QUASI-HAMILTONIAN QUOTIENTS AS DISJOINT UNIONS OF SYMPLECTIC MANIFOLDS

FLORENT SCHAFFHAUSER - KEIO UNIVERSITY, YOKOHAMA, JAPAN

Abstract. The main result of this paper is Theorem 2.13 which says that the quotient $\mu^{-1}(\{1\})/U$ associated to a quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ has a symplectic structure *even when* 1 *is not a regular value of the momentum map* μ. Namely, it is a disjoint union of symplectic manifolds of possibly different dimensions, which generalizes the result of Alekseev, Malkin and Meinrenken in [AMM98]. We illustrate this theorem with the example of representation spaces of surface groups. As an intermediary step, we give a new class of examples of quasi-Hamiltonian spaces: the isotropy submanifold M_K whose points are the points of M with isotropy group $K \subset U$.

The notion of quasi-Hamiltonian space was introduced by Alekseev, Malkin and Meinrenken in their paper [AMM98]. The main motivation for it was the existence, under some regularity assumptions, of a symplectic structure on the associated quasi-Hamiltonian quotient. Throughout their paper, the analogy with usual Hamiltonian spaces is often used as a guiding principle, replacing Lie-algebra-valued momentum maps with Lie-group-valued momentum maps. In the Hamiltonian setting, when the usual regularity assumptions on the group action or the momentum map are dropped, Lerman and Sjamaar showed in [LS91] that the quotient associated to a Hamiltonian space carries a stratified symplectic structure. In particular, this quotient space is a disjoint union of symplectic manifolds.. In this paper, we prove an analogous result for quasi-Hamiltonian quotients. More precisely, we show that for any quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$, the associated quotient $M//U := \mu^{-1}(\{1\})/U$ is a disjoint union of symplectic manifolds (Theorem 2.13):

$$\mu^{-1}(\{1\})/U = \bigsqcup_{j \in J} (\mu^{-1}(\{1\}) \cap M_{K_j})/L_{K_j}.$$

Here K_j denotes a closed subgroup of U and M_{K_j} denotes the isotropy submanifold of type K_j: $M_{K_j} = \{x \in M \mid U_x = K_j\}$. Finally, L_{K_j} is the quotient group $L_{K_j} = \mathcal{N}(K_j)/K_j$, where $\mathcal{N}(K_j)$ is the normalizer of K_j in U. As an intermediary step in our study, we show that M_{K_j} is a quasi-Hamiltonian space when endowed with the (free) action of L_{K_j}.

Acknowledgements. This paper was written during my post-doctoral stay at Keio University, which was made possible thanks to the support of the Japanese Society for Promotion of Science (JSPS). I would like to thank the referee for comments and suggestions to improve the paper.

1. Quasi-Hamiltonian spaces

1.1. **Definition.** Throughout this paper, we shall designate by U a compact connected Lie group whose Lie algebra $\mathfrak{u} = Lie(U) = T_1U$ is equipped with an Ad-invariant positive definite product denoted by $(.\,|\,.)$. We denote by χ (half) the Cartan 3-form of U, that is, the left-invariant 3-form on U defined on $\mathfrak{u} = T_1U$ by:

$$\chi_1(X, Y, Z) := \frac{1}{2}(X \,|\, [Y, Z]) = \frac{1}{2}([X, Y] \,|\, Z).$$

Recall that, since $(.\,|\,.)$ is Ad-invariant, χ is also right-invariant and that it is a closed form. Further, let us denote by θ^L and θ^R the respectively left-invariant and right-invariant Maurer-Cartan 1-forms on U: they take value in $\mathfrak{u}$ and are the identity on $\mathfrak{u}$, meaning that for any $u \in U$ and any $\xi \in T_uU$,

$$\theta^L_u(\xi) = u^{-1}.\xi \quad \text{and} \quad \theta^R_u(\xi) = \xi.u^{-1}$$

Key words and phrases. Momentum maps, quasi-Hamiltonian spaces, moduli spaces.

(where we denote by a point . the effect of translations on tangent vectors). Finally, we denote by M a manifold on which the group U acts, and by $X^{\#}$ the fundamental vector field on M defined, for any $X \in \mathfrak{u}$, by the action of U in the following way:

$$X^{\#}_x := \tfrac{d}{dt}|_{t=0} \big(\exp(tX).x\big)$$

for any $x \in M$. We then recall the definition of a quasi-Hamiltonian space, which was first introduced in [AMM98].

Definition 1.1 (Quasi-Hamiltonian space, [AMM98])**.** Let (M, ω) be a manifold endowed with a 2-form ω and an action of the Lie group $(U, (.\,|\,.))$ leaving the 2-form ω invariant. Let $\mu : M \to U$ be a U-equivariant map (for the conjugacy action of U on itself).
Then $(M, \omega, \mu : M \to U)$ is said to be a *quasi-Hamiltonian space* with respect to the action of U if the map $\mu : M \to U$ satisfies the following three conditions:

(i) $d\omega = -\mu^*\chi$
(ii) for all $x \in M$, $\ker \omega_x = \{X^{\#}_x \ :\ X \in \mathfrak{u} \,|\, (Ad\,\mu(x) + Id).X = 0\}$
(iii) for all $X \in \mathfrak{u}$, $\iota_{X^{\#}}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)$

where $(\theta^L + \theta^R \,|\, X)$ is the real-valued 1-form defined on U for any $X \in \mathfrak{u}$ by $(\theta^L + \theta^R \,|\, X)_u(\xi) := (\theta^L_u(\xi) + \theta^R_u(\xi) \,|\, X)$ (where $u \in U$ and $\xi \in T_uU$).
In analogy with the usual Hamiltonian case, the map μ is called the *momentum map*.

1.2. **Examples.** In this subsection, we recall the fundamental examples of quasi-Hamiltonian spaces. We will use them in section 3 to illustrate Theorem 2.13.

Proposition 1.2 ([AMM98])**.** *Let $\mathcal{C} \subset U$ be a conjugacy class of a Lie group $(U, (.\,|\,.))$. The tangent space to $\mathcal{C}$ at $u \in \mathcal{C}$ is $T_u\mathcal{C} = \{X.u - u.X \ :\ X \in \mathfrak{u}\}$. The 2-form ω on $\mathcal{C}$ given at $u \in \mathcal{C}$ by*

$$\omega_u(X.u - u.X, Y.u - u.Y) = \frac{1}{2}\big((Ad\,u.X \,|\, Y) - (Ad\,u.Y \,|\, X)\big)$$

is well-defined and makes $\mathcal{C}$ a quasi-Hamiltonian space for the conjugacy action with momentum map the inclusion $\mu : \mathcal{C} \hookrightarrow U$. Such a 2-form is actually unique.

The following theorem explains how to construct a new quasi-Hamiltonian U-space out of two existing quasi-Hamiltonian U-spaces.

Theorem 1.3 (Fusion product of quasi-Hamiltonian spaces, [AMM98])**.** *Let (M_1, ω_1, μ_1) and (M_2, ω_2, μ_2) be two quasi-Hamiltonian U-spaces. Endow $M_1 \times M_2$ with the diagonal action of U. Then the 2-form*

$$\omega := (\omega_1 \oplus \omega_2) + \frac{1}{2}(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$$

makes $M_1 \times M_2$ a quasi-Hamiltonian space with momentum map:

$$\begin{array}{rccl} \mu_1 \cdot \mu_2 : & M_1 \times M_2 & \longrightarrow & U \\ & (x_1, x_2) & \longmapsto & \mu_1(x_1)\mu_2(x_2) \end{array}$$

Corollary 1.4. *The product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ of l conjugacy classes of U is a quasi-Hamiltonian space for the diagonal action of U, with momentum map the product $\mu(u_1, \ldots, u_l) = u_1 \ldots u_l$.*

Proposition 1.5 ([AMM98])**.** *The manifold $\mathfrak{D}(U) := U \times U$ equipped with the diagonal conjugacy action of U, the U-invariant 2-form*

$$\omega = \frac{1}{2}(\alpha^*\theta^L \wedge \beta^*\theta^R) + \frac{1}{2}(\alpha^*\theta^R \wedge \beta^*\theta^L) + \frac{1}{2}\big((\alpha \cdot \beta)^*\theta^L \wedge (\alpha^{-1} \cdot \beta^{-1})^*\theta^R\big)$$

and the equivariant momentum map

$$\begin{array}{rccl} \mu : & \mathfrak{D}(U) = U \times U & \longrightarrow & U \\ & (a, b) & \longmapsto & aba^{-1}b^{-1} \end{array}$$

(where α and β are the projections respectively on the first and second factors of $\mathfrak{D}(U)$) is a quasi-Hamiltonian U-space, called the internally fused double *of U.*

Corollary 1.6. *The product manifold*

$$\mathcal{M}_{g,l} := \underbrace{(U \times U) \times \cdots \times (U \times U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$$

equipped with the diagonal U-action and the momentum map

$$\begin{array}{rccc} \mu_{g,l}: & (U \times U) \times \cdots \times (U \times U) \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (a_1, b_1, \ldots, a_g, b_g, u_1, \ldots, u_l) & \longmapsto & [a_1, b_1] \ldots [a_g, b_g] u_1 \ldots u_l \end{array}$$

is a quasi-Hamiltonian space.

This space plays a very important role in the description of symplectic structures on representation spaces of fundamental groups of Riemann surfaces (see [AMM98] and section 3 below).

1.3. **Properties of quasi-Hamiltonian spaces.** We now give the properties of quasi-Hamiltonian spaces that we shall need when considering the reduction theory of quasi-Hamiltonian spaces. The results in the Proposition below are quasi-Hamiltonian analogues of classical lemmas entering the reduction theory for usual Hamiltonian spaces.

Proposition 1.7 ([AMM98])**.** *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space and let $x \in M$. Then:*

(i) *The map*

$$\begin{array}{rccc} \Lambda_x : & \ker(Ad\,\mu(x) + Id) & \longrightarrow & \ker \omega_x \\ & X & \longmapsto & X^\#_x = \frac{d}{dt}|_{t=0} \big(\exp(tX).x\big) \end{array}$$

is an isomorphism.

(ii) $\ker T_x\mu \cap \ker \omega_x = \{0\}$

(iii) *The left translation*

$$\begin{array}{ccc} U & \longrightarrow & U \\ u & \longmapsto & \big(\mu(x)\big)^{-1} u \end{array}$$

induces an isomorphism

$$\mathrm{Im}\ T_x\mu \simeq \mathfrak{u}_x^\perp$$

where $\mathfrak{u}_x = \{X \in \mathfrak{u} \mid X^\#_x = 0\}$ is the Lie algebra of the stabilizer U_x of x and $\mathfrak{u}_x^\perp$ denotes its orthogonal with respect to $(.\,|\,.)$. Equivalently, $\mathrm{Im}\ (\mu^*\theta^L)_x = \mathfrak{u}_x^\perp$ *(and likewise,* $\mathrm{Im}\ (\mu^*\theta^R)_x = \mathfrak{u}_x^\perp$*).*

(iv) $(\ker T_x\mu)^{\perp_\omega} = \{X^\#_x : X \in \mathfrak{u}\}$, *where $(\ker T_x\mu)^{\perp_\omega} \subset T_xM$ denotes the subspace of T_xM orthogonal to $\ker T_x\mu$ with respect to ω_x.*

We end this subsection with a result that we will need in subsection 2.2. This theorem relates quasi-Hamiltonian spaces to usual Hamiltonian spaces and we quote it from [AMM98] (see remark 3.3, see also [HJS06]).

Theorem 1.8 (Linearization of quasi-Hamiltonian spaces, [AMM98])**.** *Let $(M_0, \omega_0, \mu_0 : M_0 \to U)$ be a quasi-Hamiltonian U-space. Suppose there exists an Ad-stable open subset $\mathcal{D} \subset \mathfrak{u}$ such that $\exp|_{\mathcal{D}} : \mathcal{D} \to \exp(\mathcal{D})$ is a diffeomorphism onto a open subset of U* containing *$\mu_0(M_0)$. Denote by $\exp^{-1} : \exp(\mathcal{D}) \to \mathcal{D}$ the inverse of $\exp|_{\mathcal{D}}$. Then, there exists a* symplectic *2-form $\widetilde{\omega_0}$ on M_0 such that $(M_0, \widetilde{\omega_0}, \widetilde{\mu_0} := \exp^{-1} \circ \mu_0 : M_0 \to \mathfrak{u})$ is a Hamiltonian U-space in the usual sense, for the* same *U-action. Furthermore, one has:*

$$\mu_0^{-1}(\{1_U\}) = \widetilde{\mu_0}^{-1}(\{0\})$$

and therefore

$$\mu_0^{-1}(\{1_U\})/U = \widetilde{\mu_0}^{-1}(\{0\})/U$$

2. Reduction theory of quasi-Hamiltonian spaces

In this section. we show that for any quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$, the associated quotient $M^{red} := \mu^{-1}(\{1\})/U$ is a disjoint union of symplectic manifolds. We begin by reviewing the usual Hamiltonian case and the reduction theorem of Alekseev, Malkin and Meinrenken for quasi-Hamiltonian spaces (Theorem 2.2). We then begin our study of the stratified case and prove the main result of this paper (Theorem 2.13). We also prove that isotropy submanifolds are always quasi-Hamiltonian spaces (Theorem 2.5).

2.1. **Symplectic reduction in the usual Hamiltonian setting.** In this subsection, we recall how to obtain a symplectic manifold from a usual Hamiltonian space by a reduction procedure, that is to say, by taking the quotient of a fiber $\mu^{-1}(\{u\})$ of the momentum map by the action of the stabilizer group U_u, which preserves the fiber $\mu^{-1}(\{u\})$ since μ is equivariant. This reduction procedure is usually called the Marsden-Meyer-Weinstein procedure.

Let us first recall how to obtain differential forms on an orbit space N/G where N is a manifold acted on by a Lie group G. We will assume that G is compact and that it acts freely on N so that N/G is a manifold and the submersion $p : N \to N/G$ is a locally trivial principal fibration with structural group G. Let $[x]$ denote the G-orbit of $x \in N$. Since p is surjective, one has $T_{[x]}(N/G) = \mathrm{Im}\ T_x p \simeq T_x N/ \ker T_x p$. And $\ker T_x p$ consists exactly of the vectors tangent to N at x which are actually tangent to the G-orbit of x in N. Those are exactly the values at x of fundamental vector fields:

$$\ker T_x p = T_x(G.x) = \{X^{\#}_x \ : \ X \in \mathfrak{g} = Lie(G)\}.$$

Let then α be a differential form on N (say, a 2-form). Under what conditions does α define a 2-form $\overline{\alpha}$ on N/G verifying $p^*\overline{\alpha} = \alpha$? This last condition amounts to saying that $\overline{\alpha}_{[x]}([v],[w]) = \alpha_x(v, w)$ for all $x \in N$ and all $v, w \in T_x N$. One then checks that the left-hand side term of this equation is well-defined by this relation if and only if the 2-form α is G-invariant. Further, since $X^{\#}_x$ is sent to 0 in $T_{[x]}(N/G)$ by the map $T_x p$, the relation $p^*\overline{\alpha} = \alpha$ implies that $\iota_{X^{\#}}\alpha = 0$ for all $X \in \mathfrak{g}$. These two conditions turn out to be enough:

Lemma 2.1. *Let $p : N \to B = N/G$ be a locally trivial principal fibration with structural group G and let α be a differential form on N. If α satisfies*

$$g^*\alpha = \alpha \quad \text{for all } g \in G \quad (G\text{−invariance})$$

and

$$\iota_{X^{\#}}\alpha = 0 \quad \text{for all } X \in \mathfrak{g} = Lie(G)$$

then there exists a unique differential form $\overline{\alpha}$ on B satisfying $p^*\overline{\alpha} = \alpha$. In such a case, the differential form α on N is said to be basic.

Observe that if G is compact and connected (so that the exponential map is surjective), the condition $g^*\alpha = \alpha$ for all $g \in G$ may be replaced by $\mathcal{L}_{X^{\#}}\alpha = 0$ for all $X \in \mathfrak{g}$ (which is always implied by the G-invariance). Further, observe that if α is basic then $d\alpha$ is also basic (the first condition is obvious and the second follows from the Cartan homotopy formula: $\iota_{X^{\#}}(d\alpha) = \mathcal{L}_{X^{\#}}\alpha - d(\iota_{X^{\#}}\alpha)$).

We can now use this result to construct differential forms on orbit spaces associated to level manifolds of the momentum map. Let (M, ω) be a symplectic manifold endowed with a Hamiltonian action of a compact connected Lie group U with momentum map $\mu : M \to \mathfrak{u}^*$, and take $N := \mu^{-1}(\{\zeta\})$ where $\zeta \in \mathfrak{u}^*$. Because of the equivariance of μ, the stabilizer $G := U_\zeta$ of ζ for the co-adjoint action of U on $\mathfrak{u}^*$ acts on $N = \mu^{-1}(\{\zeta\})$. Assuming that U_ζ (which is compact) acts freely on $\mu^{-1}(\{\zeta\})$, one has that ζ is a regular value of μ (see the proof of Theorem 2.2 for similar reasoning) and we then have a principal fibre bundle $p : \mu^{-1}(\{\zeta\}) \to \mu^{-1}(\{\zeta\})/U_\zeta$ and the following diagram:

$$\begin{array}{ccc} \mu^{-1}(\{\zeta\}) & \overset{i}{\hookrightarrow} & M \\ {\scriptstyle p}\downarrow & & \\ \mu^{-1}(\{\zeta\})/U_\zeta & & \end{array}$$

where $i : \mu^{-1}(\{\zeta\}) \hookrightarrow M$ is the inclusion map. The 2-form ω on M induces a 2-form $i^*\omega$ on $\mu^{-1}(\{\zeta\})$, which turns out to be basic (again, see the proof of Theorem 2.2 for similar reasoning). Therefore, by Lemma 2.1, there exists a unique 2-form ω^{red} on $\mu^{-1}(\{\zeta\})/U_\zeta$ such that $p^*\omega^{red} = i^*\omega$. Since ω is closed, so is ω^{red}. And one may then notice that a vector $v \in T_xN = \ker T_x\mu$ is sent by T_xp to a vector in $\ker \omega^{red}_{[x]}$ if and only if v is contained in $(T_xN)^{\perp_\omega} = (\ker T_x\mu)^{\perp_\omega} = \{X^\#_x : X \in \mathfrak{u}\}$ as well. But then $v = X^\#_x \in \ker T_x\mu \cap (\ker T_x\mu)^{\perp_\omega}$, so that by the equivariance of μ, one has, denoting by $X^\dagger$ the fundamental vector field on $\mathfrak{u}^*$ associated to X by the co-adjoint action of U: $X^\dagger_\zeta = X^\dagger_{\mu(x)} = T_x\mu.X^\#_x = 0$, so that $X \in \mathfrak{u}_\zeta = Lie(U_\zeta)$. We have thus proved that $T_xp.v \in \ker \omega^{red}_{[x]}$ if and only if $v \in \{X^\#_x : X \in \mathfrak{u}_\zeta\}$. Consequently, for such a v, one has $T_xp.v = 0$, so that ω^{red} is non-degenerate and $\mu^{-1}(\{\zeta\})/U_\zeta$ is a symplectic manifold. When $\zeta = 0 \in \mathfrak{u}^*$, $U_\zeta = U$ and one usually denotes $\mu^{-1}(\{0\})/U$ by $M//U$. This manifold is called the *symplectic quotient* of M by U. Observe that in this case $\mu^{-1}(\{0\})$ is a co-isotropic submanifold of M, since, if $\mu(x) = 0$, then for all $X \in \mathfrak{u}$, $T_x\mu.X^\#_x = X^\dagger_0 = 0$, so that $(\ker T_x\mu)^{\perp_\omega} = T_x(U.x) \subset \ker T_x\mu$. And the 2-form ω^{red} is then symplectic because the leaves of the null-foliation of $\omega|_N$ (that is, the foliation corresponding to the distribution $x \mapsto \ker(\omega|_N)_x = (T_xN)^{\perp_\omega} = (\ker T_x\mu)^{\perp_\omega}$) are precisely the U-orbits.

In [LS91], the authors study the case where regularity assumptions (such as assuming the action of U on $\mu^{-1}(\{0\})$ to be free, or the weaker assumption that 0 is a regular value of μ) are dropped. More precisely, Lerman and Sjamaar showed that when the above regularity assumptions are dropped, the reduced space $M//U$ is a union of symplectic manifolds which are the strata of a stratified space. Their proof relies on the Guillemin-Marle-Sternberg normal form for the momentum map. See subsection 2.3 for further comments.

2.2. **The smooth case.** Let us now come back to the quasi-Hamiltonian setting. In [AMM98], Alekseev, Malkin and Meinrenken showed how to construct new quasi-Hamiltonian spaces from a given quasi-Hamiltonian U-space $(M, \omega, \mu : M \to U)$ by a reduction procedure, assuming that U is a product group $U = U_1 \times U_2$ (so that μ has two components $\mu = (\mu_1, \mu_2)$). Their result says that the reduced space $\mu_1^{-1}(\{u\})/(U_1)_u$ is a quasi-Hamiltonian U_2-space. As a special case, when $U_2 = \{1\}$, they obtain a *symplectic manifold*. Since this is the case we are interested in, we will state their result in this way and give a proof that is valid in this particular situation. We refer to [AMM98] for the general case. It is quite remarkable that one can obtain symplectic manifolds from quasi-Hamiltonian spaces by a reduction procedure. As a matter of fact, this is one of the nicest features of the notion of quasi-Hamiltonian spaces: it enables one to obtain symplectic structures on quotient spaces (typically, moduli spaces) using simple finite dimensional objects as a total space. The most important example in that respect is the moduli space of flat connections on a Riemann surface Σ, first obtained (in the case of a compact surface) by Atiyah and Bott in [AB83] by symplectic reduction of an infinite-dimensional symplectic manifold. We refer to [AMM98] and to section 3 below to see how one can recover these symplectic structures using quasi-Hamiltonian spaces. Let us now state and prove the result we are interested in.

Theorem 2.2 (Symplectic reduction of quasi-Hamiltonian spaces, the smooth case, [AMM98])**.** *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space. Assume that U acts freely on $\mu^{-1}(\{1\})$. Then 1 is a regular value of μ. Further, let $i : \mu^{-1}(\{1\}) \hookrightarrow M$ be the inclusion of the level manifold $\mu^{-1}(\{1\})$ in M and let $p : \mu^{-1}(\{1\}) \to \mu^{-1}(\{1\})/U$ be the projection on the orbit space. Then there exists a unique 2-form ω^{red} on the reduced manifold $M^{red} := \mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ on $\mu^{-1}(\{1\})$ and this 2-form ω^{red} is symplectic.*

We call this case the *smooth case* because in this case the quotient is a smooth manifold. We see from the statement of the theorem that this case arises when the action of U on $\mu^{-1}(\{1\})$ is a *free action.*

Proof. Take $x \in \mu^{-1}(\{1\})$. Then, by Proposition 1.7, one has $\mathrm{Im}\ T_x\mu = \mathfrak{u}_x^{\perp}$. Since the action of U on $\mu^{-1}(\{1\})$ is free, one has $\mathfrak{u}_x = 0$ and therefore $\mathrm{Im}\ T_x\mu = \mathfrak{u}$. Consequently, $1 \in U$ is a regular value of μ and $\mu^{-1}(\{1\})$ is a submanifold of M. The end of the proof consists in showing that $i^*\omega$ is basic with respect to the principal fibration p and then verifying that the unique 2-form ω^{red} on $\mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ is indeed symplectic.

Let us first show that $i^*\omega$ is basic:

$$u^*(i^*\omega) = i^*\omega \quad \text{for all } u \in U$$

and

$$\iota_{X^\#} i^*\omega = 0 \qquad \text{for all } X \in \mathfrak{u}$$

The first condition is obvious since ω is U-invariant. Consider now $X \in \mathfrak{u}$. Then:

$$\begin{aligned} \iota_{X^\#}(i^*\omega) &= i^*(\iota_{X^\#}\omega) \\ &= i^*\big(\frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)\big) \\ &= \frac{1}{2}\big(i^* \circ \mu^*(\theta^L + \theta^R \,|\, X)\big) \\ &= \frac{1}{2}(\mu \circ i)^*(\theta^L + \theta^R \,|\, X) \\ &= 0 \end{aligned}$$

since $\mu \circ i$ is constant on $\mu^{-1}(\{1\})$ and therefore $T(\mu \circ i) = 0$, hence $(\mu \circ i)^* = 0$. Then there exists, by Lemma 2.1, a unique 2-form ω^{red} on $\mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$.

Let us now prove that ω^{red} is a symplectic form. First:

$$\begin{aligned} p^*(d\omega^{red}) &= d(p^*\omega^{red}) \\ &= d(i^*\omega) \\ &= i^*(d\omega) \\ &= i^*(-\mu^*\chi) \\ &= -\underbrace{(\mu \circ i)^*}_{=0}\,\chi \\ &= 0 \end{aligned}$$

so that $d\omega^{red} = 0$. Second, take $[x] \in \mu^{-1}(\{1\})/U$, where $x \in \mu^{-1}(\{1\})$, and $[v] \in \ker \omega^{red}_{[x]}$, where $v \in T_x\mu^{-1}(\{1\}) = \ker T_x\mu$. Then, for all $w \in T_x\mu^{-1}(\{1\}) = \ker T_x\mu$, one has:

$$(i^*\omega)_x(v, w) = (p^*\omega^{red})_x(v, w) = \omega^{red}_{[x]}([v], [w]) = 0$$

since $[v] \in \ker \omega^{red}_{[x]}$. Hence:

$$\begin{aligned} v \in \ker(i^*\omega)_x &= \{s \in \ker T_x\mu \mid \forall\, w \in \ker T_x\mu,\ \omega_x(s, w) = 0\} \\ &= \ker T_x\mu \cap (\ker T_x\mu)^{\perp_\omega} \subset T_xM \end{aligned}$$

But, by Proposition 1.7, $(\ker T_x\mu)^{\perp_\omega} = \{X_x^\# \,:\, X \in \mathfrak{u}\}$, so $v = X_x^\#$ for some $X \in \mathfrak{u}$. Hence:

$$[v] = T_xp.v = T_xp.X_x^\# = 0$$

so that ω^{red} is non-degenerate. □

2.3. **The stratified case.** What happens if we now drop the regularity assumptions of Theorem 2.2? First one may observe that if instead of assuming the action of U on $\mu^{-1}(\{1\})$ to be free one assumes that 1 is a regular value of μ, then one still has $\mathfrak{u}_x = (\mathrm{Im}\ T_x\mu)^{\perp} = \{0\}$ so that the stabilizer U_x of any $x \in \mu^{-1}(\{1\})$ is a discrete, hence finite (since U is compact), subgroup of U. Consequently, $\mu^{-1}(\{1\})/U$ is a symplectic orbifold (this is the point of view adopted in [AMM98]). Following the techniques used in [LS91] for usual Hamiltonian spaces, we will show that if we do not assume that U acts freely on

$\mu^{-1}(\{1\})$ nor that 1 is a regular value of $\mu : M \to U$ then the orbit space $\mu^{-1}(\{1\})/U$ is a disjoint union, over subgroups $K \subset U$, of symplectic manifolds $(N'_K)^{red}$:

$$\mu^{-1}(\{1\})/U = \bigsqcup_{K \subset U} (N'_K)^{red}$$

each $(N'_K)^{red}$ being obtained by applying Theorem 2.2 to a quasi-Hamiltonian space $(N'_K, \omega_K, \widehat{\mu_K}' : N'_K \to L_K)$. Actually, the study conducted in [LS91] is far more precise and ensures that the reduced space $M^{red} := \mu^{-1}(\{1\})/U$ is a *stratified space* $M^{red} = \cup_{K \subset U} S_K$(in particular, there is a notion of *smooth function* on M^{red}, and the set $\mathcal{C}^\infty(M^{red})$ of smooth functions is an algebra over the field $\mathbb{R}$), with strata $(S_K)_{K \subset U}$, such that:

- each stratum S_K is a symplectic manifold (in particular $\mathcal{C}^\infty(S_K)$ is a Poisson algebra).
- $\mathcal{C}^\infty(M^{red})$ is a Poisson algebra.
- the restriction maps $\mathcal{C}^\infty(M^{red}) \to \mathcal{C}^\infty(S_K)$ are Poisson maps.

A stratified space satisfying these additional three conditions is called a *stratified symplectic space*. In [LS91], to show that M^{red} is always a stratified symplectic space, Lerman and Sjamaar actually obtain this space as a disjoint union of symplectic manifolds in two differents ways. The first one enhances the stratified structure of M^{red} (the stratification being induced by the partition of M according to orbit types for the action of U), and relies on the Guillemin-Marle-Sternberg normal form for the momentum map. It also shows that each stratum carries a symplectic structure. The second description of M^{red} as a disjoint union of symplectic manifolds then aims at relating this reduction to the regular Marsden-Meyer-Weinstein procedure: the symplectic structure on each stratum is obtained by *symplectic reduction* from a submanifold of M endowed with a *free* action of a compact Lie group. We also refer to [OR04] for a detailed account on the stratified symplectic structure of symplectic quotients in usual Hamiltonian geometry.

Here, we shall not be dealing with the notion of stratified space and we will content ourselves with a description of $\mu^{-1}(\{1\})/U$ as a disjoint union of symplectic manifolds obtained *by reduction* from a quasi-Hamiltonian space $N'_K \subset M$. We will nonetheless call the case at hand the stratified case.

2.3.1. ***Isotropy submanifolds***. We start with a quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ and use the partition of M given by what we may call the *isotropy type*:

$$M = \bigsqcup_{K \subset U} M_K$$

where $K \subset U$ is a closed subgroup of U and M_K is the set of points of M whose stabilizer is exactly K:

$$M_K = \{x \in M \mid U_x = K\}.$$

Observe that if one wants K to be the stabilizer of some $x \in M$, one has to assume that K is closed, since a stabilizer always is. If M_K is non-empty, it is a submanifold of M (see Proposition [GS84], p.203), called the *manifold of symmetry* K in [LS91]. As for us, we will follow [OR04] and call M_K the *isotropy submanifold of type* K. The tangent space at some point $x \in M_K$ consists of all vectors in T_xM which are fixed by K:

$$T_xM_K = \{v \in T_xM \mid \text{for all } k \in K, k.v = v\}$$

where $k \in K$ acts on T_xM as the tangent map of the diffeomorphism $y \in M \mapsto k.y$ which sends x to itself by definition. The action of U does not preserve M_K but M_K is globally stable under the action of elements $n \in \mathcal{N}(K) \subset U$, where $\mathcal{N}(K)$ denotes the normalizer of K in U:

$$\mathcal{N}(K) := \{u \in U \mid \text{for all } k \in K, uku^{-1} \in K\}.$$

It is actually the largest subgroup of U leaving M_K invariant, since the stabilizer of $u.x$ for some $x \in M_K$ and some $u \in U$ is still U_x if and only if $uU_xu^{-1} = U_x$, that is, $uKu^{-1} = K$. Observe that we have:

$$Lie\big(\mathcal{N}(K)\big) \subset \{X \in \mathfrak{u} \mid \text{for all } Y \in \mathfrak{k}, [X, Y] \in \mathfrak{k}\}.$$

That is, the Lie algebra of the normalizer of K in U is included in the normalizer of $n(\mathfrak{k})$ of the Lie algebra $\mathfrak{k} := Lie(K)$ in $\mathfrak{u} = Lie(U)$. The subgroup K is normal in $\mathcal{N}(K)$ and acts trivially on M_K by definition of the isotropy submanifold of type K, so that M_K inherits an action of the quotient group $\mathcal{N}(K)/K$. It actually follows from the definition of M_K that this induced action is free: if $n \in \mathcal{N}(K)$ stabilizes some x in M_K, then $n \in K$ and so is the identity in $\mathcal{N}(K)/K$. We now wish to show that M_K is a quasi-Hamiltonian space with respect to this action. We need to find a momentum map $\mu_K : M_K \to \mathcal{N}(K)/K$ and a 2-form ω_K satisfying the axioms of definition 1.1. The natural candidates are $\mu_K := \mu|_{M_K}$ and $\omega_K := \omega|_{M_K}$, but the problem is that μ_K does not take its values in $\mathcal{N}(K)/K$. We will now show that $\mu(M_K) \subset \mathcal{N}(K)$ and that we can therefore consider the composed map $\widehat{\mu_K} := p_K \circ \mu_K : M_K \to \mathcal{N}(K)/K$, where p_K is the projection map $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K$. Denote then by L_K the group $L_K := \mathcal{N}(K)/K$. As K is closed in U, so is $\mathcal{N}(K)$, and since U is compact, $\mathcal{N}(K)$ is compact. Therefore $L_K = \mathcal{N}(K)/K$ is a compact Lie group. We will then show that $(M_K, \omega|_{M_K}, \widehat{\mu_K})$ is a quasi-Hamiltonian space. Moreover, we will show that $1 \in L_K$ is a regular value of $\widehat{\mu_K}$ and that L_K acts freely on $\widehat{\mu_K}^{-1}(\{1\})$, so that, by Theorem 2.2, the reduced space $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ is a symplectic manifold.

To do so, we start by studying $\mu(M_K)$. This whole analysis adapts the ideas of [LS91] to the quasi-Hamiltonian setting. Let us denote $\omega_K := \omega|_{M_K}$ and $\mu_K := \mu|_{M_K}$. First, for all $X \in \mathfrak{k}$, we have:

$$\iota_{X^\#}\omega_K = \tfrac{1}{2}\mu_K^*(\theta^L + \theta^R \,|\, X) \tag{1}$$

(where θ^L and θ^R denote as usual the Maurer-Cartan 1-forms of U, so that the above relationship simply follows from the fact that $(M, \omega, \mu : M \to U)$ is a quasi-Hamiltonian space). Second, since K acts trivially on M_K, we have, for all $x \in M_K$ and all $k \in K$:

$$\mu_K(x) = \mu_K(k.x) = k\mu_K(x)k^{-1}$$

so that $\mu(x)$ belongs to the centralizer of K in U:

$$\mathcal{C}(K) := \{u \in U \mid \text{for all } k \in K, uku^{-1} = k\}$$

Since $\mathcal{C}(K) \subset \mathcal{N}(K)$, we have:

$$\mu(M_K) \subset \mathcal{C}(K) \subset \mathcal{N}(K).$$

We can therefore consider the map $\widehat{\mu_K} := p_K \circ \mu_K : M_K \to L_K = \mathcal{N}(K)/K$, where $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K$. Furthermore, we may identify the Lie algebra of L_K to $Lie(\mathcal{N}(K))/\mathfrak{k}$. Under this identification, the Maurer-Cartan 1-forms $\theta^L_{L_K}$ and $\theta^R_{L_K}$ of L_K are obtained by restricting those of U to $\mathcal{N}(K)$ (which gives $Lie(\mathcal{N}(K))$-valued 1-forms) and composing by the projection $Lie(\mathcal{N}(K)) \to Lie(\mathcal{N}(K))/\mathfrak{k}$. It is then immediate from relation (1), that for all $X \in Lie(L_K)$, one has:

$$\iota_{X^\#}\omega_K = \frac{1}{2}\widehat{\mu_K}^*(\theta^L_{L_K} + \theta^R_{L_K} \mid X) \tag{2}$$

Likewise, the Cartan 3-form χ_{L_K} of L_K is obtained by restricting that of U to $\mathcal{N}(K)$ and composing the $Lie(\mathcal{N}(K))$-valued 3-form thus obtained by the projection $Lie(\mathcal{N}(K)) \to Lie(\mathcal{N}(K))/\mathfrak{k}$. Then, it follows from the fact that $d\omega = -\mu^*\chi$ that we have:

$$d\omega_K = -\mu_K^*\chi|_{\mathcal{N}(K)} = -\widehat{\mu_K}^*\chi_{L_K} \tag{3}$$

Thus, we have almost proved that $(M_K, \omega_K, \widehat{\mu_K})$ is a quasi-Hamiltonian L_K-space. In order to compute $\ker(\omega_K)_x$ for all $x \in M_K$, we observe the following two facts, the first of which is classical in symplectic geometry and the second of which is a quasi-Hamiltonian analogue:

Lemma 2.3. *Let (V, ω) be a symplectic vector space and let K be a compact group acting linearly on V preserving ω. Then the subspace*

$$V_K := \{v \in V \mid \text{for all } k \in K, k.v = v\}$$

of K-fixed vectors in V is a symplectic subspace of V.

Proof. Since K is compact, there exists a K-invariant positive definite scalar product on V, that we shall denote by $(.\,|\,.)$. Since ω is non-degenerate, there exists, for any $v \in V$, a unique vector $Av \in V$ satisfying

$$(v\,|\,w) = \omega(Av, w)$$

for all $w \in V$, and the map $A : V \to V$ thus defined is an automorphism of V. Moreover, it satisfies $A(V_K) \subset V_K$. Indeed, if $v \in V_K$, then for all $k \in K$, one has, for all $w \in V$:

$$\begin{aligned} \omega(k.Av, w) &= \omega(Av, k^{-1}.w) \\ &= (v\,|\,k^{-1}.w) \\ &= (k.v\,|\,w) \\ &= \omega(A(k.v), w) \\ &= \omega(Av, w) \end{aligned}$$

and therefore $k.Av = Av$ for all $k \in K$ (incidentally, if one forgets the last equality, which used the fact that $k.v = v$, this also proves that $Ak = kA$ for all $k \in K$), hence $Av \in V_K$. If now $v \in V_K$ satisfies $\omega(v, w) = 0$ for all $w \in V_K$, then in particular for $w = Av$, one obtains $\omega(v, Av) = 0$, that is, $(v\,|\,v) = 0$, hence $v = 0$, since $(.\,|\,.)$ is positive definite. □

Lemma 2.4. *Let (V, ω) be a vector space endowed with a possibly degenerate antisymmetric bilinear form and let K be a compact group acting linearly on V preserving w. Then the 2-form $w_K := \omega|_{V_K}$ defined on the subspace*

$$V_K := \{v \in V \mid \text{for all } k \in K, k.v = v\}$$

of K-fixed vectors of V has kernel:

$$\ker \omega_K = \ker \omega \cap V_K$$

Proof. If ω is non-degenerate then this is simply Lemma 2.3. Assume now that $\ker \omega \neq \{0\}$. Observe that $\ker \omega_K = V_K^{\perp_\omega} \cap V_K \supset \ker \omega \cap V_K$. We now consider the reduced vector space $V^{red} := V/\ker \omega$. The 2-form ω induces a 2-form ω^{red} on V^{red}, which is non-degenerate by construction. The map $V_K \hookrightarrow V \to V/\ker \omega$ induces an inclusion $V_K/(\ker \omega \cap V_K) \hookrightarrow V/\ker \omega$. Further, the action of K on V induces an action $k.[v] := [k.v]$ on V^{red}: this action is well-defined because K preserves ω and therefore if $r \in \ker \omega$ then $k.r \in \ker \omega$. The subspace $(V^{red})_K$ of K-fixed vectors for this action can be identified with $V_K/(\ker \omega \cap V_K)$. Indeed, if $[v] \in V^{red}$ satisfies, for all $k \in K$, $[k.v] = [v]$, then set:

$$w := \int_{k \in K} (k.v) d\lambda(k)$$

where λ is the Haar measure on the compact Lie group K (such that $\lambda(K) = 1$). Then for all $k' \in K$:

$$\begin{aligned} k'.w &= k'.\Big(\int_{k \in K} (k.v) d\lambda(k) \Big) \\ &= \int_{k \in K} (k'k.v) d\lambda(k) \\ &= \int_{h \in K} (h.v) d\lambda(h) \\ &= w \end{aligned}$$

since the Haar measure on K is invariant by translation. Thus $w \in V_K$ and we have:

$$\begin{aligned} [w] &= \Big[\int_{k \in K} (k.v) d\lambda(k) \Big] \\ &= \int_{k \in K} \underbrace{[k.v]}_{=[v]} d\lambda(k) \\ &= [v] \times \int_{k \in K} d\lambda(k) \\ &= [v]. \end{aligned}$$

Thus $[v] \in V_K/(\ker\omega \cap V_K) \subset V^{red}$, which proves that $(V^{red})_K \subset V_K/(\ker\omega \cap V_K)$, and therefore:

$$(V^{red})_K = V_K/(\ker\omega \cap V_K)$$

(the converse inclusion being obvious). Consequently, since V^{red} is a symplectic space, Lemma 2.3 applies and we obtain:

$$\ker\omega^{red}|_{(V^{red})_K} = \{0\}.$$

Now $\omega_K = \omega|_{V_K}$ induces a 2-form $(\omega_K)^{red}$ on $V_K/(\ker\omega \cap V_K) = (V^{red})_K$, whose kernel is, by definition:

$$\ker(\omega_K)^{red} = \ker\omega_K/(\ker\omega \cap V_K).$$

But, again by definition, $(\omega_K)^{red} = \omega^{red}|_{(V^{red})_K}$, so that $\ker(\omega_K)^{red} = \{0\}$, hence $\ker\omega_K = \ker\omega \cap V_K$, which proves the lemma. □

We then obtain a new class of examples of quasi-Hamiltonian spaces:

Theorem 2.5. *For each closed subgroup $K \subset U$, the compact Lie group $L_K := \mathcal{N}(K)/K$ acts freely on the isotropy submanifold*

$$M_K = \{x \in M \mid U_x = K\}.$$

In addition to that, $\mu(M_K) \subset \mathcal{N}(K)$ and $(M_K, \omega_K := \omega|_{M_K}, \widehat{\mu_K} := p_K \circ \mu|_{M_K})$, where p_K is the projection map $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K = L_K$, is a quasi-Hamiltonian space.

Proof. Observe first that $\widehat{\mu_K}$ is L_K equivariant because μ is U-equivariant and $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K$ is a group morphism. Second, recall that we have obtained the relations (2) and (3), so that, to prove that $(M_K, \omega_K, \widehat{\mu_K} : M_K \to L_K)$ is a quasi-Hamiltonian L_K-space, the only thing left to do is compute $\ker(\omega_K)_x \subset T_xM_K$. Since $T_xM_K = \{x \in T_xM \mid \forall k \in K, k.v = v\}$, Lemma 2.4 applies and one has:

$$\ker(\omega_K)_x = \ker\omega_x \cap T_xM_K = \{X^\#_x : X \in \mathfrak{u} \mid Ad\,\mu(x).X = -X\} \cap T_xM_K.$$

But a vector of T_xM of the form $X^\#_x$ lies in $T_xM_K \subset T_xM$ if and only of $X \in Lie(\mathcal{N}(K)) \subset \mathfrak{u}$. Further, we have seen that for all $x \in M_K$, $\mu(X) = \mu_K(x) \in \mathcal{N}(K)$. Therefore:

$$\ker(\omega_K)_x = \{X^\#_x : X \in Lie(\mathcal{N}(K)) \mid Ad\,\mu_K(x).X = -X\}.$$

Since K acts trivially on M_K and on $\mathcal{N}(K)/K$, this last statement is equivalent to:

$$\ker(\omega_K)_x = \{X^\#_x : X \in Lie(\mathcal{N}(K))/\mathfrak{k} \mid Ad\,\widehat{\mu_K}(x).X = -X\}$$

which completes the proof. □

And we then observe that:

Corollary 2.6. *$1 \in L_K$ is a regular value of $\widehat{\mu_K}$ and the reduced space $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ is a symplectic manifold.*

Proof. Since the action of L_K on M_K is free, the fact that $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ is a symplectic manifold follows from Theorem 2.2. □

2.3.2. ***Structure of quasi-Hamiltonian quotients***. We will now use the above analysis to show that, without any regularity assumptions on the action of U on M or on the momentum map $\mu : M \to U$, the orbit space $M^{red} := \mu^{-1}(\{1\})/U$ is a disjoint union of symplectic manifolds. First, in analogy with [LS91], we observe:

Lemma 2.7. *Denote by $(K_j)_{j\in J}$ a system of representatives of conjugacy classes of closed subgroups of U (every closed subgroup $K \subset U$ is conjugate to exactly one of the pairwise non-conjugate K_j). Denote by M_{K_j} the isotropy submanifold of type K_j in the quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$:*

$$M_{K_j} = \{x \in M \mid U_x = K_j\}.$$

Then, the orbit space $\mu^{-1}(\{1_U\})/U$ is the disjoint union:

$$\mu^{-1}(\{1_U\})/U = \bigsqcup_{j\in J} (\mu^{-1}(\{1_U\}) \cap U.M_{K_j})/U.$$

Proof. Take a U-orbit $U.x$ in $\mu^{-1}(\{1_U\})$. The stabilizer U_x of x is conjugate to one of the (K_j), that is: $U_x = uK_ju^{-1}$ for some $u \in U$. Therefore, the stabilizer of $y := u^{-1}.x \in \mu^{-1}(\{1_U\})$ is exactly K_j, and we then have $U.y = U.x$ with $y \in M_{K_j}$. Therefore, we have shown:

$$\mu^{-1}(\{1_U\})/U = \bigcup_{j\in J} (\mu^{-1}(\{1_U\}) \cap U.M_{K_j})/U.$$

The above union is disjoint because if $U.x$ is a U-orbit in $\mu^{-1}(\{1_U\}) \cap U.M_{K_j}$, the stabilizer of x is conjugate to K_j and therefore not conjugate to any $K_{j'}$ for $j' \neq j$. □

We will now study each one of the sets $(\mu^{-1}(\{1_U\}) \cap U.M_{K_j})/U$ separately. We will show, in analogy with the result of Lerman and Sjamaar in [LS91], that each one of these sets is a smooth manifold that carries a symplectic structure, and that this symplectic structure may be obtained *by reduction* from a quasi-Hamiltonian space endowed with a *free* action of a compact Lie group (that is, by applying Theorem 2.2). In [OR04], this procedure is called *Sjamaar's principle*. The way this principle is developped in [OR04] is way more general than what we do here: they consider the quotients $\mu^{-1}(\{\xi\})/U_\xi$ for an arbitrary $\xi \in \mathfrak{u}^*$, which also makes the situation slightly more complicated (notably to find an *equivariant* momentum map for the isotropy submanifolds M_K). Here, we we begin by observing the following fact:

Lemma 2.8. *Let $K \subset U$ be a closed subgroup of U and denote by M_K the isotropy submanifold of type K in the quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$:*

$$M_K = \{x \in M \mid U_x = K\}.$$

Denote by $\mathcal{N}(K)$ the normalizer of K in U and by L_K the quotient group $L_K = \mathcal{N}(K)/K$. Then, the map:

$$\begin{array}{rccc} f_K : (\mu^{-1}(\{1_U\}) \cap M_K)/L_K & \longrightarrow & (\mu^{-1}(\{1_U\}) \cap U.M_K)/U \\ L_K.x & \longmapsto & U.x \end{array}$$

sending the L_K-orbit of a point $x \in (\mu^{-1}(\{1_U\}) \cap M_K)$ to its U-orbit in $(\mu^{-1}(\{1_U\}) \cap U.M_K)$ is well-defined and is a bijection:

$$(\mu^{-1}(\{1_U\}) \cap M_K)/L_K \xrightarrow{\simeq} (\mu^{-1}(\{1_U\}) \cap U.M_K)/U$$

Consequently, we deduce from Lemma 2.7 that:

$$\mu^{-1}(\{1_U\})/U = \bigsqcup_{j\in J} (\mu^{-1}(\{1_U\}) \cap M_{K_j})/L_{K_j}.$$

Proof. The map f_K is well-defined because if $x, y \in \mu^{-1}(\{1_U\}) \cap M_K$ lie in a same L_K-orbit then they lie in a same U-orbit in $(\mu^{-1}(\{1_U\}) \cap U.M_K)$.
The map f_K is onto because a U-orbit in $(\mu^{-1}(\{1_U\}) \cap U.M_K)$ is of the form $U.x$ for some $x \in (\mu^{-1}(\{1_U\}) \cap M_K)$, and f_K then sends the L_K-orbit of such an x in $(\mu^{-1}(\{1_U\}) \cap M_K)$ to the U-orbit $U.x$ in $(\mu^{-1}(\{1_U\}) \cap U.M_K)$.
The map f_K is one-to-one because if $x, y \in (\mu^{-1}(\{1_U\}) \cap M_K)$ lie in a same U-orbit in $(\mu^{-1}(\{1_U\}) \cap U.M_K)$, say $y = u.x$ for some $u \in U$, then the stabilizer of y in $(\mu^{-1}(\{1_U\}) \cap U.M_K)$ is $U_y = uU_xu^{-1}$. But since $x, y \in M_K$ we have $U_x = U_y = K$, hence $u \in \mathcal{N}(K)$ and $L_K.y = L_K.x$. The rest of the Proposition follows from Lemma 2.7. □

We will now prove that each of the sets $(\mu^{-1}(\{1_U\}) \cap U.M_K)/U = (\mu^{-1}(\{1_U\}) \cap M_K)/L_K$ is a *smooth, symplectic manifold*. To do so, we will show that each of these sets is the quasi-Hamiltonian quotient $N'_K//L_K$ associated to a quasi-Hamiltonian space of the form $(N'_K, \omega_K, \widehat{\mu_K}' : N'_K \to L_K)$ (see Theorem 2.5 and Corollary 2.6). More precisely, we have to show that

$$(\mu^{-1}(\{1_U\}) \cap M_K)/L_K = (\widehat{\mu_K}')^{-1}(\{1_{L_K}\})/L_K$$

where $\widehat{\mu_K}'$ is the momentum map of a *free* action of L_K on a quasi-Hamiltonian space $(N'_K, \omega_K, \widehat{\mu_K}' : N'_K \to L_K)$. This last step is not entirely immediate. In fact, experience from the usual Hamiltonian case dealt with by Lerman and Sjamaar in [LS91] shows that in that setting too, one has to replace $(M_K, \omega_K, \widehat{\mu_K} : M_K \to Lie(L_K)^*)$ by another Hamiltonian L_K-space $(M'_K, \omega_K, \widehat{\mu_K}' : M'_K \to Lie(L_K)^*)$, that space M'_K being the union of connected components of M_K which have a non-empty intersection with $\mu^{-1}(\{0\})$. The point is that this space M'_K is in a way big enough to study the quotient $(\mu^{-1}(\{0\})\cap M_K)/L_K$ because by definition of M'_K one has $(\mu^{-1}(\{0\})\cap M_K)/L_K = (\mu^{-1}(\{0\})\cap M'_K)/L_K$. And then one can prove that $(\mu^{-1}(\{0\})\cap M'_K)/L_K = \widehat{\mu_K}'^{-1}(\{0\})/L_K = (M'_K)^{red}$ (whereas it is not true that $(\mu^{-1}(\{0\})\cap M_K)/L_K = \widehat{\mu_K}^{-1}(\{0\})/L_K$), thereby proving that $(\mu^{-1}(\{0\})\cap M_K)/L_K = (M'_K)^{red}$ is a symplectic manifold. Trying an exactly analogous approach in the quasi-Hamiltonian setting does not work: the union of connected components of M_K containing points of $\mu^{-1}(\{1_U\})$ is still too big, and one has to introduce another quasi-Hamiltonian L_K-space, which we will denote by N_K (see Lemma 2.10). This is what we do next (see also remark 2.12). We begin with the following lemma:

Lemma 2.9. *Let $\mathcal{B}\subset\mathfrak{u}$ be an Ad-stable open ball centered at $0\in\mathfrak{u}$ such that the exponential map $\exp|_{\mathcal{B}}:\mathcal{B}\to\exp(\mathcal{B})$ is a diffeomorphism onto an open subset of U containing 1_U. Denote by $N\subset M$ the U-stable open subset of M defined by*

$$N := \mu^{-1}(\exp(\mathcal{B})).$$

Then $(N,\omega|_N,\mu|_N : N\to U)$ is a quasi-Hamiltonian U-space, and one has:

$$(\mu|_N)^{-1}(\{1_U\})/U = \mu^{-1}(\{1_U\})/U.$$

Proof. Any U-stable *open* subset of a quasi-Hamiltonian space is a quasi-Hamiltonian space when endowed with the restriction of the 2-form and the restriction of the momentum map. In the above case, one has $(\mu|_N)^{-1}(\{1_U\}) = \mu^{-1}(\{1_U\})$ by construction of $N=\mu^{-1}(\exp(\mathcal{B}))$. □

We can then compare the isotropy submanifolds of M and of N:

Lemma 2.10. *Let $(N,\omega|_N,\mu|_N : N\to U)$ be the quasi-Hamiltonian U-space introduced in Lemma 2.9. Let $K\subset U$ be a closed subgroup of U and denote by*

$$M_K = \{x\in M \mid U_x = K\} \text{ and } N_K = \{x\in N \mid U_x = K\}$$

the isotropy submanifolds of type K of M and N respectively. Then one has:

$$\mu^{-1}(\{1_U\})\cap M_K = \mu^{-1}(\{1_U\})\cap N_K.$$

Proof. The equality $\mu^{-1}(\{1_U\})\cap M_K = \mu^{-1}(\{1_U\})\cap N_K$ follows from the fact that $\mu^{-1}(\{1_U\})\subset N$ by construction of $N=\mu^{-1}(\exp(\mathcal{B}))$. □

We will now show that the orbit space $(\mu^{-1}(\{1_U\})\cap N_K)/L_K$ has a symplectic structure. To do this, we apply Theorem 1.8 to the quasi-Hamiltonian space $M_0 = N = \mu^{-1}(\exp(\mathcal{B}))$ constructed in Lemma 2.9 to obtain the following result:

Lemma 2.11. *Let $(N=\mu^{-1}(\exp(\mathcal{B})),\omega|_N,\mu|_N : N\to U)$ be the quasi-Hamiltonian U-space introduced in Lemma 2.9. Let $K\subset U$ be a closed subgroup of U and let $\mathcal{N}(K)$ be its normalizer in U. Denote by L_K the quotient group $L_K := \mathcal{N}(K)/K$ and by p_K the projection $p_K:\mathcal{N}(K)\to L_K=\mathcal{N}(K)/K$. Let*

$$N_K = \{x\in N \mid U_x = K\}$$

be the istotropy submanifold of type K in N. Recall from Theorem 2.5 that $\mu(N_K)\subset\mathcal{N}(K)$ and that $(N_K,\omega|_{N_K},\widehat{\mu_K} = p_K\circ\mu|_{N_K} : N_K\to L_K)$ is a quasi-hamitonian L_K-space. Denote by N'_K the union of connected components of N_K which have a non-empty intersection with $\mu^{-1}(\{1_U\})$, and by $\widehat{\mu_K}'$ the restriction of $\widehat{\mu_K}$ to N'_K. Then: N'_K is L_K-stable and $(N'_K,\omega|_{N'_K},\widehat{\mu_K}' : N'_K\to L_K)$ is a quasi-Hamiltonian L_K-space. Furthermore, one has:

$$\mu^{-1}(\{1_U\})\cap N_K = \mu^{-1}(\{1_U\})\cap N'_K = (\widehat{\mu_K}')^{-1}(\{1_{L_K}\})$$

and consequently:

$$(\mu^{-1}(\{1_U\})\cap N_K)/L_K = (\mu^{-1}(\{1_U\})\cap N'_K)/L_K = (\widehat{\mu_K}')^{-1}(\{1_{L_K}\})/L_K = (N'_K)^{red}$$

Proof. We first show that N'_K is L_K-stable and is a quasi-Hamiltonian L_K-space. If $x \in N'_K$ and $n \in \mathcal{N}(K)$ then there exists, by definition of N'_K, a point $x_0 \in \mu^{-1}(\{1_U\}) \cap N_K$ which is connected to x by a path (x_t) in N_K. Then $(n.x_t)$ is a path from $(n.x_0)$ to $(n.x)$ in N_K. Since $\mu(n.x_0) = n\mu(x_0)n^{-1} = 1_U$ and $(n.x)$ lies in the same connected component of N_K as $(n.x_0)$, we have $(n.x) \in N'_K$. The fact that $(N'_K, \omega|_{N'_K}, \widehat{\mu_K}' : N'_K \to L_K)$ is a quasi-Hamiltonian space then follows from the fact that N'_K is an L_K-stable open subset of the quasi-Hamiltonian space $(N_K, \omega|_{N_K}, \widehat{\mu_K} : N_K \to L_K)$.
Let us now prove that $\mu^{-1}(\{1_U\}) \cap N_K = \mu^{-1}(\{1_U\}) \cap N'_K = (\widehat{\mu_K}')^{-1}(\{1_{L_K}\})$. By definition of N'_K, one has $\mu^{-1}(\{1_U\}) \cap N_K = \mu^{-1}(\{1_U\}) \cap N'_K$. Furthermore, it is obvious that $\mu^{-1}(\{1_U\}) \cap N'_K \subset (\widehat{\mu_K}')^{-1}(\{1_{L_K}\})$ since $\widehat{\mu_K}' = p_K \circ \widehat{\mu_K}|_{N'_K}$ and $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K$ is a group morphism. Let us now prove the converse inclusion. We begin by observing that since the exponential map is invertible on $\mathcal{B} \subset \mathfrak{u}$ and $N = \exp(\mathcal{B})$, Theorem 1.8 applies: the map $\widetilde{\mu} := \exp^{-1} \circ \mu|_N : N \to \mathfrak{u}$ is a momentum map in the usual sense for the action of U on N and $\mu^{-1}(\{1_U\}) = \widetilde{\mu}^{-1}(\{0\})$. In particular, one has, for all $x \in N'_K$, $\mathrm{Im}\, T_x\widetilde{\mu} = \mathfrak{u}_x^{\perp} = \mathfrak{k}^{\perp}$ and, since $0 \in \widetilde{\mu}(N'_K)$ by definition of N'_K, this implies $\widetilde{\mu}(N'_K) \subset \mathfrak{k}^{\perp}$. Take now $x \in (\widehat{\mu_K}')^{-1}(\{1_{L_K}\}) \subset N'_K$. This means that $\mu(x) \in (K \cap \mu(N'_K)) \subset \exp(\mathcal{B})$, hence $\widetilde{\mu}(x) = \exp^{-1} \circ \mu(x) \in \mathfrak{k} \cap \widetilde{\mu}(N'_K) \subset \mathfrak{k} \cap \mathfrak{k}^{\perp} = \{0\}$. Consequently, $\widetilde{\mu}(x) = 0$ and therefore $\mu(x) = 1_U$. Hence $(\widehat{\mu_K}')^{-1}(\{1_{L_K}\}) \subset \mu^{-1}(\{1_U\}) \cap N'_K$, which completes the proof. □

Remark 2.12. Lemma 2.11 is crucial in our proof of forthcoming Theorem 2.13. Although our argument is similar to the one in [LS91], where the usual Hamiltonian case is treated, extra difficulties arise to show that $\mu^{-1}(\{1_U\}) \cap N'_K = (\widehat{\mu_K}')^{-1}(\{1_{L_K}\})$. In particular, we were unable to obtain such a statement involving M_K or M'_K instead of N_K and N'_K. In the end this is not a problem because we proved that $\mu^{-1}(\{1_U\}) \cap M_K = \mu^{-1}(\{1_U\}) \cap N_K = \mu^{-1}(\{1_U\}) \cap N'_K$ (see Lemma 2.10). The point of introducing N_K (and then later N'_K) is to be able to linearize the quasi-Hamiltonian space that we are dealing with without changing the associated quotient. This idea was suggested to us by the reading of [HJS06], where a description of quasi-Hamiltonian quotients as disjoint unions of symplectic manifolds is also obtained. The main difference between Theorem 2.13 and Theorem 2.9 in [HJS06] is that in our case the symplectic structure on each component of the union is obtained by reduction from a quasi-Hamiltonian space $(N'_K, \omega_K, \widehat{\mu_K}' : N'_K \to L_K)$ endowed with a *free* action of the compact Lie group L_K. The linearization theorem enables us to reduce the case at hand to the usual Hamiltonian case and mimic the argument in [LS91] (Theorem 3.5). It would be interesting to know if this detour can be avoided.

Theorem 2.13 (Symplectic reduction of quasi-Hamiltonian spaces, the stratified case)**.** *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space. For any closed subgroup $K \subset U$, denote by M_K the isotropy manifold of type K in M:*

$$M_K = \{x \in M \mid U_x = K\}.$$

Denote by $\mathcal{N}(K)$ the normalizer of K in U and by L_K the quotient group $L_K := \mathcal{N}(K)/K$. Then the orbit space

$$(\mu^{-1}(\{1_U\}) \cap M_K)/L_K$$

is a smooth symplectic manifold.
Denote by $(K_j)_{j\in J}$ a system of representatives of closed subgroups of U. Then the orbit space $M^{red} := \mu^{-1}(\{1_U\})/U$ is the disjoint union of the following symplectic manifolds:

$$\mu^{-1}(\{1_U\})/U = \bigsqcup_{j\in J} (\mu^{-1}(\{1_U\}) \cap M_{K_j})/L_{K_j}.$$

Proof. By Lemmas 2.10 and 2.11, we have:

$$(\mu^{-1}(\{1_U\}) \cap M_K)/L_K = (\mu^{-1}(\{1_U\}) \cap N_K)/L_K = (\mu^{-1}(\{1_U\}) \cap N'_K)/L_K = (N'_K)^{red}$$

where the compact group L_K acts freely on the quasi-Hamiltonian space $(N'_K, \omega_K, \widehat{\mu_K}' : N'_K \to L_K)$, so that Theorem 2.2 shows that $(\mu^{-1}(\{1_U\}) \cap M_K)/L_K = (N'_K)^{red}$ is a symplectic manifold. By Lemmas 2.7 and 2.8, we then have:

$$\mu^{-1}(\{1_U\})/U = \bigsqcup_{j\in J} (\mu^{-1}(\{1_U\}) \cap U.M_{K_j})/U = \bigsqcup_{j\in J} (\mu^{-1}(\{1_U\}) \cap M_{K_j})/L_{K_j}.$$

□

Observe that to prove that the set $(\mu^{-1}(\{1_U\}) \cap M_K)/L_K$ is a smooth symplectic manifold, we found a quasi-Hamiltonian L_K-space $(N'_K, \omega_K, \widehat{\mu_K}' : N'_K \to L_K)$ on which L_K acts freely such that $(N'_K)^{red} = (\mu^{-1}(\{1_U\}) \cap M_K)/L_K$ and then applied quasi-Hamiltonian reduction in the smooth case (Theorem 2.2) to N'_K. One key step in this proof is to show that $(\widehat{\mu_K}')^{-1}(\{1_{L_K}\})/L_K = (\mu^{-1}(\{1_U\}) \cap N'_K)/L_K$ and it was to obtain this equality that we used the linearization Theorem 1.8. We then showed that for any quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ the reduced space $M^{red} := \mu^{-1}(\{1\})/U$ is a disjoint union of symplectic manifolds. We denote this reduced space by $M//U$, as in the usual Hamiltonian case:

Definition 2.14 (Quasi-Hamiltonian quotient)**.** The reduced space

$$M//U := \mu^{-1}(\{1_U\})/U = \bigsqcup_{j \in J} (\mu^{-1}(\{1_U\}) \cap M_{K_j})/L_{K_j}$$

associated, by means of Theorems 2.2 and 2.13, to a given quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ is called the *quasi-Hamiltonian quotient* associated to M.

Remark 2.15. Observe that when the action of U on M is free, then the only subgroup $K \subset U$ such that the isotropy submanifold M_K is non-empty is $K = \{1\}$, so that the results of Theorems 2.2 and 2.13 do coincide in this case.

As we shall see in section 3, representation spaces of surface groups naturally arise as quasi-Hamiltonian quotients. Since in this case it is known that representation spaces are stratified symplectic spaces in the sense of [LS91] (see for instance [Hue95]), it should be possible to obtain this stratified symplectic structure in the quasi-Hamiltonian framework. Following [LS91], the first step to do so should be a normal form for momentum maps on quasi-Hamiltonian spaces.

3. Application to representation spaces of surface groups

In this section, we wish to briefly explain, following [AMM98], how the notion of quasi-Hamiltonian space provides a proof of the fact that, for any Lie group $(U, (.\,|\,.))$ endowed with an *Ad*-invariant non-degenerate product and any collection $\mathcal{C} = (\mathcal{C}_j)_{1 \leq j \leq l}$ of l conjugacy classes of U, there exists a symplectic structure on the representation spaces

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$$

(see 3.1 below for a precise definition of these spaces). This will serve as an example to illustrate Theorem 2.13. Here, $\pi_{g,l} = \pi_1(\Sigma_{g,l})$ denotes the fundamental group of the surface $\Sigma_{g,l} := \Sigma_g \backslash \{s_1, \dots, s_l\}$, Σ_g being a compact Riemann surface of genus $g \geq 0$, l being an integer $l \geq 1$ and $s_1, \dots, s_l$ being l pairwise distinct points of Σ_g. When $l = 0$, we set $\mathcal{C} := \emptyset$ and $\Sigma_{g,0} := \Sigma_g$. Everything we will say is valid for any $g \geq 0$ and any $l \geq 0$ but we will not always distinguish between the cases $l = 0$ and $l \geq 1$, to lighten the presentation.

Recall that the fundamental group of the surface $\Sigma_{g,l} = \Sigma_g \backslash \{s_1, \dots, s_l\}$ has the following finite presentation:

$$\pi_{g,l} = < \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_l \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{l} \gamma_j = 1 >$$

each γ_j being the homotopy class of a loop around the puncture s_j. In particular, if $l \geq 1$, it is a free group on $(2g + l - 1)$ generators. As a consequence of this presentation, we see that, having chosen a set of generators of $\pi_{g,l}$, giving a *representation of* $\pi_{g,l}$ *in the group* U (that is, a group morphism from $\pi_{g,l}$ to U) amounts to giving $(2g + l)$ elements $(a_i, b_i, u_j)_{1 \leq i \leq g, 1 \leq j \leq l}$ of U satisfying:

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{l} u_j = 1.$$

Two representations $(a_i, b_i, u_j)_{i,j}$ and $(a'_i, b'_i, u'_j)_{i,j}$ are then called *equivalent* if there exists an element $u \in U$ such that $a'_i = u a_i u^{-1}$, $b'_i = u b_i u^{-1}$, $u'_j = u u_j u^{-1}$ for all i, j. The original approach to describing symplectic structures on spaces of representations shows that, in order to obtain *symplectic* structures, one has to prescribe the conjugacy class of each u_j, $1 \leq j \leq l$. Otherwise, one may obtain Poisson structures, but we shall not enter these considerations and refer to [Hue01] and [AKSM02] instead. We are then led to studying the space $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ of representations of $\pi_{g,l}$ in U with prescribed conjugacy classes for the $(u_j)_{1\leq j\leq l}$:

Definition 3.1. We define the space $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ to be the following set of group morphisms:

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U) = \{\rho : \pi_{g,l} \to U \mid \rho(\gamma_j) \in \mathcal{C}_j \text{ for all } j \in \{1, \dots, l\}\}.$$

Observe that this space may very well be empty, depending on the choice of the conjugacy classes $(\mathcal{C}_j)_{1\leq j\leq l}$. As a matter of fact, when $g = 0$, conditions on the $(\mathcal{C}_j)$ for this set to be non-empty are quite difficult to obtain (see for instance [AW98] for the case $U = SU(n)$). However, when $g \geq 1$ and U is semi-simple, the above set is always non-empty, as shown in [Ho04]. As earlier, giving such a morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ amounts to giving appropriate elements of U:

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U) \simeq \{(a_1, \dots, a_g, b_1, \dots, b_g, u_1, \dots u_l) \in \underbrace{U \times \cdots \times U}_{2g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{l} u_j = 1\}.$$

In particular, two representations $(a_i, b_i, u_j)_{i,j}$ and $(a'_i, b'_i, u'_j)_{i,j}$ are equivalent if and only if they are in a same orbit of the diagonal action of U on $U \times \cdots \times U \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. The *representation space* $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ is then defined to be the quotient space for this action:

$$\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) := \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U.$$

Following for instance [Hue95], the idea to obtain a symplectic structure on the representation space, or moduli space, $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ is then to see this quotient as a *symplectic quotient*, meaning that one wishes to identify $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ with the fibre of a momentum map defined on an *extended moduli space* (the expression comes from [Jef94, Hue95]). The notion of quasi-Hamiltonian space then arises naturally from the choice of

$$\underbrace{U \times \cdots \times U}_{2g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$$

as an extended moduli space, and of the map

$$\mu_{g,l}(a_1, \dots, a_g, b_1, \dots, b_g, u_1, \dots, u_l) = [a_1, b_1] \dots [a_g, b_g] u_1 \dots u_l$$

as U-valued momentum map, so that:

$$\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) = \mu_{g,l}^{-1}(\{1\})/U.$$

Actually, because of the occurrence of the commutators $[a_i, b_i]$, it is more appropriate to re-arrange the arguments of the map $\mu_{g,l}$ in the following way:

$$\mu_{g,l}(a_1, b_1, \dots, a_g, b_g, , u_1, \dots, u_l) = [a_1, b_1] \dots [a_g, b_g] u_1 \dots u_l = 1$$

and to write the extended moduli space:

$$\underbrace{(U \times U) \cdots \times (U \times U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l.$$

In the case where $g = 0$, one simply has:

$$\begin{array}{rccc} \mu_{0,l}: & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \dots, u_l) & \longmapsto & u_1 \dots u_l \end{array}$$

When $g = 1$ and $l = 0$, one has:

$$\begin{array}{rccc} \mu_{1,0}: & U \times U & \longrightarrow & U \\ & (a, b) & \longmapsto & aba^{-1}b^{-1} \end{array}$$

These two particular cases correspond to the examples we recalled in Propositions 1.4 and 1.5, and motivate the notion of quasi-Hamiltonian space. Thus, in general, the extended moduli space is the following quasi-Hamiltonian space:

$$\mathcal{M}_{g,l} := \underbrace{\mathfrak{D}(U) \times \cdots \times \mathfrak{D}(U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l.$$

(where $\mathfrak{D}(U)$ is the internally fused double of U of Proposition 1.5) equipped with the diagonal U-action and the momentum map

$$\begin{array}{rccc} \mu_{g,l}: & \mathfrak{D}(U) \times \cdots \times \mathfrak{D}(U) \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (a_1, b_1, \ldots, a_g, b_g, u_1, \ldots, u_l) & \longmapsto & [a_1, b_1] \ldots [a_g, b_g] u_1 \ldots u_l \end{array}$$

The representation space $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ is then the associated quasi-Hamiltonian quotient (see definition 2.14):

$$\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) = \mathcal{M}_{g,l}//U = (\underbrace{\mathfrak{D}(U) \times \cdots \times \mathfrak{D}(U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l)//U.$$

In particular, in the case of an l-punctured sphere ($g = 0$), we have:

$$\mathrm{Hom}_{\mathcal{C}}\big(\pi_1(S^2 \backslash \{s_1, \ldots, s_l\}), U\big)/U = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_l)//U.$$

We also spell out the case of torus:

$$\mathrm{Hom}\big(\pi_1(\mathbb{T}^2), U\big)/U = \mathfrak{D}(U)//U$$

(there are no conjugacy classes necessary here, as the surface $\mathbb{T}^2$ is closed) and of the punctured torus:

$$\mathrm{Hom}_{\mathcal{C}}\big(\pi_1(\mathbb{T}^2 \backslash \{s\}), U\big)/U = (\mathfrak{D}(U) \times \mathcal{C})//U.$$

We then know from Theorems 2.2 and 2.13 that these representation spaces $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) = \mathcal{M}_{g,l}//U$ carry a symplectic structure, obtained by reduction from the quasi-Hamiltonian space $\mathcal{M}_{g,l}$. More precisely, the representation spaces $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ are *disjoint unions of symplectic manifolds*. Observe that one essential ingredient to obtain this symplectic structure was the fact that $\pi_{g,l}$ admits a finite presentation with a single relation, which was used as a momentum relation.

References

[AB83] M.F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[AKSM02] A. Alekseev, Y. Kosmann-Schwarzbach, and E. Meinrenken. Quasi-Poisson manifolds. *Canad. J. Math.*, 54(1):3–29, 2002.

[AMM98] A. Alekseev, A. Malkin, and E. Meinrenken. Lie group valued moment maps. *J. of Differential Geom.*, 48(3):445–495, 1998.

[AW98] S. Agnihotri and C. Woodward. Eigenvalues of products of unitary matrices and quantum Schubert calculus. *Math. Res. Lett.*, 5(6):817–836, 1998.

[GS84] V. Guillemin and S. Sternberg. *Symplectic techniques in physics.* Cambridge University Press, 1984.

[HJS06] J. Hurtubise, L. Jeffrey, and R. Sjamaar. Group-valued implosion and parabolic structures. *Amer. J. Math.*, 128(1):167–214, 2006.

[Ho04] N-K. Ho. The real locus of an involution map on the moduli space of flat connections on a Riemann surface. *Int. Math. Res. Not.*, 61:3263–3285, 2004.

[Hue95] J. Huebschmann. Symplectic and Poisson structures of certain moduli spaces. I. *Duke Math. J.*, 80(3):737–756, 1995.

[Hue01] J. Huebschmann. On the variation of the Poisson structures of certain moduli spaces. *Math. Ann.*, 319(2):267–310, 2001.

[Jef94] L. Jeffrey. Extended moduli spaces of flat connections on Riemann surfaces. *Math. Ann.*, 298(4):667–692, 1994.

[LS91] E. Lerman and R. Sjamaar. Stratified symplectic spaces and reduction. *Ann. of Math.*, 134(2):375–422, 1991.

[OR04] J.P. Ortega and T. Ratiu. *Momentum maps and Hamiltonian reduction.* Number 222 in Progress in Mathematics. Birkhäuser, 2004.