$ depends on ϕ) such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\phi"'] & \psi(X) \arrow[dl, "\overline{\phi}"] \\ Y & \end{tikzcd} .$$

By means of the dictionary between affine algebraic sets and finitely reduced k-algebras, we should have the following diagram

$$\begin{tikzcd} A(X) & A(\psi(X)) \arrow[l, "\psi^*"'] \\ A(Y) \arrow[u, "\phi*"] \arrow[ur, "\overline{\phi}^*"'] & \end{tikzcd} .$$

But then lemma 1 says that $A(\psi(X)) := k[x_1, \dots, x_n]/I(\psi(X)) \cong A(X)^G$. Then, we only need to prove that such a $\overline{\phi}$ can be defined.

Theorem 3. *Let G be reductive, X an affine algebraic variety and ψ defined as above. Then,*

1. *$\psi(X)$ is Zariski closed,*

2. *$\psi : X \to \psi(X)$ is a categorical quotient: i.e. for every $\phi : X \to Y$ G-invariant there is a well defined morphism $\overline{\phi}$ which makes $\psi(X)$ a categorical quotient.*

Proof. 1. We want to prove that $\overline{\psi(X)} := V(I(\psi(X)) = \psi(X)$. Let $a = (a_1, \dots, a_n) \in \overline{\psi(X)}$ and define the following homorphism

$$\begin{aligned} \pi :& A(X) \oplus \cdots \oplus A(X) \to A(X) \\ &(g_1, \dots, g_n) \mapsto \sum_{i=1}^{n} g_i(f_i - a_i) \end{aligned}$$

Note that in this morphism $f_i \in A(X)^G$ and a_i are G-invariant. Then this makes π a morphism of k-algebras. If one takes $\pi^G : \oplus_{i=1}^n A(X)^G \to A(X)^G$, then $\pi(\oplus_{i=1}^n A(X)^G)$ under the isomorphism α with $A(\psi(X))$ is precisely the maximal ideal I_a in $A(X)^G$ that repesents a. This means that π^G is not surjective. But G is reductive, and then π is not surjective. Let I be the maximal ideal in $A(X)$ that contains I_a. Then $A(X)^G \bigcap I$ is a maximal ideal in $A(X)^G$ and contains the maximal ideal I_a. Then $I \bigcap A(X)^G = I_a$. Then $a = \psi(I)$.

2. Let $\overline{x} \in \psi(X)$. Then $\overline{x} = \psi(x)$ for some $x \in X$. Define $\overline{\phi}(\overline{x}) := \phi(x)$. Let us prove it is well defined. Let $\tilde{x} \in X$ such that $\psi(\tilde{x}) = \overline{x}$. By our analysis above $\overline{G \cdot x} \bigcap \overline{(G \cdot \tilde{x})} \neq \varnothing$. But ϕ is G-invariant (hence $\phi(G \cdot x) = y$) and continuous, hence $\phi(\overline{G \cdot x}) = \phi(\overline{G \cdot \tilde{x}}) = y$.

□

The following fact is fundamental when dealing with particular examples, and is a key point to understand the last chapter of this thesis. Roughly speaking, it gives geometric intuition about the quotient.

Theorem 4. *Let G be a complex algebraic group acting on an affine variety X. Then:*

1. *for any $x \in X$, $G \cdot x$ is a variety (then it makes sense asking what is its dimension),*
2. *any orbit of minimal dimension in a closure-equivalence class is closed,*
3. *every closure equivalence class contains a closed orbit of minimal dimension,*
4. *if G is reductive then the class is unique.*

Proof. 1. Recall that the image of a morphism is a constructible set (a finite union of sets which are open in their closure). This implies that the map $\phi_x : G \to X$ defined by $\phi_x(g) = g \cdot x$ (which is obviously a morphism), has as image $G \cdot x$, and therefore is a constructible subset of X. Then $\overline{G \cdot x} = \overline{\bigcup_{i=1}^n U_i} = \bigcup_{i=1}^n \overline{U_i}$ where U_i are open in $\overline{U_i}$. This implies that $G \cdot x$ is an open set of finite union of closed sets (and then closed), and hence a variety.

   *Remark:*We should know a few things. If G is a group, the connected component of the identity G^0 is a subgroup, and G/G^0 has finitely many cosets. On the other hand, for any element $x \in X$, the stablizer G_x of x is a closed subgroup. Then, we have the formula $dim(G \cdot x) = dim(G) - dim(G_x)$ which makes sense since $G \cdot x$ is a variety.

2. Now, let $G \cdot x$ be of minimal dimension and $x \in \bigcap_{G \cdot y \in \mathcal{C}} G \cdot y$ where $\mathcal{C}$ is a closure equivalence class. Suppose it is not closed. Then for every $y \in \overline{G \cdot x} \backslash G \cdot x$, $G \cdot y$ is a closed set of strictly smaller dimension. Indeed, if not, then one $G \cdot y$ must have the same dimension of $G \cdot x$ (because $dim(\overline{G \cdot x}) = dim(G \cdot x)$ and then any other orbit in $\overline{G \cdot x}$ has at most the same dimension of $G \cdot x$). Recall that both orbits are open in their closure $\overline{G \cdot x}$ (by the way we picked x, the closures of the orbit of x and y coincide). Then $G \cdot y$ is a union of connected components of $\overline{G \cdot x}$ which do not intersect $G \cdot x$. But $G \cdot x$ is dense in $\overline{G \cdot x}$, which is a contradiction.

3. Is a consequence of 2.

4. If G is reductive, and a closure equivalence class has two closed orbits, then they must intersect, otherwise there is a G-invariant polynomial that separates the two closed orbits (by Proposition 2). Then they are the same orbit.

□

Definition 4. *A point is polystable if its orbit is closed. The set of polystable points of X is denoted by X^{ps}.*

After theorem 3 it is easy to assert that there is a bijection between the points of $\psi(X)$, the orbits of polystable points and closure equivalence classes. The image of X under ψ is denoted as $X/\!/G$.

Example 7 (Ex. 2 revisited again)**.** *We can finally give a conclusion to the example above. Recall that the image of* $\mathbb{C}^2$ *under* ψ *is our quotient variety. But it is obvious that* ψ *is surjective. This implies that our quotient variety is* $\mathbb{C}$.

Let us give another

Example 8. *Let us take* $G = GL(\mathbb{C}^n)$ *acting on* $M_n(\mathbb{C})$ *by conjugation. Then the orbits of this action are given by the classes of similar matrices. Any matrix* $M \in M_n(\mathbb{C})$ *has a Jordan form, and hence its orbit has an element that can be written as the sum of a diagonal matrix* D *and a nilpotent matrix* N *(*N *is just a matrix with zeros in the diagonal, 1's and zeros in the second upper diagonal and zeros in the other components). Then suppose that* $M = D + N$. *One can easily find a matrix* A_λ *such that* $A_\lambda \cdot M = A_\lambda \cdot D + A_\lambda \cdot N = D + \lambda N$. *If we take* $t \to 0$, *then* $D \in \overline{G \cdot M}$. *This implies that any closure equivalence orbit has a diagonal matrix.*

Remark: it is formal to take the limit as we did even if we are working with the Zariski topology. Indeed, the complex topology is finer than the Zariski topology. Hence, because the closure of a set is the intersection of all the closed sets containing it, the complex closure is smaller than the Zariski closure. But taking a limit is just calculating the closure of a sequence of elements.

If M *is not diagonalizable we can repeat the process above in order to obtain a diagonal matrix in the closure of the orbit, which is not in the orbit itself (otherwise it would be diagonalizable). However, this process cannot be applied to a diagonalizable matrix. Since a closed orbit has to exist in the closure of every orbit, this makes clear the fact that the polystable points are exactly the diagonalizable matrices.*

On the other hand, any polystable orbit has only one diagonal form modulo permutations. This implies that our quotient is in bijection with $\mathbb{C}^n/S_n$, *where* S_n *stands for the permutation group of a set of* n *elements. The algebra of invariants is given hence by* $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$.

On the other hand, the algebra of invariants in this case is generated by the symmetric polynomials (including the trace and the determinant). Then one has a map $\psi : M_n \to \mathbb{C}^n$, *because there are* n *symmetric polynomials. This is obviously surjective, hence the quotient* $M_n(\mathbb{C})/\!\!/GL(\mathbb{C}^n)$ *is precisely* $\mathbb{C}^n$.

Chapter 2

Symplectic Quotients

2.1 Hamiltonian Actions

Recall that given a manifold M and a Lie group G, a smooth action of G over M is a homomorphism $\psi : G \to Diff(M)$ such that its "evaluation map" $ev_\psi : G \times M \to M$ is C^∞. A manifold with an action of a group G is often called a G-manifold.

There is a correspondence:

$$\{\text{actions of } \mathbb{R} \text{ on } M\} \longleftrightarrow \{\text{complete vector fields over } M\}$$

The correspondence can be written as:

$$\psi \longmapsto X \text{ such that } X_p = \frac{d}{dt}_{t=0} \psi(t, p)$$

Given any G-manifold M and X an element of the Lie algebra $\mathfrak{g}$, we could repeat the last procedure for the one parameter subgroup determined by the subspace generated by X under the exponential map: hence, there is an associated field over M such that if $p \in M$ then

$$X^{\#}(p) := \frac{d}{dt}_{t=0} (exp(-tX) \cdot p)$$

(it is called the *fundamental field* of X or infinitesimal action of X on p).

Given a Lie group G, it can be considered as a G-manifold by considering the conjugacy action: $g \cdot m = int_g(m) := gmg^{-1}$. G not only acts on itself, but also on $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) by the adjoint (resp. co-adjoint) actions. Recall that the adjoint action is defined by

$$g \cdot X = Ad_g(X) := d(int_g)(X)$$

The co-adjoint action is given by the transpose of the adjoint action.

A symplectic manifold M is an even-dimensional manifold with a non-degenerate closed 2-form ω.

Example 9 (Complex space)**.** *Let us consider the space $\mathbb{R}^{2n}$. A natural symplectic form for this space is given by*

$$\omega_{\mathbb{R}^{2n}} = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Now let us see $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ as a complex manifold. Then we could write the form in the following way (it is an easy calculation)

$$\omega_{\mathbb{C}^n} = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \frac{i}{2} \partial \overline{\partial} \left| z \right|^2$$

By a theorem due to Darboux, all the symplectic forms are locally like $\omega_{\mathbb{R}^{2n}}$.

A symplectomorphism between two symplectic manifolds $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is a diffeomorphism such that $\psi^* \omega_2 = \omega_1$. The group of symplectomorphisms of M is denoted by $\mathrm{Sympl}(M, \omega)$ and is a subgroup of $\mathrm{Diff}(M)$. A symplectic smooth action of a Lie group G over M is a homomorphism $\psi : G \to \mathrm{Sympl}(M)$ such that its evaluation map is smooth. This means, it is an action that preserves the symplectic form.

Now, observe that the non-degeneracy of ω gives rise to an isomorphism $T_p M \to T_p^* M$. In terms of vector fields we have the one-to-one correspondence $Vect(M) \to \Omega^1(M)$ such that if X is a vector field, the corresponding form is $\iota(X)\omega$. Recall that $\mathcal{L}_X \omega = \iota(X) d\omega + d(\iota(X)\omega) = d(\iota(X)\omega)$ by closedness of ω. By the correspondence between $\mathbb{R}$-actions and complete fields described above, this yields:

Definition 5. *We say that the vector field X in M is symplectic if the following equivalent conditions are satisfied:*

1. *its associated 1-form is closed,*
2. *its associated $\mathbb{R}$-action is symplectic,*
3. $\mathcal{L}_X \omega = 0$.

Remark: Symplectic vector fields are closed under the Lie bracket.

The last example can be brought a little bit further.

Example 10. *We now assume that we are working with a complex vector space V with a Hermitian product $\langle , \rangle_H$. Recall that for vector spaces we can identify the tangent space at each point with V itself. Then, a natural symplectic form on V is given by*

$$\omega_p(v, w) := Im \left\langle v, w \right\rangle_H .$$

Indeed, it is skew symmetric, since $\langle v, w \rangle_H = \overline{\langle w, v \rangle_H}$ and then $Im \langle v, w \rangle_H = -Im \langle w, v \rangle_H$. Non-degeneracy comes from the non-degeneracy of the Hermitian product. This means that we can choose a basis of V as a real vector space, that *makes it linearly symplectomorphic to $\mathbb{R}^{dim_{\mathbb{R}}V}$ with its standard symplectic form. From this we conclude that our form is closed.*

Suppose moreover that we have a complex linear action of a Lie group K on V (this only means that K acts through a complex representation of K on V). Then any Hermitian product can be brought to a K-invariant Hermitian product via the Haar's integral.

Then suppose that $\langle , \rangle_H$ is a K-invariant Hermitian product over V. Then the action of K is symplectic with respect to the form defined by it.

The last definitions make natural to ask: when is a (symplectic) $\mathbb{R}$-action exact? The vector fields generated by such actions and the $\mathbb{R}$-action itself are called *Hamiltonian*. In other words: if ψ is a Hamiltonian $\mathbb{R}$-action, there is a function $H : M \to \mathbb{R}$ such that $\iota_{X^\#}\omega = dH$ (H is *not* unique because two functions differing from a constant have the same derivative). Moreover, if $H : M \to \mathbb{R}$ then dH is a 1-form and (by the 1-1 correspondence between vector fields and 1-forms described above) there is only one field X^H such that $\iota_{X^H}\omega = dH$. Then the action is Hamiltonian if $X^H = X^\#$. Such a function is called a *Hamiltonian* of the field or the action, and it should be thought of as a *moment map.*

One would like to have a general notion of Hamiltonian G-actions. A possible first attempt would be defining a Hamiltonian G-action as a symplectic G-action such that for each $X \in \mathfrak{g}$ one has that $\frac{d}{dt}|_{t=0} exp(-tX)\cdot$ is a Hamiltonian vector field: this means that for any one-parameter subgroup generated by the elements of the Lie algebra, the action is Hamiltonian. These actions are called *weakly Hamiltonian.* In other words, we are choosing a function H^X for every $X \in \mathfrak{g}$. By linearity of the derivative and the symplectic form, one can choose H^X in order to have a well defined linear map

$$\mathfrak{g} \to C^\infty(M).$$

This is called a *comoment map* if it is a Lie algebra homomorphism ($\mathfrak{g}$ with the Lie bracket and $C^\infty(M)$ with the Poisson bracket defined by $\{f, g\} := \omega(X^f, X^g)$).

Finally, our objective is to define a function over M that characterizes the Hamiltonian actions. This is useful because gives rise to level sets in M that allow one to define symplectic quotients. The definition is the following:

Definition 6. *Let G be a Lie group and (M, ω) a symplectic G-manifold. The action of G is Hamiltonian if there exists a map $\mu : M \to \mathfrak{g}^*$ such that:*

- *For every $X \in \mathfrak{g}$, if $\mu^X : M \to \mathbb{R}$ defined by $\mu^X(p) := \langle \mu(p), X \rangle$ then*

$$\iota_{X^\#}\omega = d\mu^X$$

- *μ is equivariant with respect to the action of G on M and the co-adjoint action of G on $\mathfrak{g}^*$.*

The function μ above is called a *moment map*. Every comoment ϕ map defines a moment map by $\langle \mu(p), X\rangle = \phi(X)(p)$. In the same manner, a moment map gives rise to a comoment map. It is easy but not completely trivial how moment and comoment maps are related. More precisely, the first condition for a moment map is equivalent to being weakly Hamiltonian, and then to have the correspondence $\mathfrak{g} \to C^\infty(M)$. However, it is not obvious that equivariance in the moment map is equivalent to the co-moment map being a Lie algebra homomorphism. For this see [6], page 163, lemma 5.16.

Example 11. *Let K be a compact Lie group acting on a vector space V. Now let $\langle , \rangle_H$ be a K-invariant Hermitian product on V. In example 10 we gave a symplectic structure on V. We saw that the action of K is symplectic with respect to this symplectic structure. Let us show that such an action is actually Hamiltonian. However, let us see how the linear K-action can be expressed explicitly.*

First, recall that K acts through a homomorphism $\rho : G \to GL(V)$. This means that $g \cdot v = \rho(g)v$ which is simply a product of a matrix by a vector. Then this map has as derivative in the identity a Lie algebra homomorphism $d\rho(e) : \mathfrak{g} \to \mathfrak{gl}(V)$. Moreover, recall that for a vector space one identifies the tangent space at a point with the vector space itself. Then if $\xi \in \mathfrak{g}$, $\xi \cdot v := \xi_v^\# = \frac{d}{dt}|_{t=0}\rho(exp(t\xi))v = \frac{d}{dt}|_{t=0}exp(td\rho(e)(\xi))v = d\rho(e)(\xi)v$ which is a product of a matrix and a vector.

Now we are ready. To prove that our action is Hamiltonian we give a moment map explicitly:

$$\langle \mu(v), \xi\rangle := \frac{1}{2} Im \langle v, \xi \cdot v\rangle_H .$$

We have to prove that this satisfies:

1. *$d\mu^\xi(v)(w) = \omega(\xi_v^\#, w) = Im\langle \xi \cdot v, w\rangle$:*

$$\begin{aligned} d\mu^\xi(v)(w) &= d(\frac{1}{2}\langle \mu(v), \xi\rangle(w)) = \frac{1}{2}\frac{d}{dt}_{t=0} Im\langle v + tw, (\xi)\cdot(v+tw)\rangle_H \\ &= \frac{1}{2} Im \frac{d}{dt}_{t=0}\langle v + tw, \xi \cdot v + t\xi \cdot w\rangle \\ &= \frac{1}{2} Im\langle w, \xi\cdot v\rangle_H + \frac{1}{2} Im\langle v, \xi \cdot w\rangle_H . \end{aligned}$$

Now note that since K is compact, then its image under ρ is a subgroup of $U(V, \langle , \rangle_H)$ and hence its Lie algebra is the set of skew-Hermitian matrices. This means that:

$$\langle v, \xi \cdot w\rangle_H = \langle d\rho(e)(\xi)^* \cdot v, w\rangle_H = -\langle \xi \cdot v, w\rangle_H = -\overline{\langle w, \xi \cdot v\rangle_H}.$$

This completes our computation

$$\begin{aligned} d\mu^\xi(v)(w) &= \frac{1}{2} Im(\langle w, \xi \cdot v\rangle_H - \overline{\langle w, \xi \cdot v\rangle_H}) = \frac{1}{2} Im2(Im\langle w, \xi \cdot v\rangle_H) \\ &= Im(\langle w, \xi \cdot v\rangle_H) = \omega(\xi_v^\#, w) \end{aligned}$$

2. Now we need to check that μ is K-equivariant:

$$\begin{aligned}\langle \mu(g\cdot v), \xi\rangle &= Im\,\langle g\cdot v, \xi\cdot(g\cdot v)\rangle_H = Im\,\langle v, \rho(g)^*\cdot(\xi\cdot(g\cdot v))\rangle_H \\ &= Im\,\langle v, \rho(g)^* d\rho(e)(\xi)\rho(g)v))\rangle_H\end{aligned}$$

Since $\rho(g)$ is unitary, then $\rho(g)^* = \rho(g)^{-1}$. Moreover, $\rho(g)^* d\rho(e)(\xi)\rho(g)(v)$ is linear in v, and hence its derivative is itself. Then $\rho(g)^* d\rho(e)(\xi)\rho(g)(v) = Ad_g(\xi)\cdot v$. This means,

$$\begin{aligned}\langle \mu(g\cdot v), \xi\rangle &= Im\,\langle v, Ad_g(\xi)v\rangle_H = \langle \mu(v), Ad_g(\xi)\rangle_H \\ &= Ad_g(\mu(v))(\xi).\end{aligned}$$

Thus $12Im\,\langle v, \xi\cdot v\rangle_H$ defines a moment map $\mu : V \to \mathfrak{k}^*$

2.2 Symplectic reduction

In general, if the action of a Lie group over a manifold is free and proper, its space of orbits inherits a manifold structure. In the category of symplectic manifolds, in particular, the orbit space does not have a symplectic structure: the new manifold does not even have even dimension. However, at least in the case of Hamiltonian actions, there is a natural notion of quotient. It is called reduction because in physics (where it has its origins) one looks for the reduction of the phase space by the symmetries (Lie group actions) of the phase space itself. The construction is due to Meyer, Marsden and Weinstein.

At the linear level we have the following:

Lemma 2 (Linear symplectic reduction)**.** *Let (V, ω) a symplectic vector space, and $W \leq V$ a co-isotropic (this means that $W^\omega \subseteq W$) subspace. Then there is a symplectic form ω_{red} in W/W^ω*

Proof. It is natural to define $\omega_{red}([u],[v]) := \omega(u,v)$, for $u, v \in W$. Hence one has to check that this form is well defined and that it is non-degenerate (if it is well defined it is clear that it is alternate).

- (well-defined:) let us write $\omega_{red}([u],[v]) = \omega(u+W^\omega, v+W^\omega) = \omega(u,v) + \omega(u, W^\omega) + \omega(W^\omega, v) + \omega(W^\omega, W^\omega)$. It is obvious that $\omega(W^\omega, -) = \omega(-, W^\omega) = 0$ because W is co-isotropic. Hence $\omega_{red}([u],[v]) = \omega(v,w)$.
- (non-degenerate:) let $u \in W$ and $\omega_{red}([u],[v]) = 0$ for every $w \in W$. Then $w \in (W)^\omega$. Hence $w \in W = [0]$.

□

In order to give a motivation let us see how symplectic reduction works when we have a 1-periodic $\mathbb{R}$-action:

Example 12 (S^1 reduction)**.** *Let (M, ω) be a symplectic manifold with an S^1-action. Now let $H : M \to \mathbb{R}$ be the Hamiltonian, and λ a regular value of H, such that the action is free in the level hypersurface $H^{-1}(\lambda)$. Then the quotient manifold $H^{-1}(\lambda)/S^1$ inherits a symplectic structure ω_{red}. This means that $i^*\omega = \pi^*\omega_{red}$ where $i : H^{-1}(\lambda) \to M$ is the inclusion and π is the projection to the quotient.*

Freeness and regularity are enough to ensure that the quotient is a manifold, because S^1 is compact. Moreover, the integral curves of a Hamiltonian action are tangent to the level sets (cfr.[6]). This means that the tangent space is coisotropic. By the lemma 1 above, the quotient inherits a symplectic form in each tangent space $\omega_{red,\overline{p}}$. Closedness of ω_{red} follows from

$$\pi^* d\omega_{red} = d\pi^*\omega_{red} = di^*\omega = i^*d\omega = 0$$

But π is surjective, then π^* is injective and hence $\omega_{red} = 0$

In practice S^1 reduction works as follows:

Example 13. *Let us study example 11, for some S^1-actions on $\mathbb{C}^{n+1}$. One says that S^1 acts with weights $l_1, \ldots, l_{n+1} \in \mathbb{Z}$ if the action is given by*

$$e^{2\pi it} \cdot (z_1, \ldots, z_{n+1}) = (e^{2\pi i l_1 t} z_1, \ldots, e^{2\pi i l_{n+1} t} z_{n+1})$$

Note that a representation of S^1 would be through diagonal unitary matrices with the same entry in the diagonal. Then it has as Lie algebra $i\mathbb{R}$ (or better skew-hermitian matrices with the same purely imaginary element in the diagonal). The exponential of this group is the usual exponential, and then:

$$\begin{aligned} \frac{d}{dt}_{t=0} exp(-irt) \cdot z &= \frac{d}{dt}_{t=0} (e^{-2\pi i l_1 t} z_1, \ldots, e^{-2\pi i l_{n+1} t} z_{n+1}) \\ &= -(2\pi i l_1 z_1, \ldots, 2\pi i l_{n+1} z_{n+1}) \end{aligned}$$

Then,

$$\langle \mu(z), X \rangle = \frac{Im \, \langle z, (-2\pi i l_1 z_1, \ldots, 2\pi i l_{n+1} z_{n+1}) \rangle}{2}$$

and then the moment map is:

$$\langle \mu(z), X \rangle = -\sum_{j=1}^{n} l_j \left| z_j \right|^2$$

Then for instance, one could pick $n = 2$, $l_1 = 1$ and $l_2 = -1$:first note that $\mu^{-1}(0) = \{(x, y) \in \mathbb{C} : |x| = |y|\}$. Clearly S^1 acts on this set, because if $\lambda \in \mathbb{C}$ then

$$|\lambda x| = |\lambda| \, |x| = |x| = |y| = \left|\lambda^{-1}\right| |y| = \left|\lambda^{-1} y\right|$$

The action is free as you can easily check. Now, note that given an element $x \in \mathbb{C}$, if $y \in \mathbb{C}$ such that $|x| = |y|$ then it is clear that there is only one element $\lambda \in S^1$, such that $\lambda x = y$. Then there is a bijection between element of $\mathbb{C}$ and S^1-classes in $\mu^{-1}(0)$. Hence our reduced manifold is $\mathbb{C}$.

Theorem 5 (Meyer, Marsden-Weinstein)**.** *Let* (M, ω, G) *a Hamiltonian* G*-manifold with moment map* μ*. Suppose that the action of* G *is free and proper on* $\mu^{-1}(0)$ *(for instance, the action of any compact group is proper). Then:*

1. $M_{red} = X/\!\!/G := \mu^{-1}(0)/G$ *is a manifold,*
2. *the projection* $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ *is a principal* G*-bundle,*
3. *there is a symplectic form* ω_{red} *on* M_{red} *such that:* $i^*\omega = \pi^*\omega_{red}$ *(where* i *stands for the inclusion of* $\mu^{-1}(0)$ *in* M*).*

The proof of the theorem is based on the following lemma:

Lemma 3. *Under the assumptions of the theorem, the following are true:*

1. $\ker(d\mu_x) = T_x(G_x)^{\omega_x}(= \{X^\#(x) : X \in \mathfrak{g}\}^{\omega_x})$,
2. $Im(d\mu_x) = \mathfrak{g}_x^0$ *(the anihilator of* $\mathfrak{g}_x$*).*

Proof. The proof is based on the following formula:

$$\langle d_x\mu(v), X\rangle = \omega_x(v, X_x^\#)$$

1. $d\mu_x(v) = 0 \Leftrightarrow \langle d\mu_x(v), X\rangle = 0$ for every $X \in \mathfrak{g} \Leftrightarrow \omega_x(v, X_x^\#) = 0$ which means that v is in the symplectic complement of $\{X^\#(x) : X \in \mathfrak{g}\}$. Since we have an equivalence at every step, the fact is proved.

2. Let us first prove ($\subseteq$): let $X \in \mathfrak{g}_x = Lie(G_x) \leq \mathfrak{g}$, where G_x stands for the stabilizer of x in G. Thus, $X^\#(x) = \frac{d}{dt}_{t=0}(exp(-tX) \cdot x)$. But $exp(-tX) \in G_x$ for every t, which means that it is constant. Then $exp(-tX) \cdot x$ is constant as well, and hence its derivative is 0. Then $\langle d\mu_x(v), X\rangle = \omega_x(v, X^\#(x)) = 0$.

   In order to prove the equality, it is enough to calculate dimensions:

$$\begin{aligned}\dim(Imd\mu_x) &= \dim(M) - \dim(\ker d\mu_x) = \dim M - (\dim M - \dim G \cdot x) \\ &= \dim G - \dim G_x = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_x) = \dim(\mathfrak{g}_x^0)\end{aligned}$$

□

Proof of theorem 5. The proof is based on lemma 5.

1. Let us prove first that $\mu^{-1}(0)$ is a manifold: by lemma 3 (part 2) $Im(d\mu_x) = \mathfrak{g}_x^0$. But $\dim \mathfrak{g}_x^0 = \dim \mathfrak{g} - \dim \mathfrak{g}_x^0 = \dim G - 0$ for any $x \in \mu^{-1}(0)$ since the action is free there and hence any point has trivial stablizer. Then $\dim Imd\mu_x = \dim G$ for any $x \in \mu^{-1}(0)$, which means that $d\mu_x$ is surjective. This says precisely that 0 is a regular value of μ. Then $\mu^{-1}(0)$ is a manifold. Since the action is free and proper, then it follows that $\mu^{-1}(0)/G$ is a manifold as well.

2. Follows from the first item.

3. This can be divided into two parts.

   First we need to prove that there is a form ω_{red} in M_{red} such that $i^*\omega = \pi^*\omega_{red}$. This has to be done in every orbit first. Let us define $\omega_{red}([X],[Y]) := \omega(X,Y)$ in $T_x(\mu^{-1}(0)/G) = T_x\mu^{-1}(0)/T_x\mathcal{O}_x$. To check it is well defined, we need to use lemma 2. In order to do so, it is necessary to verify that $T_x\mathcal{O}_x$ is isotropic. If $T_x\mathcal{O}_x = T_x\mu^{-1}(0)^\omega$, since $T_x\mathcal{O}_x \subset T_x\mu^{-1}(0)$ then the statement follows. Thus we prove that $T_x\mathcal{O}_x = T_x\mu^{-1}(0)^\omega$: if $\mu(x) = 0$ then $\mu^X(x) = 0$ for any $X \in \mathfrak{g}$. This means that μ^X is constant and hence has trivial derivative. By the moment map condition, for any $v \in T_x\mu^{-1}(0)$, $\omega(X_x^{\#}, v) = 0$. To prove the equality it is enough to notice that $\dim(T_x\mathcal{O}_x) = \dim(\mathcal{O}_x) = \dim(G)$ because the action is free. Moreover $\dim(G) = \dim(M) - \dim(\mu^{-1}(0)) = codim(\mu^{-1}(0)) = \dim(T_x\mu^{-1}(0))^\omega$.

   Finally, we would like to see that ω_{red} is symplectic. Non-degeneracy follows inmediately from 2) in lemma 2. On the other hand, the form is closed because $\pi^*(d\omega_{red}) = d\pi^*(\omega_{red}) = di^*\omega = i^*(d\omega) = 0$. Since π is surjective, π^* is injective and hence $d\omega_{red} = 0$.

□

Let us give an example of reduction where the acting group is $U(n)$.

Example 14. *Let $X = M_n(\mathbb{C})$ and $K = U(n)$ acting on X by conjugation. Any matrix $M \in X$ has a unique decomposition as the sum of a Hermitian and a skew-Hermitian matrices as follows:*

$$M = \left(\frac{M+M^*}{2}\right) + \left(\frac{M-M^*}{2}\right)$$

where $\frac{M+M^*}{2} \in \mathcal{H}_n$(Hermitian $n \times n$ matrices) and $\frac{M-M^*}{2} \in i\mathcal{H}_n$ (skew-Hermitian $n\times n$ matrices). Moreover, any hermitian and skew-Hermitain matrix can be diagonalized through a unitary matrix. In our terms, every K-orbit of a Hermitian or a skew-Hermitian matrix contains a diagonal matrix.

If $M = A+B$ is a decompostion of this kind, then a way to ensure that both A, B be diagonalizable by the same matrix is saying that they conmute. Let us write this down:

$$AB = \left(\frac{M+M^*}{2}\right)\left(\frac{M-M^*}{2}\right) = \left(\frac{M-M^*}{2}\right)\left(\frac{M+M^*}{2}\right) = BA$$

Then

$$M^2 - MM^* + M^*M - M^{*2} = M^2 + MM^* - M^*M - M^{*2}$$

This means that:

$$[M, M^*] = MM^* - M^*M = 0$$

This is equivalent to say that M is a normal matrix. The important thing here, is the fact that we have just found a candidate for a moment map. However, $MM^* - M^*M$ is a Hermitian matrix and then it is not in $\mathfrak{u}(n)$. But $i(MM^* - M^*M)$ is skew-Hermitian and then it is in $\mathfrak{u}(n)$. To have an element in the dual $\mathfrak{u}(n)^*$ we use the isomorphism determined by the Hermitian product. For reasons that will be clear in the calculation, we will divide $i(MM^* - M^*M)$ by 2. Then the moment map is the composite map:

$$\begin{aligned} \mu : & M_n(\mathbb{C}) \longrightarrow \mathfrak{u}(n) \xrightarrow{\cong} \mathfrak{u}(n)^* \\ & M \longmapsto (i/2)[M, M^*] \longmapsto \langle (i/2)[M, M^*], - \rangle_H \end{aligned}$$

where the Hermitian product is defined as $\langle A, B \rangle_H := -tr(AB^*)$. If this is a moment map, then $\mu^{-1}(0) = \{normal\ matrices\}$. This means that in the decomposition $M = A + B$, A and B can be diagonalized simultaneously, and then the diagonal form of M is $D + \tilde{D}$, where D is diagonal Hermitian (hence real) and $\tilde{D}$ is diagonal skew-Hermitian (hence purely imaginary). Obviously any diagonal matrix is normal and any permutation matrix is unitary, and then $X//K$ is in bijection with the set $\{diagonal\ matrices\}/\{permutation\ matrices\}$ and hence with $\mathbb{C}^n/S_n$.

It remains to check that μ above is a moment map. We will show some simple computations before:

- *whenever we have an identification via a Hermitian product it is clear that: $\langle \langle X, - \rangle_M, Y \rangle = \langle X, Y \rangle_M$ where $X, Y \in \mathfrak{u}(n)$*
- *$X_p^{\#}$ is easy to compute in the case of matrices. Indeed,*

$$\begin{aligned} X_p^{\#} &= \frac{d}{dt}_{t=0} e^{tX} \cdot p = \frac{d}{dt}_{t=0} e^{tX} p e^{-tX} \\ &= \frac{d}{dt}_{t=0} e^{tX} p + p \frac{d}{dt}_{t=0} e^{-tX} = Xp - pX \end{aligned}$$

This makes clear that we want the following expressions to be equivalent in order to verify the first condition of a moment map:

- $\omega_p(\xi_p^{\#}, Y) = \omega_p(\xi p - p\xi, Y) = -Im(tr((\xi p - p\xi)Y^*))$.
- $d\mu(p)Y$:

$$\begin{aligned} d\mu(p)Y &= \frac{d}{dt}_{t=0} \mu(p + tY) \\ &= \frac{d}{dt}_{t=0} [(p + tY)(p + tY)^* - (p + tY)^*(p + tY)] \\ &= \frac{d}{dt}_{t=0} [pp^* + tpY^* + tYp^* + t^2 YY^* - (p^*p + tY^*p + tp^*Y + t^2 Y^*Y)] \\ &= pY^* + Yp^* - (Y^*p + p^*Y) \end{aligned}$$

The latter is skew-Hermitian.

- *We are ready to prove the equality $\langle d\mu(p), \xi \rangle = \omega_p(\xi_p^{\#}, Y)$ Be the two items above, we only need to realize that*

$$-Im(tr((\xi p - p\xi)Y^*)) = \frac{1}{2}\langle pY^* + Yp^* - (Y^*p + p^*Y), \xi\rangle$$
$$= -\frac{1}{2}Im(tr(i(pY^* + Yp^* - (Y^*p + p^*Y))\xi^*)).$$

It is clear that $\xi^* = \xi$. Then $-(pY^*\xi + Yp^*\xi - Y^*p\xi - p^*Y\xi)$ and

$$tr(pY^*\xi + Yp^*\xi - Y^*p\xi - p^*Y\xi)$$

By using the commutation properties of the trace and conjugating, the equality follows.

Chapter 3

The Kempf-Ness theorem

In this chapter we explore the relation between quotients in algebraic geometry and symplectic reduction. The Kempf-Ness theorem states an explicit correspondence between closure-equivalence classes and points in the symplectic quotient. This relation can be formulated in an explicit way in some cases. However, the proof of the theorem in its general version relies in the properties of certain funtions (Kempf-Ness functions), which allow one to study the polystability of a point.

3.1 Some examples

First let us treat the case of $G = \mathbb{C}^*$ acting on $\mathbb{C}^2$ given in the example 2 of chapter 1. Let us calculate the polystable points. Recall that there are three kinds of orbits: orbits of points (a, b) such that $a, b \neq 0$, the x-axis without $(0, 0)$, the y-axis whithout $(0, 0)$ and $(0, 0)$.

- If $a, b \neq 0$, then any point (x, y) is in the orbit of(a, b) iff $\lambda \cdot (a, b) = (x, y)$. Equivalently $\lambda^{-1} \cdot (x, y) = (a, b)$, or even better $\lambda^{-1} x = a, \lambda y = b$. This means that $xy = ab$ gives the points of the orbit. Then all these orbits are closed.
- If $a = 0$ and $b \neq 0$, then obviously $\lim_{\lambda \to 0} \lambda^{-1}(a, b) = (0, 0)$. Hence $(0, 0)$ is in the closure but not in the orbit. This means that the orbit is not closed and hence (a, b) is not polystable. (The same works for $a \neq 0$ and $b = 0$).
- Evidently, the orbit of the point $(0, 0)$ is closed.

Then the polystable points modulo G are $\mathbb{C}$.

On the other hand, $K = S^1$ is a maximal compact subgroup of $\mathbb{C}^*$, and $\mathbb{C}^*$ is its complexification. We already treated the case of reduction in this case. We obtained precisely $\mathbb{C}$ as well.

Then the theorem of Kempf and Ness says several things (recall that $\mu^{-1}(0) = \{(x, y) \in \mathbb{C}^2 : |x| = |y|\}$):

- $\mu^{-1}(0) \subseteq X^{ps}$: this is clear since the only points of $(x, y) \in \mathbb{C}^2$ which are not polystable are are those in the axes except for the origin. None of these points has the property that $|x| = |y|$ since one is zero but not both are 0.

- $X^{ps} \subseteq G \cdot \mu^{-1}(0)$: if (x, y) is polystable then $x, y \neq 0$ or $x, y = 0$. The second case is trivial. In the first we want to find $\lambda \in \mathbb{C}^*$ such that $|\lambda x| = |\lambda^{-1} y|$. A solution for this equation in λ is simply $|\lambda| = \sqrt{|y/x|}$.

- Any G-orbit of a polystable point contains only one K-orbit of $\mu^{-1}(0)$. Existence follows from the item above. To prove the uniqueness we observe that the only solutions of the last equation are $S^1 \cdot \lambda$. This determines the only S^1-orbit.

- There is a bijection

$$X /\!\!/ G = X^{ps}/G \to X /\!\!/ K = \mu^{-1}(0)/K.$$

  It is a direct consequence of the last statements.

Let us treat the second example that we have worked with throughout the text: $G = GL(\mathbb{C}^n)$, $K = U(n)$ and $X = M_n(\mathbb{C})$. The polystable points of X are the diagonalizable matrices. On the other hand $\mu^{-1}(0)$ are the normal matrices. Repeating the statements above we have:

- $\mu^{-1}(0) \subseteq X^{ps}$: it is clear that any normal matrix is diagonalizable. Indeed, if M is normal then $M = A+B$ where A is Hermitian, B is skew-Hermitian and both commute. Then they are simultaneously diagonalizable by a unitary matrix. Thus M is similar to the sum of a Hermitian and a skew-Hermitian diagonal matrices. In other words, the sum is a complex diagonal matrix.

- $X^{ps} \subseteq G \cdot \mu^{-1}(0)$: every polystable element M has a diagonal D in its orbit. Diagonal matrices are normal, and then $D \in \mu^{-1}(0)$. Hence $D, M \in G \cdot D \subseteq G \cdot \mu^{-1}(0)$.

- Any G-orbit of a polystable point contains only one K-orbit of $\mu^{-1}(0)$. Let M be polystable. This means that it is diagonalizable, and then there is a diagonal, and then normal, matrix D in its orbit. Then the K-orbit of D is contained in the G-orbit of M. Uniqueness: if two normal matrices A, B are such that $G \cdot A = G \cdot B$ then they are unitarily diagonalizable, with diagonal matrix D. This means that $D \in K \cdot A \bigcap K \cdot B$ and then A and B are K-similar or equivalently $K \cdot A = K \cdot B$.

- As a consequence there is a bijection $X /\!\!/ G = X^{ps}/G \to X /\!\!/ K = \mu^{-1}(0)/K$.

In these two cases, it was easy to check the bijection between the affine GIT quotient and the symplectic quotient. The construction relied all the time on the specific structure of each example. The theorem of Kempf and Ness says that the facts listed above are true in general.

3.2 The statement and proof of the Kempf-Ness theorem

The following theorem in its most general form is about complex projective varieties. However, it is easier to understand it first in the case of a complex vector space.

Theorem 6. *[Kempf-Ness] Let G be the complexification of a compact real Lie group K acting on a finite dimensional complex vector space V through a representation $\rho : G \to GL_{\mathbb{C}}(V)$. Suppose that the action of K is Hamiltonian with respect to the symplectic form in V induced by a K-invariant Hermitian product $\langle , \rangle_H$. Let $X \subseteq V$ be a smooth G-invariant affine variety. Then:*

1. $\mu^{-1}(0) \subseteq X^{ps}$,
2. $X^{ps} \subseteq G \cdot \mu^{-1}(0)$,
3. *every G-orbit in X^{ps} contains only one K-orbit of $\mu^{-1}(0)$.*
4. *The last two statements induce a bijection*

$$X /\!/ G \cong X^{ps}/G \to X /\!/ K := \mu^{-1}(0)/K.$$

Remark: If one wants to have a smooth structure on the reduced space (and by Marsden-Weinstein-Meyer a symplectic structure), then the action of K on $\mu^{-1}(0)$ should be free.

The proof is based on the properties of the so called *Kempf-Ness functions* defined as follows:

$$\begin{aligned} \psi_v :& G \to \mathbb{R} \\ & g \mapsto ||g \cdot v||^2 \end{aligned}$$

These actually can be seen as the composition of the "norm" function $N : X \to \mathbb{R}$ defined by $N(x) = ||x||^2$ and $\alpha_v : G \to X$ defined by $\alpha_v(g) = g \cdot v$. It is clear that $N \circ \alpha_v = \psi_v$. The importance of these functions will become apparent as we prove the following lemmas.

Remark: Note that the function ψ_v is defined by means of the norm function N. The latter is defined using the K-invariant Hermitian product in V, and hence ψ_v is K-invariant for every $v \in V$. This means that we can actually define ψ_v in the cosets of $K\backslash G$. Finally, since the group G is **reductive** then $\mathfrak{g}$ admits a splitting $\mathfrak{k} \oplus i\mathfrak{k}$.

Lemma 4. *If $G \cdot v$ is closed (i.e. v is polystable) then ψ_v attains a minimum.*

Proof. Let $l := inf\{\psi_v(g) : g \in G\} \in \mathbb{R}$, which exists because $\{\psi_v(g) : g \in G\}$ is positive and then bounded below. Let $\{g_i\} \subseteq G$ be a sequence such that $\psi_v(g_i) \to l$. Now it is clear that $l \in \overline{N(G \cdot v)}$.

On the other hand, N is a proper function. Indeed, it is continuous (and then the preimage of a closed set is closed) and the preimage of a bounded set in $\mathbb{R}$ under N is a bounded set by definition. Moreover, $N(G \cdot v) \subseteq N(\overline{G \cdot v})$. This yields

$$l \in \overline{N(G \cdot v)} \subseteq \overline{N(\overline{G \cdot v})} = N(\overline{G \cdot v}) = N(G \cdot v),$$

where the first equality follows from the fact that if N is proper then it is closed (this holds for manifolds), and the second equality comes from the assumption on v. $\square$

Remark: the converse of this lemma is also true. We will prove it later.

Now we want to study the behaviour of the Kempf-Ness functions by simple considerations of differential calculus. The important observation at this point is that the Kempf-Ness functions are the integral of the moment map in V (and then in X).

Lemma 5 (Study of ψ_v)**.** *Let $\lambda \in \mathfrak{k}$ and $v \in V$. Then the following are true:*

1. *Moment map: $d(\psi_v)(g)(i\lambda) = 2\langle \mu(g \cdot v), \lambda\rangle$,*
2. *First derivative: $d_g\psi(v) = 0$ if and only if for every $\lambda \in \mathfrak{k}$ $\langle \mu(v), \lambda\rangle = 0$,*
3. *Second derivative: $\frac{d^2}{dt^2}|_{t=0}\psi_v(it\lambda) \geq 0$ (this means that the Kempf-Ness functions are convex). Moreover, $\frac{d^2}{dt^2}|_{t=0}\psi_v(it\lambda) > 0$ if and only if $\lambda \in \mathfrak{k}\backslash\mathfrak{k}_x$ (this says that the Kempf-Ness functions are strictly convex in the directions not in the infinitesimal stabilizer).*

Proof. Suppose $\lambda \in \mathfrak{k}$ and $v \in V$.

1. $$\begin{aligned} d\psi_v(g)(i\lambda) &= \frac{d}{dt}_{t=0}||exp(it\lambda)g \cdot v||^2 = \frac{d}{dt}_{t=0}\langle exp(it\lambda)g \cdot v, exp(it\lambda)g \cdot v\rangle_H \\ &= \left\langle \frac{d}{dt}_{t=0} exp(it\lambda)g \cdot v, g \cdot v\right\rangle_H + \left\langle g \cdot v, \frac{d}{dt}_{t=0} exp(it\lambda)g \cdot v\right\rangle_H \\ &= \langle i\lambda g \cdot v, g \cdot v\rangle_H + \langle g \cdot v, i\lambda g \cdot v\rangle)_H \\ &= i(\overline{\langle g \cdot v, \lambda g \cdot v\rangle_H} - \langle g \cdot v, \lambda g \cdot v\rangle_H) \\ &= i(-2i)Im(\langle g \cdot v, i\lambda g \cdot v\rangle_H) = 2Im(\langle g \cdot v, i\lambda g \cdot v\rangle_H) = 2\langle \mu(g \cdot v), \lambda\rangle_H . \end{aligned}$$

2. This follows easily from (1).

3. This is again a simple calculation:

$$\begin{aligned}
\frac{d^2}{dt^2}_{t=0}\psi_v(e) &= 2\frac{d}{dt}_{t=0}\langle \mu(exp(i\lambda)\cdot v), \lambda\rangle \\
&= 2\frac{d}{dt}_{t=0} Im\,\langle (exp(i\lambda)\cdot v), \lambda(exp(i\lambda)\cdot v)\rangle_H \\
&= 2Im(\langle (i\lambda\cdot v), \lambda\cdot v\rangle_H + \langle \cdot v, \lambda i\lambda\cdot v\rangle_H) \\
&= 2Im(i\,\langle \lambda\cdot v, \lambda\cdot v\rangle_H - \langle \lambda\cdot v, i\lambda\cdot v\rangle_H) \\
&= 2Im(i\,\langle \lambda\cdot v, \lambda\cdot v\rangle_H + i\,\langle \lambda\cdot v, \lambda g\cdot v\rangle_H) \\
&= 2Im(i(2||\lambda\cdot v||^2)) = 4||\lambda\cdot v||^2 \geq 0.
\end{aligned}$$

To prove the second part of the statement, let us define the infinitesimal stabilizer $\mathfrak{k}_v := \{\lambda \in \mathfrak{k} : \lambda\cdot v = 0\}$. On the other hand, $||\lambda\cdot v|| = 0$ only if $\lambda\cdot v = 0$, and hence only if $\lambda \in \mathfrak{k}_v$.

□

Remark: In this case convexity was proved along the flow lines $exp(it\lambda)\cdot v$ determined by the action of G.

So far we have that if $v \in X^{ps}$ (i.e. $G\cdot v$ is closed), then (lemma 4) ψ_v attains a minimum, say $g\cdot v$. By lemma 5, it is a critical point and hence a zero of the moment map, i.e. $\mu(g\cdot v) = 0$. Then $v = g^{-1}(g\cdot v)$. In other words $v \in G\cdot\mu^{-1}(0)$, so we proved part (2) of the theorem.

To prove part (1) of the theorem we need the next lemma. To understand what it says, we should mention a few things before. If the group G is reductive, then it is the complexification of a compact real Lie group. Then, G has a *Cartan decomposition*, i.e. a diffeomorphism

$$K \times \mathfrak{k} \to G$$

such that $(k,\lambda) \mapsto kexp(i\lambda)$. This implies that if we take the quotient $K\backslash G$, it is diffeomorphic to the vector space $\mathfrak{k}$. Now let us see this more concretely. Recall that G acts through a representation and since our Hermitian product is K-invariant, K acts through unitary matrices. On the other hand, any invertible matrix A admits a unique polar decomposition $A = Ue^{iH}$, where e^{iH} is a positive definite Hermitian matrix (and H is skew-Hermitian) and U is unitary. But skew-Hermitian matrices form a vector space, and have a well defined norm that inherit from the space of Hermitian matrices by means of the exponential map. In conclusion, it makes sense if we write $|g| \to \infty$.

Lemma 6. *1. If ψ_v attains a minimum then $\lim_{|g|\to\infty} ||g\cdot v||^2 = \infty$,*

2. If $\lim_{|g|\to\infty} ||g\cdot v||^2 = \infty$ then $G\cdot v$ is closed.

Proof. 1. This is a consequence of the convexity of ψ_v. Indeed, convex functions tend to infinity at infinity.

2. Suppose that $G \cdot v$ is not closed. Then there is $l \in V$ such that $g_i \cdot v \to l$ for a sequence g_i in G. Since $||l|| < \infty$ then $|g_i|$ cannot go to infinity by hypothesis. Then $|g_i|$ is bounded, and hence there is a subsequence such that $g_{i_k} \to g$ for some $g \in G$. Thus $g_{i_k} \cdot v \to g \cdot v$, and then $l = g \cdot v$. This implies that $l \in G \cdot v$.

$\square$

By the first part of lemma 5, we know that if the moment map is 0 in v then the identity is a critical point of ψ_v. Then by lemma 6 the orbit $G \cdot v$ is closed, and then v is a polystable point. This proves part 1 of the theorem.

Proof of the theorem 6. To finish the proof of the theorem, we need to show that if v is polystable, then $G \cdot v$ contains only one K-orbit of a zero of the moment map. In other words, if $a, b \in G \cdot v$ are such that $\mu(a) = \mu(b) = 0$, then $a = k \cdot b$ for some $k \in K$.

We already know that $a = g \cdot b$ and g can be written as $ke^{i\lambda}$ (polar form or Cartan's decomposition), where $k \in K$ and $\lambda \in \mathfrak{k}$. We know that

$$\psi_a(g) = ||g \cdot v||^2 = ||ke^{i\lambda} \cdot v||^2 = ||e^{i\lambda} \cdot v||^2$$

Moreover, since we proved convexity along $exp(i\lambda t) \cdot v$, for any $t \in [0, 1]$. This means that in particular $\psi_a(exp(i\lambda t) \cdot v) \leq \psi_a(e) = \psi_a(g)$. But $\psi_a(e) = \psi_a(g)$ are global minima, and then $\psi_a(exp(i\lambda t) \cdot v) = \psi_a(e) = \psi_a(g)$. Hence $exp(it\lambda)$ is constant, and then $\lambda \in \mathfrak{k}_a$ by the third part of lemma 5. This says that $i\lambda \in i\mathfrak{k}_a \subseteq \mathfrak{g}_a$. Then $exp(i\lambda) \in G_a$. In other words

$$a = g \cdot b = kexp(i\lambda) \cdot b = k \cdot b.$$

$\square$

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