Representations of the fundamental group of an l-punctured sphere generated by products of Lagrangian involutions

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Abstract

In this paper, we characterize unitary representations of $\pi := \pi_1(S^2\backslash\{s_1, \dots, s_l\})$ whose generators $u_1, \dots, u_l$ (lying in conjugacy classes fixed initially) can be decomposed as products of two Lagrangian involutions $u_j = \sigma_j\sigma_{j+1}$ with $\sigma_{l+1} = \sigma_1$. Our main result is that such representations are exactly the elements of the fixed-point set of an anti-symplectic involution defined on the moduli space $\mathcal{M}_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\pi, U(n))/U(n)$. Consequently, as this fixed-point set is non-empty, it is a Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}}$. To prove this, we use the quasi-Hamiltonian description of the symplectic structure of $\mathcal{M}_{\mathcal{C}}$ and give conditions on an involution defined on a quasi-Hamiltonian U-space $(M, \omega, \mu : M \to U)$ for it to induce an anti-symplectic involution on the reduced space $M//U := \mu^{-1}(\{1\})/U$.

1 Introduction

The fundamental group $\pi := \pi_1(S^2\backslash\{s_1, \dots, s_l\})$ of an l-punctured 2-sphere ($l \geq 1$) has finite presentation $< g_1, g_2, \dots, g_l \mid g_1 g_2 \dots g_l = 1 >$, where g_j stands for the homotopy class of a loop around s_j. Therefore, giving a unitary representation of this surface group (i.e. a group morphism ρ from π to $U(n)$) amounts to giving l unitary matrices $u_1, u_2, \dots, u_l$ satisfying the relation $u_1 u_2 \dots u_l = 1$ (we always identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and endomorphisms of $\mathbb{C}^n$ with their matrices in the canonical basis). One may then want to study representations with prescribed conjugacy classes of generators : given l conjugacy classes $\mathcal{C} = (\mathcal{C}_j = \{u \exp(i\lambda_j)u^{-1} \ : \ u \in U(n)\})_{1\leq j\leq l}$, do there exist l unitary matrices $u_1, u_2, \dots, u_l$ satisfying $u_j \in \mathcal{C}_j$ and $u_1 u_2 \dots u_l = 1$. The answer to this problem was given by Agnihotri and Woodward in [AW98], by Belkale in [Bel01] and by Biswas in [Bis99] : they gave necessary and sufficient conditions on the $\lambda_j \in \mathbb{R}^n$ for the above question to have a positive answer (before that, the case of $SU(2)$ was discussed in [JW92], in [Gal97], in [KM99] and in [Bis98]). In the following, we will always focus our interest on representations with prescribed conjugacy classes of generators and denote by $\mathrm{Hom}_{\mathcal{C}}(\pi, U(n))$ the set of such representations (i.e. group morphisms $\rho : \pi \to U(n)$ such that $\rho(g_j) \in \mathcal{C}_j$ for all j). Coming back to the relation $u_1 \dots u_l = 1$, one may notice that if we decompose each rotation $u_j \in U(n)$ as a product of two orthogonal symmetries (which are no longer unitary transformations, since they reverse orientation, see section 2 for a precise definition of these symmetries) in the following way $u_1 = \sigma_1\sigma_2$, $u_2 = \sigma_2\sigma_3, \dots, u_l = \sigma_l\sigma_1$, then the relation $u_1 \dots u_l = 1$ is automatically satisfied, since orthogonal symmetries are elements of order 2. The appropriate orthogonal symmetries to consider turn out to be orthogonal symmetries with respect to a Lagrangian subspace of $\mathbb{C}^n$, which are just real lines of $\mathbb{C}$ when $n = 1$. A unitary representation of $\pi_1(S^2\backslash\{s_1, \dots, s_l\})$ whose generators $u_1, \dots, u_l$ admit a decomposition $u_j = \sigma_j\sigma_{j+1}$, where σ_j is the orthogonal symmetry with respect to a Lagrangian subspace L_j of $\mathbb{C}^n$, and where $\sigma_{l+1} = \sigma_1$, will be called a *Lagrangian* representation. The natural question to ask is then the following one : when is a given representation a Lagrangian one ? Further, two unitary representations of π with respective generators $(u_1, \dots, u_l)$ and $(u'_1, \dots, u'_l)$ being equivalent if there exists a unitary map $\varphi \in U(n)$ such that $u'_j = \varphi u_j \varphi^{-1}$ for all j, what can one say about the set of Lagrangian representations in the moduli space $\mathcal{M}_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\pi, U(n))/U(n)$ of unitary representations ?

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In this paper, we address these two questions. First, we denote by L_0 the horizontal Lagrangian $L_0 := \mathbb{R}^n \subset \mathbb{C}^n$ of $\mathbb{C}^n$ and we call a representation σ_0-Lagrangian if it is Lagrangian with $L_1 = L_0$. We will see in subsection 6.6 that a given represemtation is Lagrangian if and only if it is equivalent to a σ_0-Lagrangian one. We then obtain the following characterization of σ_0-Lagrangian representations :

Theorem 1. *Given $l \geq 1$ conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l \subset U(n)$ of unitary matrices such that there exist $(u_1, u_2, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ satisfying $u_1 u_2 \ldots u_l = 1$, the representation of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ corresponding to such a $(u_1, u_2, \ldots, u_l)$ is σ_0-Lagrangian if and only if $u_l = u_l^t$, $u_{l-1} = \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l$, ..., and $u_1 = \overline{u}_l^{-1} \overline{u}_{l-1}^{-1} \ldots \overline{u}_2^{-1} u_1^t \overline{u}_2 \ldots \overline{u}_{l-1} \overline{u}_l$.*

Theorem 1 will be proved in subsection 6.6 (theorem 6.10). Second, we recall that the moduli space $\mathcal{M}_\mathcal{C}$ of unitary representations of the surface group π with prescribed conjugacy classes of generators is a symplectic manifold (actually a stratified symplectic space, see [LS91], since we have to take into account the singularities in the manifold structure, see subsections 2.4 and 6.2 in [Jef94]). This symplectic structure, first investigated in [AB83] and in [Gol84], can be obtained in a variety of ways (see for instance [GHJW97, AMM98, AM95, MW99] and the references therein). For our purposes, we will use the one given by Alekseev, Malkin and Meinrenken in [AMM98] and think of our moduli space as a symplectic quotient obtained by reduction of a quasi-Hamiltonian manifold. We then have the following description of the set of equivalence classes of Lagrangian representations of π :

Theorem 2. *The set of equivalence classes of Lagrangian representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ is a Lagrangian submanifold of the moduli space $\mathcal{M}_\mathcal{C} = \mathrm{Hom}_\mathcal{C}(\pi, U(n))/U(n)$ of unitary representations of π (in particular, it is always non-empty).*

Theorem 2 will be proved in subsection 6.7 (theorem 6.12). The fact that there always exist Lagrangian representations was first proved in [FW06], where the dimension of the submanifold of (equivalence classes of) Lagrangian representations was shown to be half the dimension of the moduli space. For a proof of the non-emptiness using ideas from (quasi-) Hamiltonian geometry, we refer to [Sch05] or to the forthcoming paper [Sch] (see also theorem 5.3). For now, we will use the quasi-Hamiltonian description of the symplectic structure of the moduli space to prove theorem 2.

The main intuition to tackle the aforementioned problems is the use of momentum maps to solve questions of linear algebra (see [Knu00]), which first seemed relevant for this problem after studying the case $n = 2$ (see [FMS04]), and which fits right into place with the important idea of thinking of the space of equivalence classes of representations (that is, the moduli space $\mathcal{M}_\mathcal{C}$) as a symplectic quotient. In this framework, the key idea to solve our problem is to obtain the set of Lagrangian representations as the fixed-point set of an involution β, which is first used to give the explicit necessary and sufficient conditions for a representation to be σ_0-Lagrangian appearing in theorem 1 and then turns out to induce an anti-symplectic involution on the moduli space.

After reviewing some background material on Lagrangian involutions (that will later explain how the involution β is obtained), we shall proceed with recalling the notion of quasi-Hamiltonian space introduced in [AMM98] and then use it to obtain the symplectic structure of the moduli space $\mathcal{M}_\mathcal{C}$ (we will restrict ourselves to representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ and give an explicit description of the symplectic 2-form in the case $l = 3$). Then we will show how to obtain Lagrangian submanifolds of a quasi-Hamiltonian symplectic quotient, in a way which theorem 2 will later provide a concrete example of. Finally, we will obtain σ_0-Lagrangian representations of π as the fixed-point set of an involution on the product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ of the prescribed conjugacy classes, and therefrom deduce theorem 1 and theorem 2. Along the way, we will have proved another result which is worth mentioning here and which will be proved in section 5 (theorem 5.2) :

Theorem 3. *Let U be a compact connected Lie group and let (M, ω) be a quasi-Hamiltonian U-space with momentum map $\mu : M \to U$. Let τ be an involutive automorphism of U, denote by τ^- the involution defined on U by $\tau^-(u) = \tau(u^{-1})$ and let β be an involution on M such that :*

(i) $\forall u \in U, \forall x \in M, \beta(u.x) = \tau(u).\beta(x)$

(ii) $\forall x \in M, \mu \circ \beta(x) = \tau^- \circ \mu(x)$

(iii) $\beta^* \omega = -\omega$

then β induces an anti-symplectic involution $\hat{\beta}$ on the reduced space $M^{red} := \mu^{-1}(\{1\})/U$. If $\hat{\beta}$ has fixed points, then $Fix(\hat{\beta})$ is a Lagrangian submanifold of M^{red}.

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2 Background on Lagrangian involutions and angles between Lagrangian subspaces

We give here the properties of Lagrangian involutions that we shall need in the following. Recall that $\mathbb{C}^n$ is endowed with the symplectic form $\omega = -\mathrm{Im}\, h$ where h is the canonical Hermitian product $h = \sum_{k=1}^n dz_k \otimes d\overline{z}_k$, for which it is symplectomorphic to $\mathbb{R}^{2n}$ endowed with the canonical symplectic form $\omega = \sum_{k=1}^n dx_k \wedge dy_k$. Mutiplication by $i \in \mathbb{C}$ in $\mathbb{C}^n$ corresponds to an $\mathbb{R}$-endomorphism J of $\mathbb{R}^{2n}$ satisfying $J^2 = -Id$. Denoting $g = \mathrm{Re}\, h = \sum_{k=1}^n (dx_k \otimes dx_k + dy_k \otimes dy_k)$ the canonical Euclidean product on $\mathbb{R}^{2n}$, we have $g = \omega(., J.)$ (J is called a complex structure and is said to be compatible with ω). A real subspace L of $\mathbb{C}^n$ is said to be Lagrangian if $\omega|_{L\times L} = 0$ and if $\dim_{\mathbb{R}} L = n$ (that is, L is maximal isotropic with respect to ω). One may then check that L is Lagrangian if and only if its g-orthognal complement is $L^{\perp_g} = JL$. We may then define, for any Lagrangian subspace L of $\mathbb{C}^n$, the $\mathbb{R}$-linear map

$$\begin{array}{rccc} \sigma_L : & \mathbb{C}^n = L \oplus JL & \longrightarrow & \mathbb{C}^n \\ & x + Jy & \longmapsto & x - Jy \end{array}$$

called the *Lagrangian involution associated to L*. Observe that σ_L is *anti-holomorphic* : $\sigma_L \circ J = -J \circ \sigma_L$. In the following, we denote by $\mathcal{L}(n)$ the set of all Lagrangian subspaces of $\mathbb{C}^n$ (the Lagrangian Grassmannian of $\mathbb{C}^n$). Finally, recall that, under the identification $(\mathbb{C}^n, h) \simeq (\mathbb{R}^{2n}, J, \omega)$, we have $U(n) = O(2n) \cap Sp(n)$. Furthermore, the action of $U(n)$ on $\mathcal{L}(n)$ is transitive and the stabilizer of the horizontal Lagrangian $L_0 := \mathbb{R}^n \subset \mathbb{C}^n$ is the orthogonal group $O(n) \subset U(n)$, giving the usual homogeneous description $\mathcal{L}(n) = U(n)/O(n)$. Observe that $O(n) = Fix(\tau)$ where $\tau : u \mapsto \overline{u}$ is complex conjugation on $U(n)$, so that $\mathcal{L}(n)$ is a compact symmetric space.

Proposition 2.1. [FMS04] *Let $L \in \mathcal{L}(n)$ be a Lagrangian subspace of $\mathbb{C}^n$. Then:*

(i) There exists a unique anti-holomorphic map σ_L whose fixed point set is exactly L.

(ii) If L' is a Lagrangian subspace such that $\sigma_L = \sigma_{L'}$, then $L = L'$: there is a one-to-one correspondence between Lagrangian subspaces and Lagrangian involutions.

(iii) σ_L is anti-unitary *: for all $z, z' \in \mathbb{C}^n$, $h(\sigma_L(z), \sigma_L(z')) = \overline{h(z, z')}$.*

(iv) For any $\varphi \in U(n)$, $\sigma_{\varphi(L)} = \varphi \sigma_L \varphi^{-1}$.

Denote then by $\mathcal{LI}nv(n) := \{\sigma_L \ : \ L \in \mathcal{L}(n)\}$ the subset of $O(2n)$ consisting of Lagrangian involutions. Observe that it is not a subgroup, as it is not stable by composition of maps. Statement (iv) of the above proposition then shows that the subgroup $\widehat{U(n)} :=< U(n) \cup \mathcal{LI}nv(n) > \subset O(2n)$ generated by Lagrangian involutions and unitary transformations is in fact generated by $U(n)$ and σ_{L_0} : $\widehat{U(n)} =< U(n) \cup \{\sigma_{L_0}\} >$. As a word in $< U(n) \cup \{\sigma_{L_0}\} >$ contains either an even or an odd number of occurrences of σ_{L_0} (depending only on whether it represents a holomorphic or an anti-holomorphic transformation of $(\mathbb{R}^{2n}, J) \simeq \mathbb{C}^n$), it can be written uniquely under the reduced form $u\varepsilon$ where $u \in U(n)$ and $\varepsilon = 1$ or $\varepsilon = \sigma_{L_0}$. Consequently, we have $< U(n) \cup \{\sigma_{L_0}\} > = U(n) \sqcup U(n)\sigma_{L_0}$, so that $U(n)$ is indeed a subgroup of index 2 of $\widehat{U(n)}$. Further, if we write $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma_{L_0}\}$ and consider the action of this group on $U(n)$ given by $\sigma_{L_0}.u = \sigma_{L_0} u \sigma_{L_0} = \overline{u} = \tau(u)$, then the map

$$\begin{array}{ccc} U(n) \rtimes \mathbb{Z}/2\mathbb{Z} & \longrightarrow & U(n) \sqcup U(n)\sigma_{L_0} \\ (u, \varepsilon) & \longmapsto & u\varepsilon \end{array}$$

(where $\varepsilon = 1$ or $\varepsilon = \sigma_{L_0}$) is a group isomorphism. Finite subgroups of $U(2) \rtimes \mathbb{Z}/2\mathbb{Z}$ generated by Lagrangian involutions are studied in [Fal01]. As for us, one of the major interests of Lagrangian involutions will be that they measure angles of Lagrangian subspaces of $\mathbb{C}^n$ under the action of the unitary group :

Theorem 2.2. [Nic91, FMS04] *Let (L_1, L_2) and (L'_1, L'_2) be two pairs of Lagrangian subspaces of $\mathbb{C}^n$. Then there exists a unitary map $\varphi \in U(n)$ such that $\varphi(L_1) = L'_1$ and $\varphi(L_2) = L'_2$ if and only if $\sigma_{L'_1}\sigma_{L'_2}$ is conjugate to $\sigma_{L_1}\sigma_{L_2}$ in $U(n)$.*

The following series of results will be useful to us in the proof of theorem 6.10. The underlying idea is that the elements of the symmetric space $\mathcal{L}(n) = U(n)/O(n)$ can be identified with the *symmetric* elements of $U(n)$ (that is, elements of $U(n)$ satisfying $\tau(u) = u^{-1}$, see [Hel01, Loo69]), all of them being of the form $\varphi^t\varphi$, where $\varphi \in U(n)$ and φ^t denotes the transpose of φ (so that the symmetric elements of $U(n)$ are indeed symmetric unitary matrices).

Proposition 2.3. *Let $W(n) := \{w \in U(n) \mid w^t = w\}$ be the set of symmetric unitary matrices.*

(i) Let $u \in U(n)$. Then $u \in W(n)$ if and only if there exists $k \in O(n)$ such that kuk^{-1} is diagonal.

(ii) If $w \in W(n)$, then there exists $\varphi \in W(n)$ such that $\varphi^2 = w$.

(iii) For any $w \in W(n)$, define $L_w := \{z \in \mathbb{C}^n \mid z - w\overline{z} = 0\}$. Then, if φ is any element in $W(n)$ such that $\varphi^2 = w$, we have $\varphi(L_0) = L_w$. Consequently, L_w is a Lagrangian subspace of $\mathbb{C}^n$. Furthermore, $\sigma_{L_w}\sigma_{L_0} = w$.

(iv) The map $w \in W(n) \mapsto L_w \in \mathcal{L}(n)$ is a diffeomorphism whose inverse is the well-defined map

$$\begin{array}{ccc} \mathcal{L}(n) = U(n)/O(n) & \longrightarrow & W(n) \\ L = u(L_0) & \longmapsto & uu^t \end{array}$$

(v) For any $L \in \mathcal{L}(n)$, we have $\sigma_{L_0}\sigma_L = v^t v$, where v is any unitary map such that $v(L) = L_0$.

(vi) For any $u \in U(n)$, there exist two Lagrangian subspaces $L_1, L_2 \in \mathcal{L}(n)$ such that $u = \sigma_{L_1}\sigma_{L_2}$.

Proof. (i) Observe that, alternatively, $W(n) = \{w \in U(n) \mid w^{-1} = \overline{w}\}$. Now take $w \in W(n)$ and write $w = x + iy$ where x, y are real matrices. Then $w^t = w$ implies $x^t = x$ and $y^t = y$, and $w\overline{w} = Id$ implies $x^2 + y^2 = Id$ and $xy - yx = 0$. Thus x and y are commuting real symmetric matrices, so there exists $k \in O(n)$ such that $d_x := kxk^{-1}$ and $d_y = kyk^{-1}$ are both diagonal. Therefore, $kwk^{-1} = d_x + id_y$ is diagonal. The converse is obvious. One may observe that since $d_x^2 + d_y^2 = k(x^2 + y^2)k^{-1} = Id$, one has $d_x + id_y = \exp(iS)$ where S is a real symmetric matrix.

(ii) is an immediate consequence of (i).

(iii) Take $\varphi \in W(n) \mid \varphi^2 = w$. Then $z - w\overline{z} = 0$ iff $z - \varphi^2\overline{z} = 0$, that is, $\varphi^{-1}z - \varphi\overline{z} = 0$. But $\varphi^{-1} = \overline{\varphi}$ so that $z \in L_w$ is equivalent to $\varphi^{-1}z = \overline{\varphi^{-1}z}$ hence to $\varphi^{-1}z \in L_0$, hence to $z \in \varphi(L_0)$, which shows that $L_w = \varphi(L_0)$ is a Lagrangian subspace of $\mathbb{C}^n$. Furthermore, $\sigma_{L_w}\sigma_{L_0} = \varphi\sigma_{L_0}\varphi^{-1}\sigma_{L_0}$. But since σ_{L_0} is complex conjugation in $\mathbb{C}^n$ and since φ is both symmetric and unitary, we have $\varphi^{-1}\sigma_{L_0} = \overline{\varphi^t}\sigma_{L_0} = (\sigma_{L_0}\varphi^t\sigma_{L_0})\sigma_{L_0} = \sigma_{L_0}\varphi$, therefore $\sigma_{L_w}\sigma_{L_0} = \varphi\sigma_{L_0}^2\varphi = \varphi^2 = w$.

(iv) Observe that if u, v are two unitary maps sending L_0 to $L \in \mathcal{L}(n)$ then $v^{-1}u \in Stab(L_0) = O(n)$ so that $uu^t = vv^t$. Then, if $L = u(L_0) \in \mathcal{L}(n)$, one has $L_{uu^t} = \{z - uu^t\overline{z} = 0\}$. But $z - uu^t\overline{z} = 0$ iff $u^{-1}z = \overline{u}^{-1}\overline{z}$, that is, $u^{-1}z \in L_0$ so $L_{uu^t} = u(L_0)$. Conversely, we know that $L_w = \varphi(L_0)$ where $\varphi \in W(n) \mid \varphi^2 = w$ so that indeed $\varphi\varphi^t = \varphi^2 = w$.

(v) For a given $L \in \mathcal{L}(n)$, take $v \in U(n)$ such that $v(L) = L_0$. Then $L = v^{-1}(L_0)$ and so we know from (iii) and (iv) that $L = \{z - (v^{-1})(v^{-1})^t\overline{z} = 0\}$ and that $\sigma_L\sigma_{L_0} = v^{-1}(v^{-1})^t$. Hence $\sigma_{L_0}\sigma_L = (\sigma_L\sigma_{L_0})^{-1} = v^t v$.

(vi) Let $d = \mathrm{diag}\,(\alpha_1, \dots, \alpha_l) \in U(n)$ be a diagonal matrix such that $u = \varphi d^2\varphi^{-1}$ and set $L = d(L_0)$. Then we know from (iii) and (iv) that $\sigma_L\sigma_{L_0} = d^2$, hence $u = \varphi\sigma_L\sigma_{L_0}\varphi^{-1} = \sigma_{\varphi(L)}\sigma_{\varphi(L_0)}$.

□

Statement (v) may seem a bit useless at this point as it is just a way of rephrasing (ii), but it will prove useful to us when formulating the *centered Lagrangian problem* (see subsection 6.2).

3 Quasi-Hamiltonian spaces

We recall here the definition of quasi-Hamiltonian spaces and the examples that shall be useful to us in the following. We follow [AMM98] (see also [GHJW97] and [AKSM02] for related constructions). Let U be a compact connected Lie group acting on a manifold M endowed with a 2-form ω. We denote by $(\,.\,|\,.\,)$ an Ad-invariant Euclidean product on $\mathfrak{u} = Lie(U) = T_1 U$. Let χ be (half) the Cartan 3-form of U, that is, the left-invariant 3-form defined on $\mathfrak{u} = T_1 U$ by $\chi_1(X, Y, Z) = \frac{1}{2}\,(X\,|\,[Y, Z]) = \frac{1}{2}\,([X, Y]\,|\,Z)$, where the last equality follows from the Ad-invariance property. Since $(\,.\,|\,.\,)$ is Ad-invariant, χ is actually bi-invariant and therefore closed : $d\chi = 0$. Further, denote by θ^L and θ^R the left and right-invariant Maurer-Cartan 1-forms on U : they take values in $\mathfrak{u}$ and are the identity on $\mathfrak{u}$, meaning that for any $u \in U$ and any $\xi \in T_u U$, $\theta^L_u(\xi) = u^{-1}.\,\xi$ and $\theta^R_u(\xi) = \xi.u^{-1}$ (where we denote by a point . the effect of translations on tangent vectors). Finally, denote by $X^\sharp$ the fundamental vector field on M defined, for any $X \in \mathfrak{u}$, by the action of U : $X^\sharp_x = \frac{d}{dt}|_{t=0}(\exp(tX).x)$ for any $x \in M$. Throughout this paper, we will follow the conventions in [Mor01] to compute exterior products and exterior differentials of differential forms.

Definition 3.1 (Quasi-Hamiltonian space)**.** [AMM98] In the above notations, (M, ω) is called a *quasi-Hamiltonian space* if :

(i) The 2-form ω is U-invariant : $\forall u \in U$, the associated diffeomorphism of M, denoted by φ_u, satisfies $\varphi_u^*\omega = \omega$.

(ii) There exists a map $\mu : M \to U$, called the *momentum map*, such that :

   (a) μ is equivariant with respect to the U-action on M and conjugation in U

   (b) $d\omega = -\mu^*\chi$

   (c) $\forall x \in M$, $\ker \omega_x = \{X^\sharp_x \,:\, X \in \mathfrak{u} \mid Ad\,\mu(x).X = -X\}$

   (d) $\forall X \in \mathfrak{u}$, the interior product of $X^\sharp$ and ω is

$$\iota_{X^\sharp}\omega = \frac{1}{2}\,\mu^*(\theta^L + \theta^R\,|\,X)$$

   where $(\theta^L + \theta^R\,|\,X)$ is the real-valued 1-form defined on U by $(\theta^L + \theta^R\,|\,X)_u(\xi) = (\theta^L_u(\xi) + \theta^R_u(\xi)\,|\,X)$ for any $u \in U$ and any $\xi \in T_u U$.

The examples of quasi-Hamiltonian space that will be of most interest to us are the conjugacy classes of U.

Proposition 3.1. [AMM98] *Let $\mathcal{C} \subset U$ be a conjugacy class of a compact connected Lie group U. The tangent space to $\mathcal{C}$ at $u \in \mathcal{C}$ is $T_u\,\mathcal{C} = \{X.u{-}u.X \,:\, X \in \mathfrak{u}\}$. For a given $X \in \mathfrak{u}$, denote $[X]_u := X.u{-}u.X$. Then the 2-form ω on $\mathcal{C}$ given at $u \in \mathcal{C}$ by*

$$\omega_u([X]_u, [Y]_u) = \frac{1}{2}\left((Ad\,u.X\,|\,Y) - (Ad\,u.Y\,|\,X)\right)$$

is well-defined and makes $\mathcal{C}$ a quasi-Hamiltonian space for the conjugation action and with momentum map the inclusion $\mu : \mathcal{C} \hookrightarrow U$. Such a 2-form is actually unique.

Observe that $[X]_u = X^\sharp_u$, that is : the fundamental vector fields generate the tangent space to $\mathcal{C}$. It is also useful to write this quantity $[X]_u = (X - Ad\,u.X).u = u.(Ad\,u^{-1}.X - X)$. In order to describe the symplectic structure on the moduli space $\mathcal{M}_\mathcal{C} = \mathrm{Hom}_\mathcal{C}(\pi, U(n))/U(n)$, we will have to consider the product space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ where the $\mathcal{C}_j$ are conjugacy classes in $U(n)$, endowed with the diagonal action of $U(n)$. To make this a quasi-Hamiltonian space with momentum map the product map $\mu(u_1, \ldots, u_l) = u_1 \ldots u_l$, one has to endow it with a form that is *not* the product form but has extra terms. The product space thus obtained is called the *fusion product* and usually denoted $\mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_l$. The general result is the following :

Theorem 3.2 (Fusion product of quasi-Hamiltonian spaces)**.** [AMM98] *Let (M_1, ω_1, μ_1) and (M_2, ω_2, μ_2) be two quasi-Hamiltonian U-spaces. Endow $M_1 \times M_2$ with the diagonal action of U. Then the 2-form*

$$\omega := (\omega_1 \oplus \omega_2) + (\mu_1^* \theta^L \wedge \mu_2^* \theta^R)$$

makes $M_1 \times M_2$ a quasi-Hamiltonian space with momentum map

$$\begin{array}{cccc} \mu_1 \cdot \mu_2 : & M_1 \times M_2 & \longrightarrow & U \\ & (x_1, x_2) & \longmapsto & \mu_1(x_1)\mu_2(x_2) \end{array}$$

Here, the 2-form $\omega_1 \oplus \omega_2$ is the product form $(\omega_1 \oplus \omega_2)_{(x_1,x_2)}((v_1, v_2), (w_1, w_2)) = (\omega_1)_{x_1}(v_1, w_1) + (\omega_2)_{x_2}(v_2, w_2)$ and $(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$ is the 2-form defined on $M_1 \times M_2$ by

$$(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big) = \frac{1}{2}\Big(\big((\mu_1^*\theta^L)_{x_1}.v_1 \,|\, (\mu_2^*\theta^R)_{x_2}.w_2\big) - \big((\mu_1^*\theta^L)_{x_1}.w_1 \,|\, (\mu_2^*\theta^R)_{x_2}.v_2\big)\Big)$$

The above result shows that $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ is indeed a quasi-Hamiltonian space for the diagonal action of $U(n)$, with momentum map the product $\mu(u_1, \ldots, u_l) = u_1 \ldots u_l$. For a product of three factors, one can write down the fusion product form explicitly in the following way :

Corollary 3.3. *The fusion product form on $M_1 \times M_2 \times M_3$ is the 2-form*

$$\omega = (\omega_1 \oplus \omega_2 \oplus \omega_3) + \big((\mu_1^*\theta^L \wedge \mu_2^*\theta^R) \oplus (\mu_2^*\theta^L \wedge \mu_3^*\theta^R) \oplus (\mu_1^*\theta^L \wedge (\mu_2^*Ad).\mu_3^*\theta^R)\big)$$

Proof. To obtain the above expression, one applies theorem 3.2 successively to $M_1 \times M_2$ and to $(M_1 \times M_2) \times M_3$. One can then also check that the fusion product is associative, as shown in [AMM98] :

$$\begin{aligned} \omega &= \Big(\big((\omega_1 \oplus \omega_2) + (\mu_1^*\theta^L \wedge \mu_2^*\theta^R)\big) \oplus \omega_3\Big) + \big((\mu_1 \cdot \mu_2)^*\theta^L \wedge \mu_3^*\theta^R\big) \\ &= \Big(\omega_1 \oplus \big((\omega_2 \oplus \omega_3) + (\mu_2^*\theta^L \wedge \mu_3^*\theta^R)\big)\Big) + \big(\mu_1^*\theta^L \wedge (\mu_2 \cdot \mu_3)^*\theta^R\big) \end{aligned}$$

□

4 The symplectic structure on the moduli space of unitary representations of surface groups

The theory of quasi-Hamiltonian spaces provides a very nice description of the symplectic structure of moduli spaces of unitary representations of surface groups. We refer to [AMM98] for the general description of these moduli spaces as quasi-Hamiltonian quotients and we will now concentrate on the space of representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\}) = < g_1, g_2, \ldots, g_l \mid g_1 g_2 \ldots g_l = 1 >$. Giving such a representation with prescribed conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l$ of generators amounts to giving l unitary matrices $u_1, \ldots, u_l$ such that $u_j \in \mathcal{C}_j$ and $u_1 \ldots u_l = 1$. But we know from section 3 that this amounts to saying that $(u_1, \ldots, u_l) \in \mu^{-1}(\{1\})$ where

$$\begin{array}{cccc} \mu : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U(n) \\ & (u_1, \ldots, u_l) & \longmapsto & u_1 \ldots u_l \end{array}$$

is the momentum map of the diagonal $U(n)$-action. The moduli space of unitary representations is then $\mathcal{M}_\mathcal{C} = \mathrm{Hom}_\mathcal{C}(\pi, U(n))/U(n) = \mu^{-1}(\{1\})/U(n)$, which is the symplectic manifold obtained from $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ by *quasi-Hamiltonian reduction*, a procedure which we now recall, stating theorem 5.1 from [AMM98] in a particular case to apply it more directly to our setting.

Theorem 4.1 (Symplectic reduction of quasi-Hamiltonian manifolds)**.** [AMM98] *Let (M, ω) be a quasi-Hamiltonian U-space with momentum map $\mu : M \to U$. Let $i : \mu^{-1}(\{1\}) \hookrightarrow M$ be the inclusion of the level set $\mu^{-1}(\{1\})$ in M and let $p : \mu^{-1}(\{1\}) \to \mu^{-1}(\{1\})/U$ be the projection on the orbit space. Assume that U acts freely on $\mu^{-1}(\{1\})$. Then there exists a unique symplectic form ω^{red} on the reduced space $M^{red} := \mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ on $\mu^{-1}(\{1\})$.*

The proof consists in showing that $i^*\omega$ is basic with respect to the fibration p and then verifying that the corresponding form ω^{red} on $\mu^{-1}(\{1\})/U$ is indeed a symplectic form. In virtue of the above theorem, describing the symplectic structure of $\mathcal{M}_{\mathcal{C}} = \mu^{-1}(\{1\})/U$ amounts to giving the 2-form defining the quasi-Hamiltonian structure on the product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. We now give the description of this 2-form in the case where $l = 3$.

Proposition 4.2. *Let $(u_1, u_2, u_3) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$. Take $X_j, Y_j \in \mathfrak{u}$ and write $[X_j], [Y_j] \in T_{u_j}\mathcal{C}_j$ for the corresponding tangent vectors (see section 3). The 2-form ω making $\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ a quasi-Hamiltonian space with momentum map $\mu(u_1, u_2, u_3) = u_1u_2u_3$ is given by :*

$$\begin{aligned}
\omega_u([X],[Y]) \;=\; \frac{1}{2}\big(& (Ad\, u_1.X_1 \,|\, Y_1) - (Ad\, u_1.Y_1 \,|\, X_1) + (Ad\, u_2.X_2 \,|\, Y_2) - (Ad\, u_2.Y_2 \,|\, X_2) \\
& + (Ad\, u_3.X_3 \,|\, Y_3) - (Ad\, u_3.Y_3 \,|\, X_3) + (Ad\, u_1^{-1}.X_1 - X_1 \,|\, Y_2 - Ad\, u_2.Y_2) \\
& - (Ad\, u_1^{-1}.Y_1 - Y_1 \,|\, X_2 - Ad\, u_2.X_2) + (Ad\, u_2^{-1}.X_2 - X_2 \,|\, Y_3 - Ad\, u_3.Y_3) \\
& - (Ad\, u_2^{-1}.Y_2 - Y_2 \,|\, X_3 - Ad\, u_3.X_3) + (Ad\, u_1^{-1}.X_1 - X_1 \,|\, Ad\, u_2.Y_3 - Ad\,(u_2u_3).Y_3) \\
& - (Ad\, u_1^{-1}.Y_1 - Y_1 \,|\, Ad\, u_2.X_3 - Ad\,(u_2u_3).X_3)\big)
\end{aligned}$$

The above expression is obtained by applying corollary 3.3. Observe that the fusion product 2-form on $\mathcal{C}_1 \times \mathcal{C}_2$ consists exactly of terms of the above expression which do not contain vectors X_3 or Y_3. See also remark 5.3 in [Tre02] for expressions of fusion product forms on products of conjugacy classes.

5 Lagrangian submanifolds of a quasi-Hamiltonian quotient

The purpose of this section is to give a way of finding Lagrangian submanifolds in a symplectic manifold obtained by reduction from a quasi-Hamiltonian space. It mainly consists in carrying over a standard procedure for usual symplectic quotients to the quasi-Hamiltonian setting. To that end, we recall the following result from [OS00] (proposition 2.3), which concerns Hamiltonian spaces. Let U be a compact connected Lie group acting on a symplectic manifold (M, ω) in a Hamiltonian fashion with momentum map $\Phi : M \to \mathfrak{u}^*$. Let τ denote an involutive automorphism of U and still denote by τ the involution

$$\begin{array}{rccc} (T_1\,\tau)^* : & \mathfrak{u}^* & \longrightarrow & \mathfrak{u}^* \\ & \lambda & \longmapsto & \lambda \circ T_1\,\tau \end{array}$$

that it induces on the dual $\mathfrak{u}^*$ of the Lie algebra $\mathfrak{u} = T_1\,U$ of U. Let β be an anti-symplectic involution on M (that is, such that $\beta^*\omega = -\omega$ and $\beta^2 = Id_M$). In the above notations, β is said to be *compatible with the action* of U if $\forall u \in U$, $\forall x \in M$, $\beta(u.x) = \tau(u).\beta(x)$ and β is said to be *compatible with the momentum map* $\Phi : M \to \mathfrak{u}^*$ if $\forall x \in M$, $\Phi \circ \beta(x) = -\tau \circ \Phi(x)$.

Proposition 5.1. [OS00] *If $M^\beta := Fix(\beta)$ is non-empty, it is a Lagrangian submanifold of M, stable by the action of the subgroup $U^\tau := Fix(\tau)$ of U.*

O'Shea and Sjamaar then proceed to studying the reduced space $M^{red} = \Phi^{-1}(\{0\})/U$, on which β induces an involution $\hat{\beta}$. To obtain analogous results for a symplectic manifold $M^{red} = \mu^{-1}(\{1\})/U$ obtained by reduction of a *quasi*-Hamiltonian space M, we wish to define an involution β on M such that β induces an anti-symplectic involution $\hat{\beta}$ on M^{red}. This is done the following way :

Theorem 5.2. *Let U be a compact connected Lie group and let (M, ω) be a quasi-Hamiltonian U-space with momentum map $\mu : M \to U$. Let τ be an involutive automorphism of U, denote by τ^- the involution defined on U by $\tau^-(u) = \tau(u^{-1})$ and let β be an involution on M such that :*

(i) $\forall u \in U$, $\forall x \in M$, $\beta(u.x) = \tau(u).\beta(x)$ (β is said to be *compatible with the action* of U)

(ii) $\forall x \in M$, $\mu \circ \beta(x) = \tau^- \circ \mu(x)$ (β is said to be *compatible with the momentum map* $\mu : M \to U$)

(iii) $\beta^*\omega = -\omega$ (β reverses the 2-form ω)

then β induces an anti-symplectic involution $\hat{\beta}$ on the reduced space $M^{red} := \mu^{-1}(\{1\})/U$. If $\hat{\beta}$ has fixed points, then $Fix(\hat{\beta})$ is a Lagrangian submanifold of M^{red}.

Remark. See the end of this section for comments on the condition $Fix(\hat{\beta}) \neq \emptyset$.

Proof. Compatibility with the momentum map (condition (ii)) shows that β maps $\mu^{-1}(\{1\})$ into $\mu^{-1}(\{1\})$ (since $\tau^-(1) = 1$). Compatibility with the action (condition (i)) then shows that $\beta(u.x)$ and $\beta(x)$ lie in the same U-orbit, so that we have a map

$$\begin{array}{rccc} \hat{\beta} : & \mu^{-1}(\{1\})/U & \longrightarrow & \mu^{-1}(\{1\})/U \\ & U.x & \longmapsto & U.\beta(x) \end{array}$$

We know from quasi-Hamiltonian reduction (see theorem 4.1) that there exists a unique symplectic form ω^{red} on $M^{red} = \mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ where $i : \mu^{-1}(\{1\}) \hookrightarrow M$ and $p : \mu^{-1}(\{1\}) \to M^{red}$. To show that $\hat{\beta}^*\omega^{red} = -\omega^{red}$, let us first prove that $i^*(\beta^*\omega)$ is basic with respect to the fibration p. Then there will exist a unique 2-form γ on M^{red} such that $p^*\gamma = i^*(\beta^*\omega)$. Since both $\gamma = -\omega^{red}$ and $\gamma = \hat{\beta}^*\omega^{red}$ satisfy this condition, they have to be equal. The last part of the theorem then follows from proposition 5.1, as the fixed-point set of an anti-symplectic involution, if it is non-empty, is always a Lagrangian submanifold. Let us now write this explicitly.
Verifying that $i^*(\beta^*\omega)$ is basic is easy since $\beta^*\omega = -\omega$ and $i^*\omega$ is basic (see [AMM98]) but it is actually true without this assumption so we prove it for β satisfying only conditions (i) and (ii) above. We have to show that $i^*(\beta^*\omega)$ is U-invariant and that for every $X \in \mathfrak{u} = Lie(U)$, we have $\iota_{X^\sharp}(i^*(\beta^*\omega)) = 0$, where $X^\sharp$ is as usual the fundamental vector field $X^\sharp_x = \frac{d}{dt}|_{t=0}(\exp(tX).x)$ (for any $x \in M$) associated to $X \in \mathfrak{u}$ by the action of U on M. Let $u \in U$ and denote by φ_u the corresponding diffeomorphism of M. The map μ being equivariant φ_u sends $\mu^{-1}(\{1\})$ into itself, hence $i \circ \varphi_u = \varphi_u \circ i$ on $\mu^{-1}(\{1\})$. Furthermore, compatibility with the action yields $\beta \circ \varphi_u = \varphi_{\tau(u)} \circ \beta$. We then have, on $\mu^{-1}(\{1\})$,

$$\begin{aligned} \varphi_u^*(i^*(\beta^*\omega)) &= (\beta \circ i \circ \varphi_u)^*\omega \\ &= (\varphi_{\tau(u)} \circ \beta \circ i)^*\omega \\ &= i^*(\beta^*(\underbrace{\varphi^*_{\tau(u)}\omega}_{=\omega})) \end{aligned}$$

where the very last equality follows from the U-invariance of ω. Further, let $X \in \mathfrak{u}$. Since β is compatible with the action, one has $\beta(\exp(tX).x) = \tau(\exp(tX)).\beta(x) = \exp(t\tau(X)).\beta(x)$ (where we still denote by τ the involution $T_1\,\tau$ on $\mathfrak{u} = T_1\,U$), hence $T_x\beta.X^\sharp_x = (\tau(X))^\sharp_{\beta(x)}$, hence $\iota_{X^\sharp}(\beta^*\omega) = \beta^*(\iota_{(\tau(X))^\sharp}\omega)$. Since $\iota_{X^\sharp}(i^*(\beta^*\omega)) = i^*(\iota_{X^\sharp}(\beta^*\omega))$, we can compute, using the fact that β is compatible with μ,

$$\begin{aligned} \iota_{X^\sharp}(\beta^*\omega) &= \beta^*(\iota_{(\tau(X))^\sharp}\omega) \\ &= \beta^*\Big(\frac{1}{2}\,\mu^*\big(\theta^L + \theta^R \,|\, \tau(X)\big)\Big) \\ &= \frac{1}{2}\,(\mu \circ \beta)^*\big(\theta^L + \theta^R \,|\, \tau(X)\big) \\ &= \frac{1}{2}\,(\tau^- \circ \mu)^*\big(\theta^L + \theta^R \,|\, \tau(X)\big) \\ &= \frac{1}{2}\,\mu^*\Big((\tau^-)^*\big(\theta^L + \theta^R \,|\, \tau(X)\big)\Big) \end{aligned}$$

hence $i^*(\iota_{X^\sharp}(\beta^*\omega)) = \frac{1}{2}\,i^* \circ \mu^*(\ldots) = \frac{1}{2}\,(\mu \circ i)^*(\ldots)$. But $\mu \circ i : \mu^{-1}(\{1\}) \to U$ is a constant map, therefore $T(\mu \circ i)$ and consequently $(\mu \circ i)^*$ are zero, which completes the proof that $i^*(\beta^*\omega)$ is basic. Finally, let us show that $p^*(\hat{\beta}\omega^{red}) = i^*(\beta^*\omega) = p^*(-\omega^{red})$ (this is where we really use $\beta^*\omega = -\omega$). We have, on $\mu^{-1}(\{1\})$, $p^*(\hat{\beta}^*\omega^{red}) = (\hat{\beta} \circ p)^*\omega^{red} = (p \circ \beta)^*w^{red} = \beta^*(p^*\omega^{red}) = \beta^*(i^*\omega) = (i \circ \beta)^*\omega = (\beta \circ i)^*\omega = i^*(\beta^*\omega) = i^*(-\omega) = -i^*\omega = -p^*\omega^{red} = p^*(-\omega^{red})$. This completes the proof, as indicated above. $\square$

In the next section, we will give an example of a map β satisfying the hypotheses of theorem 5.2. In the case we will then be dealing with, it will be important to us that the map τ under consideration be actually an isometry for the Euclidean product $(\,.\,|\,.\,)$ on $\mathfrak{u}$ (recall that this scalar product is part of the initial data to define quasi-Hamiltonian U-spaces). So far, we did not need that hypothesis. Before ending this section, we would like to say that, in fact, if β satisfies the conditions of theorem 5.2 and has fixed points then $\hat{\beta}$ necessarily has fixed points. Indeed, observe first that $Fix(\hat{\beta}) \neq \emptyset$ if and only if $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$. We then have the following result, which is a convexity result concerning momentum maps in the quasi-Hamiltonian framework and which is adapted from (a special case of) the convexity theorem of O'Shea and Sjamaar (see [OS00]) :

Theorem 5.3. [Sch05] *Let β be an involution defined on a quasi-Hamiltonian (U,τ)-space $(M, \omega, \mu : M \to U)$ such that β is compatible with the action and the momentum map and such that $\beta^*\omega = -\omega$. Assume that $Fix(\beta) \neq \emptyset$ and that there exists a maximal torus T of U which is fixed pointwise by τ^- and let $\mathcal{W} \subset \mathfrak{t} = Lie(T)$ be a Weyl alcove. Then $\mu(M^\beta) \cap \exp \mathcal{W} = \mu(M) \cap \exp \mathcal{W}$.*

The proof of this theorem is too long to be presented here, all the more so as it calls for techniques which are very different from the ones we have made use of so far. A proof of this result is available in [Sch05] and will appear in a forthcoming paper ([Sch]). The fact that $Fix(\hat{\beta}) \neq \emptyset$ is then a corollary of this theorem :

Corollary 5.4. [Sch05] *If $Fix(\beta) \neq \emptyset$ and $\mu^{-1}(\{1\}) \neq \emptyset$ then $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$, in which case the involution $\hat{\beta}$ induced by β on $\mu^{-1}(\{1\})/U$ satisfies $Fix(\hat{\beta}) \neq \emptyset$.*

Proof of the corollary. Since $Fix(\beta) \neq \emptyset$, the above claim applies. Since $\mu^{-1}(\{1\}) \neq \emptyset$ and $1 \in \exp \mathcal{W}$, we then have $1 \in \mu(M) \cap \exp \mathcal{W} = \mu(M^\beta) \cap \exp \mathcal{W}$, which means that $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$, which in turn is equivalent to $Fix(\hat{\beta}) \neq \emptyset$. □

We will not use theorem 5.3, nor its corollary, in the following.

6 Lagrangian representations as fixed point set of an involution

In this section, we will state our main result, which is the characterization of a σ_0-Lagrangian representations of $\pi = \pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ as the elements of the fixed point set of an involution β defined on the product of l conjugacy classes of the unitary group (satisfying the condition $\exists\ (u_1, \dots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \dots u_l = 1$). Using theorem 5.2 proved in the preceding section, we will deduce that the set of (equivalence classes of) Lagrangian representations is a Lagrangian submanifold of the moduli space $\mathcal{M}_\mathcal{C} = \mathrm{Hom}_\mathcal{C}(\pi, U(n))/U(n)$. The first five subsections explain how these results (in particular the involution β) were obtained but may be skipped if one wants to go straight to the actual theorems (whose proofs may also be read without knowledge of the previous subsections).

6.1 The infinitesimal picture and the momentum map approach

Let us recall our problem : given l unitary matrices $u_1, \dots, u_l \in U(n)$ satisfying $u_j \in \mathcal{C}_j$ and $u_1 \dots u_j = 1$, do there exist l Lagrangian subspaces $L_1, \dots, L_l$ of $\mathbb{C}^n$ such that $\sigma_j\sigma_{j+1} = u_j$ (where σ_j is the Lagrangian involution associated with L_j and $\sigma_{l+1} = \sigma_1$) ? As was recalled in section 2, the condition $\sigma_j\sigma_{j+1} \in \mathcal{C}_j$, which lies on the spectrum of the unitary map $\sigma_j\sigma_{j+1}$, can be interpreted geometrically as the measure of an angle between Lagrangian subspaces. The Lagrangian problem above can therefore be thought of as a configuration problem in the Lagrangian Grassmannian $\mathcal{L}(n)$ of $\mathbb{C}^n$: given eigenvalues $\exp(i\lambda_j)$, $\lambda_j \in \mathbb{R}^n$, do there exist l Lagrangian subspaces $L_1, \dots, L_l$ such that measure$(L_j, L_{j+1}) = \exp(i\lambda_j)$? Under this geometrical form, the Lagrangian problem is slightly more general than our original representation theory problem. It is very much linked to the unitary problem studied for instance in [JW92, Gal97, Bis98, AW98, KM99, Bis99, Bel01], which is the following : given $\lambda_j \in \mathbb{R}^n$, do there exist l unitary matrices $u_1, \dots, u_l$ satisfying $\mathrm{Spec}\, u_j = \exp(i\lambda_j)$ and $u_1 \dots u_l = 1$? In fact, a solution $(L_1, \dots, L_l)$ to the Lagrangian problem (second version) provides a solution $u_j = \sigma_j\sigma_{j+1}$ to the unitary problem. As was shown in [FMS04], it is possible to use this approach to give an interpretation of the inequalities found by Biswas in [Bis98] (which are necessary and sufficient conditions on the λ_j for the

unitary problem to have a solution in the case $n = 2$ and $l = 3$) in terms of the inequalities satisfied by the angles of a spherical triangle.
The fact that the unitary problem admits a symplectic description was our first motivation to study the Lagrangian problem from a symplectic point of view. The second motivation is derived from the above-given geometrical formulation of the problem. To better understand this, let us try and formulate an infinitesimal version of the Lagrangian problem. Take three Lagrangian subspaces L_1, L_2, L_3 close enough so that we can think of these points in $\mathcal{L}(n)$ as tangent vectors to $\mathcal{L}(n)$ at some point L_0 representing the center of mass of L_1, L_2, L_3. Tangent vectors to the Lagrangian Grassmannian are identified with real symmetric matrices S_1, S_2, S_3 and the center of mass condition then turns into $S_1 + S_2 + S_3 = 0$. It seems reasonable in this context to translate the angle condition $\mathrm{mes}(L_j, L_{j+1}) = \exp(i\lambda_j)$ (that is, $\mathrm{Spec}\,\sigma_j\sigma_{j+1} = \exp(i\lambda_j)$) into the spectral condition $\mathrm{Spec}\,S_j = \lambda_j \in \mathbb{R}^n$. We then recognize a real version (replacing complex Hermitian matrices with real symmetric ones) of a famous problem in mathematics (see [Ful98] for a review of this problem and those related to it) : given $\lambda_j \in \mathbb{R}^n$, do there exist Hermitian matrices H_1, H_2, H_3 such that $\mathrm{Spec}\,H_j = \lambda_j$ and $H_1 + H_2 + H_3 = 0$? In fact, these last two problems are equivalent (meaning that, for given $(\lambda_j)_j$, one of them has a solution if and only if the other one does) and this can be shown in a purely symplectic framework (see [AMW01]) using momentum maps to translate the condition $H_1 + H_2 + H_3 = 0$ into $(H_1, H_2, H_3) \in \mu^{-1}(\{0\})$. Therefrom, it seems promising to try to think of the Lagrangian problem as a real version, in a sense that will be made precise in subsection 6.5, of the unitary problem (since a solution to the Lagrangian problem provides an obvious solution to the unitary problem).

6.2 The centered Lagrangian problem

As a consequence of the above infinitesimal picture, we replace our Lagrangian problem with a centered problem, meaning that instead of measuring the angles (L_j, L_{j+1}), we measure the angles (L_0, L_j) where L_0 is the horizontal Lagrangian $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$ (playing the role of an origin in $\mathcal{L}(n)$). Recall from section 2 (theorem 2.2 and proposition 2.3) that this angle is measured by the spectrum of $\sigma_0\sigma_j = u_j^t u_j$, where u_j is any unitary map sending L_j to L_0. We then ask the following question : given l conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_l \subset U(n)$, does there exist l unitary matrices $u_1, \dots, u_l$ such that $u_j^t u_j \in \mathcal{C}_j$ and $u_1 \dots u_l = 1$? The main observation here is then to see that the condition $\mathrm{Spec}\,u^t u = \exp(i\lambda)$, for some $\lambda \in \mathbb{R}^n$ (that is, $u^t u$ lies in some fixed conjugacy class of $U(n)$) means that u belongs to a fixed orbit of the action of $O(n) \times O(n)$ on $U(n)$ given by $(k_1, k_2).u = k_1 u k_2^{-1}$, as is shown by the following elementary result :

Lemma 6.1. *For any $u, v \in U(n)$, $\mathrm{Spec}\,u^t u = \mathrm{Spec}\,v^t v$ if and only if there exist $(k_1, k_2) \in O(n) \times O(n)$ such that $v = k_1 u k_2^{-1}$.*

Since we think of the above problem as a real version of some complex problem, we now wish to find this complex version, which is done by abstracting a bit our situation to put it in the appropriate framework.

6.3 Complexification of the centered Lagrangian problem

Let us formulate the centered Lagrangian problem in greater generality. For everything regarding the theory of Lie groups and symmetric spaces, especially regarding real forms and duality, we refer to [Hel01]. We start with a real Lie group H. Let $G = H^{\mathbb{C}}$ be its complexification and let τ be the Cartan involution on G associated to H, that is to say, the involutive automorphism of G such that $Fix(\tau) = H$. Let U be a compact connected real form of G such that the associated Cartan involution θ satisfies $\theta\tau = \tau\theta$. Such a compact group always exists and is stable under τ. The group H is then stable under θ and U and H are said to be dual to each other (when H is non-compact, they indeed define dual symmetric spaces $U/(U \cap H)$ and $H/(U \cap H)$). Moreover, because of the fact that τ is the Cartan involution associated to the non-compact dual H of U, the compact connected group U contains a maximal torus T such that $\tau(t) = t^{-1}$ for all $t \in T$ ((U, τ) is said to be *of maximal rank*, see [Loo69] pp. 72-74 and 79-81). Let $K := U \cap H$. Then $K = Fix(\tau|_U) \subset U$ and $K = Fix(\theta|_H) \subset H$. We consider the action of $K \times K$ on U given by $(k_1, k_2).u = k_1 u k_2^{-1}$. Notice that if H is compact to start with, then $K = U = H$ and the above action defines congruence in U. As for us though, we are interested in the case where H is non-compact.

For $H = Gl(n,\mathbb{R})$, we have $U = U(n)$ and $K = O(n)$, and we are then led to ask the following question, which is a generalized version of our centered Lagrangian problem : given l orbits $\mathcal{D}_1, \dots, \mathcal{D}_l$ of the action of $K \times K$ on U, do there exist $u_1, \dots, u_l \in U$ such that $u_j \in \mathcal{D}_j$ and $u_1 \dots u_l = 1$? Observe that, as a generalization of lemma 6.1, these orbits are in one-to-one correspondence with the conjugacy classes in U of elements of the form $\tau^-(u)u$, where u is any element in a given orbit $\mathcal{D}$ and $\tau^-(u) = \tau(u^{-1})$. Indeed, this is a corollary of theorem 8.6 in chapter VII (p. 323) of [Hel01], which we now state under a form most convenient for our purposes :

Theorem 6.2 (Cartan decomposition of U)**.** [Hel01] *Let U be a compact connected Lie group and let τ be an involutive automorphism of U. Let $K = Fix(\tau) \subset U$. Still denote by τ the involutive automorphism $T_1\tau : \mathfrak{u} = T_1 U \to \mathfrak{u}$. Then there exist a subset $\mathfrak{q}_0 \subset \mathfrak{u}$ such that :*

(i) $\forall X \in \mathfrak{q}_0$, $\tau(X) = -X$

(ii) each $u \in U$ can be written $u = k_1 \exp(X) k_2^{-1}$ for some $k_1, k_2 \in K$ and for a unique $X \in \mathfrak{q}_0$.

Further, if $X, Y \in \mathfrak{u}$ satisfy $\tau(X) = -X$ and $\tau(Y) = -Y$, and if there exist $u \in U$ such that $Ad\, u.X = Y$, then there exists $k \in K \subset U$ such that $Ad_U\, k.X = Y$.

Corollary 6.3. *Let $u, v \in U$. Then there exist $(k_1, k_2) \in K \times K$ such that $v = k_1 u k_2^{-1}$ if and only if $\tau^-(v)v$ and $\tau^-(u)u$ lie in a same conjugacy class in U.*

Proof. The first implication is obvious. Conversely, write $u = k_1 \exp(X) k_2^{-1}$ as in the above theorem. Then $\tau^-(u) = k_2 \exp(X) k_1^{-1}$ (since $\tau(k_j) = k_j$ in U and $\tau^-(X) = -\tau(X) = X$ in $\mathfrak{u}$) and therefore $\tau^-(u)u = k_2 \exp(2X) k_2^{-1}$. Likewise, we can write $v = k_1' \exp(Y)(k_2')^{-1}$ and therefore $\tau^-(v)v = k_2' \exp(2Y)(k_2')^{-1}$. Since $\tau^-(v)v$ is conjugate to $\tau^-(u)u$ in U, we see that $2Y$ is $Ad_U\, U$-conjugate to $2X+H$ where $H \in \mathfrak{u}$ satisfies $\exp(H) = 1$. We then necessarily have $\tau(H) = -H$. By using theorem 8.5 in chapter VII of [Hel01], we can then write $H = 2Z$ with $Z \in \mathfrak{u}$ satisfying $\tau(Z) = -Z$ and $\exp(Z) \in K$. Then Y is $Ad_U\, U$-conjugate to $(X+Z)$. But $\tau(Y) = -Y$ and $\tau(X+Z) = -(X+Z)$, therefore, by the above theorem, Y and $(X+Z)$ are $Ad_U\, K$-conjugate. Then we have $Y = k.(X+Z).k^{-1}$ in $\mathfrak{u}$ for some $k \in K$, so that $v = k_1' \exp(Y)(k_2')^{-1} = k_1' k \exp(X) \exp(Z) k^{-1} (k_2')^{-1} = \underbrace{k_1' k k_1^{-1}}_{\in K} \underbrace{(k_1 \exp(X) k_2^{-1})}_{=u} \underbrace{k_2 \exp(Z) k^{-1} (k_2')^{-1}}_{\in K}$ □

Now, to find the complex version of our problem, we apply the same construction to the complex Lie group $G = H^{\mathbb{C}}$ viewed as a real Lie group. Then $G^{\mathbb{C}} = G \times G$ is the complexification of G and $\widetilde{U} = U \times U \subset G \times G = G^{\mathbb{C}}$ is a compact real form of $G^{\mathbb{C}}$. Its non-compact dual (which needs to be a subgroup of $G^{\mathbb{C}} = G \times G$) is then $\widetilde{H} = \{(g, \theta(g)) \,:\, g \in G\} \simeq G$ where θ is the Cartan involution associated to U. The Cartan involution associated to $\widetilde{U}$ is $\widetilde{\theta} : (g_1, g_2) \in \widetilde{G} \mapsto (\theta(g_1), \theta(g_2))$ and the Cartan involution associated to $\widetilde{H}$ is $\widetilde{\tau} : (g_1, g_2) \mapsto (\theta(g_2), \theta(g_1))$. Indeed, $Fix(\widetilde{\theta}) = \widetilde{U}$, $Fix(\widetilde{\tau}) = \widetilde{H}$ and $\widetilde{\theta}\widetilde{\tau} = \widetilde{\tau}\widetilde{\theta}$. Then we define :

$$\begin{aligned} \widetilde{K} &:= \widetilde{U} \cap \widetilde{H} \\ &= \Big\{ \big(g, \theta(g)\big) \mid \widetilde{\theta}\big(g, \theta(g)\big) = \big(g, \theta(g)\big) \Big\} \\ &= \Big\{ \big(g, \theta(g)\big) \mid \theta(g) = g \Big\} \\ &= \big\{ (u, u) \,:\, u \in U \big\} \end{aligned}$$

(we will also use the notation $U_\Delta := \{(u,u) \,:\, u \in U\}$ instead of $\widetilde{K}$) and we consider the action of $\widetilde{K} \times \widetilde{K} = U_\Delta \times U_\Delta$ on $\widetilde{U} = U \times U$ defined by :

$$\big((u_1, u_1), (u_2, u_2)\big).(u, v) = (u_1 u u_2^{-1}, u_1 v u_2^{-1})$$

Our problem then states : given l orbits $\widetilde{\mathcal{D}}_1, \dots, \widetilde{\mathcal{D}}_l$ of the above action, do there exist l pairs $(u_1, v_1), \dots, (u_l, v_l) \in \widetilde{U} = U \times U$ such that $(u_j, v_j) \in \widetilde{\mathcal{D}}_j$ and $(u_1, v_1). \dots .(u_l, v_l) = 1$, that is, $u_1 \dots u_l = 1$ and $v_1 \dots v_l = 1$?

Before passing on to the next subsection, we wish to point out that if we consider the action of $K \times K$

not on U but rather on its dual H, then the orbits of this action are characterized by the singular values ($\mathrm{Sing}\, h = \mathrm{Spec}\,(\theta^-(h)h)$ where $h \in H$ and $\theta^-(h) = \theta(h^{-1})$) of any of their elements. As a consequence, our (centered) Lagrangian problem appears as a compact version of the (real) Thompson problem, replacing θ with τ in the latter to formulate the former (see [AMW01] and [EL05] for a proof of the Thompson conjecture in the real case).

6.4 Equivalence between the complexification of the centered Lagrangian problem and the unitary problem

From now on, the initial data is a compact connected Lie group U. For such a group, we can formulate : (i) the centered Lagrangian problem (concerning $K \times K$-orbits in U, where $K = U \cap H$ with H the non-compact dual of U) (ii) a complex version of this (concerning $U_\Delta \times U_\Delta$-orbits in $U \times U$) (iii) the unitary problem (concerning conjugacy classes in U). To show the equivalence of these last two problems, the main observation to make is the following one :

Lemma 6.4. *The map*

$$\begin{array}{cccc} \eta : & U \times U & \longrightarrow & U \\ & (u,v) & \longmapsto & u^{-1}v \end{array}$$

sends a $U_\Delta \times U_\Delta$-orbit $\widetilde{\mathcal{D}}$ in $U \times U$ onto a conjugacy class $\mathcal{C}$ in U.

Proof. If $(u,v) = (u_1 u_0 u_2^{-1}, u_1 v_0 u_2^{-1})$ then $u^{-1}v = u_2(u_0^{-1}v_0)u_2^{-1}$ so $\eta(\widetilde{\mathcal{D}}) \subset \mathcal{C}$ where $\mathcal{C}$ is a conjugacy class in U. Further, let $(u,v) \in \widetilde{\mathcal{D}}$ and take any $w \in \mathcal{C}$. Then $\exists\, u_2 \mid w = u_2 u^{-1} v u_2^{-1}$ so that $(1,w) = (1, u_2 u^{-1} v u_2^{-1}) = \underbrace{((u_2u^{-1}, u_2u^{-1}), (u_2,u_2))}_{\in U_\Delta \times U_\Delta} . \underbrace{(u,v)}_{\in \widetilde{\mathcal{D}}}$, hence $(1,w) \in \widetilde{\mathcal{D}}$, therefore $w = \eta(1,w) \in \eta(\widetilde{\mathcal{D}})$. ☐

It is nice to observe that this map η may be used to show that a compact connected Lie group U is a symmetric space $U = (U \times U)/U_\Delta$ (see [Hel01]). Coming back to the matter at hand, we have the following result, that says that the complexification of the centered Lagrangian problem has a solution if and only if the unitary problem has a solution (that is, these two problems are equivalent).

Proposition 6.5. *Let $\widetilde{\mathcal{D}}_1, \dots, \widetilde{\mathcal{D}}_l$ be l orbits of $U_\Delta \times U_\Delta$ in $U \times U$ and let $\mathcal{C}_1, \dots, \mathcal{C}_l \subset U$ be the corresponding conjugacy classes : $\mathcal{C}_j = \eta(\widetilde{\mathcal{D}}_j)$. Then there exists $((u_1,v_1), \dots, (u_l,v_l)) \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ such that $u_1 \dots u_l = 1$ and $v_1 \dots v_l = 1$ if and only if there exist $(w_1, \dots, w_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $w_1 \dots w_l = 1$.*

Proof. Setting $(u_j,v_j) := (1,w_j)$ for every j, we see that the second condition implies the first one. Conversely, assume that $((u_1,v_1), \dots, (u_l,v_l)) \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ satisfy $u_1 \dots u_l = 1$ and $v_1 \dots v_l = 1$. Then $(u_1 \dots u_l)^{-1} v_1 \dots v_l = 1$, hence $u_l^{-1} \dots u_2^{-1}(u_1^{-1}v_1)v_2 \dots v_l = 1$, with $u_1^{-1}v_1 \in \mathcal{C}_1$. Hence :

$$\underbrace{u_l^{-1} \dots u_2^{-1}(u_1^{-1}v_1)u_2 \dots u_l}_{\in \mathcal{C}_1} \underbrace{u_l^{-1} \dots u_3^{-1}(u_2^{-1}v_2)u_3 \dots u_l}_{\in \mathcal{C}_2} \dots \underbrace{(u_l^{-1}v_l)}_{\in \mathcal{C}_l} = 1$$

Setting $w_1 = u_l^{-1} \dots u_2^{-1}(u_1^{-1}v_1)u_2 \dots u_l$, $w_2 = u_l^{-1} \dots u_3^{-1}(u_2^{-1}v_2)u_3 \dots u_l$, $\dots$, and $w_l = u_l^{-1}v_l$ then gives a solution $(w_1, \dots, w_l)$ to the unitary problem. ☐

In analogy with a result on double cosets of $U(n)$ in $Gl(n,\mathbb{C})$ (which are characterized by the singular values $\mathrm{Sing}\, g = \mathrm{Spec}\,(\theta^-(g)g)$ of any of their elements) and dressing orbits of $U(n)$ in $(U(n))^* = \{b \in Gl(n,\mathbb{C}) \mid b \text{ is upper triangular and } diag(b) \in (\mathbb{R}^{*+})^n\}$ appearing in [AMW01], the above proposition can be formulated more precisely in the following way. Consider the action of U^l on $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ given by

$$(\varphi_1, \dots, \varphi_l).\big((u_1,v_1), \dots, (u_l,v_l)\big) = \big(\underbrace{\varphi_1.(u_1,v_1).\varphi_2^{-1}}_{=(\varphi_1 u_1 \varphi_2^{-1}, \varphi_1 v_1 \varphi_2^{-1})}, \varphi_2.(u_2,v_2).\varphi_3^{-1}, \dots, \varphi_l.(u_l,v_l).\varphi_l^{-1}\big)$$

and the diagonal action of $U(n)$ on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$: $\varphi.(w_1, \dots, w_l) = (\varphi w_1 \varphi^{-1}, \dots, \varphi w_l \varphi^{-1})$. These actions respectively preserve the relations $u_1 \dots u_l = v_1 \dots v_l = 1$ and $\omega_1 \dots \omega_l = 1$. We may then define the orbit spaces

$$\mathcal{M}_{\widetilde{\mathcal{D}}} := \left\{ \big((u_j, v_j)\big)_j \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \mid u_1 \dots u_l = v_1 \dots v_l = 1 \right\} \Big/ U^l$$

and

$$\mathcal{M}_{\mathcal{C}} = \left\{ (w_j)_j \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid w_1 \dots w_l = 1 \right\} \Big/ U$$

And we then have

Proposition 6.6. *The map*

$$\begin{array}{rccc} \eta^{(l)} : & \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & ((u_1, v_1), \dots, (u_l, v_l)) & \longmapsto & (u_l^{-1} \dots u_2^{-1}(u_1^{-1} v_1) u_2 \dots u_l, \dots, u_l^{-1}(u_{l-1}^{-1} v_{l-1}) u_l, u_l^{-1} v_l) \end{array}$$

induces a homeomorphism $\mathcal{M}_{\widetilde{\mathcal{D}}} \simeq \mathcal{M}_{\mathcal{C}}$.

We will not use this result in the following so we do not give the proof, which is but a consequence of the above. We point out the fact that this result reinforces the analogy between our problem and the Thompson problem. We now wish to explain in what precise sense the Lagrangian problem is a real version of these two equivalent problems.

6.5 Solutions to real problems as fixed point sets of involutions

The important idea of thinking of possible solutions to a real problem as the fixed point set of an involution defined on the set of possible solutions to a corresponding complex problem is well-established in symplectic geometry and is due to Michael Atiyah and Alan Weinstein (see [Ati82, Dui83] and [LR91]). In fact, the idea is that the set of possible solutions to a complex problem carries a symplectic structure and that the corresponding real problem is formulated for elements of the fixed point set of an anti-symplectic involution defined on this symplectic manifold. Examples of results obtained using this idea include the (linear and non-linear) real Kostant convexity theorems (see [Dui83, LR91]) and the real Thompson conjecture (see [AMW01, EL05]). Although we will have to replace symplectic manifolds with quasi-Hamiltonian spaces for technical considerations, the above idea plays a key role in our approach. Keeping this in mind, we will eventually define an involution $\beta^{(l)}$ on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. But, to explain how this involution is obtained, we will first work on the product $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ of l $U_\Delta \times U_\Delta$-orbits in $U \times U$.

The key here is to try and see the $K \times K$-orbit of $w \in U$ as a subset of some $U_\Delta \times U_\Delta$-orbit $\widetilde{\mathcal{D}} \subset U \times U$. This is done by observing that $w \in \mathcal{D}$ is equivalent to $\tau^-(w)w \in \mathcal{C}$ which in turn is equivalent to $(\tau(w), w) \in \widetilde{\mathcal{D}}$. Indeed, the first equivalence is corollary 6.3, where $\mathcal{C}$ is defined as the conjugacy class of $\tau^-(w)w$ for any $w \in \mathcal{D}$. Then we know from lemma 6.4 that $\mathcal{C} = \eta(\widetilde{\mathcal{D}})$ where $\widetilde{\mathcal{D}}$ is the $U_\Delta \times U_\Delta$-orbit of $(1, \tau^-(w)w) \sim (\tau(w), w)$, which gives the second equivalence. In order to obtain elements of the form $(\tau(w), w)$ as fixed points of an involution, we set

$$\begin{array}{rccc} \alpha : & U \times U & \longrightarrow & U \times U \\ & (u, v) & \longmapsto & \big(\tau(v), \tau(u)\big) \end{array}$$

Then $\alpha^2 = Id$ and $Fix(\alpha) = \{(\tau(v), v) \mid v \in U\} \simeq U$. In particular, $Fix(\alpha)$ is always non-empty. Moreover, we have :

Lemma 6.7. $\alpha(\widetilde{\mathcal{D}}) = \widetilde{\mathcal{D}}$, *so that* α *defines an involution on* $\widetilde{\mathcal{D}}$, *whose fixed point set is isomorphic to* $\mathcal{D}$ *and therefore non-empty.*

Proof. If $(u, v) \in \widetilde{\mathcal{D}}$, we have $\eta(\alpha(u, v)) = \tau^-(v)\tau(u) = \tau(v^{-1}u) = \tau^-(u^{-1}v)$. But if $w \in U$, then $\tau^-(w)$ is conjugate to w. Indeed, since τ comes from the Cartan involution defining the non-compact dual of U, there always exists a maximal torus of U which is fixed pointwise by τ^- (see [Loo69], pp. 72-74 and 79-80), and w is conjugate to an element in such a torus : $w = \varphi t \varphi^{-1}$ with $\tau^-(t) = t$ so that $\tau^-(w) = \tau(\varphi)t\tau(\varphi^{-1}) = \tau(\varphi)\varphi^{-1}w\varphi\tau(\varphi^{-1})$ (observe that when $U = U(n)$ then $\tau^-(w) = w^t$ and all of this becomes clear). Thus $\eta(\alpha(u,v)) = \tau(u^{-1}v)$ and $u^{-1}v = \eta(u,v)$ lie in the same conjugacy class $\mathcal{C} = \eta(\widetilde{\mathcal{D}})$, so, by lemma 6.4, we have indeed $\alpha(u,v) \in \widetilde{\mathcal{D}}$. From the remark preceding lemma 6.7 we see that $Fix(\alpha|_{\widetilde{\mathcal{D}}}) \simeq \mathcal{D} \neq \emptyset$. $\square$

On the product $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ of l $U_\Delta \times U_\Delta$-orbits in $U \times U$, we can therefore define the involution :

$$\begin{array}{rccc} \alpha^{(l)} : & \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l & \longrightarrow & \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \\ & \big((u_1,v_1), \dots, (u_l,v_l)\big) & \longmapsto & \Big(\big(\tau(v_1),\tau(u_1)\big), \dots, \big(\tau(v_l),\tau(u_l)\big)\Big) \end{array}$$

Observe that its fixed point set satisfies $Fix(\alpha^{(l)}) \simeq \mathcal{D}_1 \times \cdots \times \mathcal{D}_l$ and is therefore non-empty. We then have the following result, which says that the centered Lagrangian problem has a solution if and only if there exists a solution of the complexified problem which is fixed by $\alpha^{(l)}$:

Proposition 6.8. *Let $\mathcal{D}_1, \dots, \mathcal{D}_l$ be l $K \times K$-orbits in U. For every $j \in \{1, \dots, l\}$, let $\mathcal{C}_j$ be the conjugacy class of $\tau^-(w)w$ where w is any element in $\mathcal{D}_j$, and let $\widetilde{\mathcal{D}}_j$ be the corresponding $U_\Delta \times U_\Delta$-orbit in $U \times U$ (i.e., such that $\eta(\widetilde{\mathcal{D}}_j) = \mathcal{C}_j$, where $\eta(u,v) = u^{-1}v$). Then there exist $(w_1, \dots, w_l) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_l$ such that $w_1 \dots w_l = 1$ if and only there exist $((u_1,v_1), \dots, (u_l,v_l)) \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ such that $u_1 \dots u_l = 1$, $v_1 \dots v_l = 1$ and $u_j = \tau(v_j)$ for all $j \in \{1, \dots, l\}$ (that is, $((u_1,v_1), \dots, (u_l,v_l)) \in Fix(\alpha^{(l)})$).*

Proof. For a given $(w_1, \dots, w_l) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_l \mid w_1 \dots w_l = 1$, set $(u_j, v_j) := (\tau(w_j), w_j)$. By lemma 6.4, (u_j, v_j) then belongs to $\widetilde{\mathcal{D}}_j$ and we have indeed $u_1 \dots u_l = v_1 \dots v_l = 1$. Conversely, for $((u_j,v_j))_j \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \mid u_1 \dots u_l = v_1 \dots v_l = 1$ and such that $u_j = \tau(v_j)$ for all j, set $w_j := v_j$. Then $w_1 \dots w_l = 1$ and $\tau^-(w_j)w_j = u_j^{-1}v_j \in \mathcal{C}_j$, so that, by corollary 6.3, $w_j \in \mathcal{D}_j$. $\square$

This type of result is exactly why some given problem (A) is called a real version of another problem (B) : if $\mathcal{S}_\mathbb{C}$ denotes the set of solutions to problem (B) (we assume that $\mathcal{S}_\mathbb{C} \neq \emptyset$) and $\mathcal{S}_\mathbb{R}$ the set of solutions to problem (A), then there exists an involution α on some space $M \supset \mathcal{S}_\mathbb{C}$, whose fixed point set is non-empty, such that $\mathcal{S}_\mathbb{R} \neq \emptyset$ iff $\mathcal{S}_\mathbb{C} \cap Fix(\alpha) \neq \emptyset$.
The question then is : what is the real version of the unitary problem ? Given what we have done so far, we see that giving an answer to this question amounts to defining an involution $\beta^{(l)}$ on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $\beta^{(l)} \circ \eta^{(l)} = \eta^{(l)} \circ \alpha^{(l)}$, where $\eta^{(l)} : \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ is defined as in proposition 6.6, so that $\eta(Fix(\alpha^{(l)})) \subset Fix(\beta^{(l)})$, which in particular implies that $Fix(\beta^{(l)}) \neq \emptyset$. The only possibility is then to set, for any $(w_1, \dots, w_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\beta^{(l)}(w_1, \dots, w_l) \;=\; \big(\tau^-(w_l)\dots\tau^-(w_2)\tau^-(w_1)\tau(w_2)\dots\tau(w_l), \dots, \tau^-(w_l)\tau^-(w_{l-1})\tau(w_l), \tau^-(w_l)\big) \quad (1)$$

We then have the following result (proposition 6.9), along the lines of proposition 6.6. As earlier, we see that the group K^l acts on $Fix(\alpha^{(l)})$ and preserves the relations $u_1 \dots u_l = v_1 \dots v_l = 1$. Likewise, K acts diagonally on $Fix(\beta^{(l)})$, preserving the relation $w_1 \dots w_l = 1$. We may therefore define :

$$\mathcal{M}^\alpha_{\widetilde{\mathcal{D}}} := \Big\{ \big((u_j,v_j)\big)_j \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \mid u_1 \dots u_l = v_1 \dots v_l = 1 \text{ and } \big((u_j,v_j)\big)_j \in Fix(\alpha^{(l)}) \Big\} \Big/ K^l$$

and

$$\mathcal{M}^\beta_{\mathcal{C}} = \big\{ (w_j)_j \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \,|\, w_1 \dots w_l = 1 \text{ and } (w_j)_j \in Fix(\beta^{(l)}) \big\} \big/ K$$

We then have :

Proposition 6.9. *The map $\eta^{(l)} : \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ induces a homeomorphism $\mathcal{M}^\alpha_{\widetilde{\mathcal{D}}} \simeq \mathcal{M}^\beta_{\mathcal{C}}$.*

Again, this is an analog of a result in [AMW01], which justifies that we may consider our Lagrangian problem a compact version of the Thompson problem. We may now move on to the main results of this paper.

6.6 The set of σ_0-Lagrangian representations

Let $\mathcal{C}_1, \dots, \mathcal{C}_l$ be l conjugacy classes in $U(n)$ such that there exist $(u_1, \dots, u_l) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_l$ satisfying $u_1 \dots u_l = 1$.

Definition 6.1. The representation of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ corresponding to such a $(u_1, \dots, u_l)$ is said to be *Lagrangian* if there exist l Lagrangian subspaces $L_1, \dots, L_l$ of $\mathbb{C}^n$ such that, denoting by σ_j the Lagrangian involution associated to L_j, we have $u_j = \sigma_j \sigma_{j+1}$ for all $j \in \{1, \dots, l\}$ (with $\sigma_{l+1} = \sigma_1$). It is said to be σ_0-*Lagrangian* if it is Lagrangian with $L_1 = L_0 := \mathbb{R}^n \subset \mathbb{C}^n$.

Recall that two representations $(u_1, \dots, u_l)$ and $(v_1, \dots, v_l)$ of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ are equivalent if and only if there exists a unitary map $\varphi \in U(n)$ such that $\varphi u_j \varphi^{-1} = v_j$ for all $j \in \{1, \dots, l\}$. Since $\sigma_{\varphi(L)} = \varphi \sigma_L \varphi^{-1}$, we have that any representation equivalent to a Lagrangian one is itself Lagrangian. In particular, since for any Lagrangian $L \in \mathcal{L}(n)$ there exists a unitary map $\varphi \in U(n)$ such that $\varphi(L) = L_0$, we see that a given representation is Lagrangian if and only if it is equivalent to a σ_0-Lagrangian one. We now define the map :

$$\begin{array}{rccl} \beta: & \mathcal{C}_1 \times \dots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \dots \times \mathcal{C}_l \qquad (2) \\ & (u_1, \dots, u_l) & \longmapsto & (\overline{u}_l^{-1} \dots \overline{u}_2^{-1} u_1^t \overline{u}_2 \dots \overline{u}_l, \dots, \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l, u_l^t) \qquad (3) \end{array}$$

(see equation (1) in the previous subsection for motivation : when $U = U(n)$, $\tau(u) = \overline{u}$). Observe that β is an involution (for $l = 3$ one easily sees that $\beta^2 = Id$) and that $Fix(\beta) \neq \emptyset$ (one may for instance pick a diagonal element u_j in every $\mathcal{C}_j$ and then $\beta(u_1, \dots, u_l) = (u_1, \dots, u_l)$). Also, we have the compatibility relations (see theorem 5.2) $\beta(\varphi.(u_1, \dots, u_l)) = \overline{\varphi}.\beta(u_1, \dots, u_l)$ and $\mu \circ \beta(u_1, \dots, u_l) = \overline{u}_l^{-1} \dots \overline{u}_1^{-1} = (\overline{\mu(u_1, \dots, u_l)})^{-1}$, where μ is the product map $\mu(u_1, \dots u_l) = u_1 \dots u_l$ on $\mathcal{C}_1 \times \dots \times \mathcal{C}_l$. Finally, we consider the Euclidean product on the Lie algebra $\mathfrak{u}(n)$ given by $(X \,|\, Y) = tr(XY^*) = -tr(XY)$. In particular, the map $\tau : X \in \mathfrak{u}(n) \mapsto \overline{X} \in \mathfrak{u}(n)$ is an isometry for this scalar product. We may now state and prove the following characterization of σ_0-Lagrangian representations :

Theorem 6.10. *Given l conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_l$ of unitary matrices such that there exist $(u_1, \dots, u_l) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_l$ satisfying $u_1 \dots u_l = 1$, the representation of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ corresponding to such a $(u_1, \dots, u_l)$ is σ_0-Lagrangian if and only if $\beta(u_1, \dots, u_l) = (u_1, \dots, u_l)$ (see equation (3) for a definition of β).*

We could as well have defined β on $U(n) \times \dots \times U(n)$ and obtained a similar result but we deliberately stated our result this way, as it will be more appropriate to work with the quasi-Hamiltonian space $\mathcal{C}_1 \times \dots \times \mathcal{C}_l$ in the following.

Proof of theorem 6.10. Let us start with $(u_1, \dots, u_l) \in Fix(\beta)$, that is :

$$\begin{array}{rcl} \overline{u}_l^{-1} \dots \overline{u}_2^{-1} u_1^t \overline{u}_2 \dots \overline{u}_l & = & u_1 \\ \overline{u}_l^{-1} \dots \overline{u}_3^{-1} u_2^t \overline{u}_3 \dots \overline{u}_l & = & u_2 \\ & \vdots & \\ \overline{u}_l^{-1} \dots \overline{u}_{j+1}^{-1} u_j^t \overline{u}_{j+1} \dots \overline{u}_l & = & u_j \\ & \vdots & \\ \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l & = & u_{l-1} \\ u_l^t & = & u_l \end{array}$$

Then we have $u_l^t = u_l$ (so that $\overline{u}_l = u_l^{-1}$), $(u_{l-1} u_l)^t = (\overline{u}_l^{-1} u_{l-1}^t \overline{u}_l u_l)^t = (u_l^t u_{l-1}^t)^t = u_{l-1} u_l, \dots,$ $(u_j \dots u_l)^t = (\overline{u}_l^{-1} \dots \overline{u}_{j+1}^{-1} u_j^t \overline{u}_{j+1} \dots \overline{u}_l \dots \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l u_l)^t = (u_l^t u_{l-1}^t \dots u_{j+1}^t u_j^t)^t = u_j \dots u_l, \dots,$ and $(u_1 \dots u_l)^t = (\overline{u}_l^{-1} \dots \overline{u}_2^{-1} u_1^t \overline{u}_2 \dots \overline{u}_l \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l u_l)^t = (u_l^t u_{l-1}^t \dots u_2^t u_1^t)^t = u_1 \dots u_l$. To these l symmetric unitary matrices we can associate, by proposition 2.3, l Lagrangian subspaces :

$$\begin{array}{rcl} L_1 & := & \{z \in \mathbb{C}^n \mid z - (u_1 \dots u_l)\overline{z} = 0\} \\ L_2 & := & \{z \in \mathbb{C}^n \mid z - (u_2 \dots u_l)\overline{z} = 0\} \\ & \vdots & \\ L_j & := & \{z \in \mathbb{C}^n \mid z - (u_j \dots u_l)\overline{z} = 0\} \\ & \vdots & \\ L_{l-1} & := & \{z \in \mathbb{C}^n \mid z - (u_{l-1} u_l)\overline{z} = 0\} \\ L_l & := & \{z \in \mathbb{C}^n \mid z - u_l\overline{z} = 0\} \end{array}$$

and denote by σ_j the Lagrangian involution associated to L_j. Let us now assume that $(u_1, \dots, u_l)$ satisfy the full hypotheses of the theorem, that is, that we have $u_1 \dots u_l = 1$. Then $L_1 = L_0$. Therefore, by proposition 2.3, since $L_l = \{z - u_l\overline{z} = 0\}$, we have $\sigma_l\sigma_0 = u_l$, that is, $\sigma_l\sigma_1 = u_l$. Further, since $L_2 = \{z - (u_2 \dots u_l)\overline{z} = 0\}$, we have $\sigma_2\sigma_0 = u_2 \dots u_l = u_1^{-1}$ hence $u_1 = \sigma_1\sigma_2$. Finally, for all $j \in \{2, \dots, l-1\}$, since $(u_j \dots u_l)^t = u_j \dots u_l$, there exists, by proposition 2.3, a unitary map $\varphi_j \in U(n) \mid \varphi_j^t = \varphi_j$ and $\varphi_j^2 = u_j \dots u_l$, and we then have $\varphi_j(L_0) = L_j$. Set $L'_j = \varphi_2^{-1}(L_j) = L_0$ and $L'_{j+1} = \varphi_j^{-1}(L_{j+1})$, and denote by σ'_j and σ'_{j+1} the associated involutions. Then :

$$\begin{array}{rcl} L'_{j+1} & = & \{z \mid \varphi_j(z) \in L_{j+1}\} \\ & = & \{z \mid \varphi_j(z) - u_{j+1} \dots u_l\overline{\varphi_j(z)} = 0\} \\ & = & \{z \mid \varphi_j(z) - u_{j+1} \dots u_l \underbrace{\overline{\varphi_j}}_{=\varphi_j^{-1}}(\overline{z}) = 0\} \\ & = & \{z \mid z - (\varphi_j^{-1}u_{j+1} \dots u_l\varphi_j^{-1})\overline{z} = 0\} \end{array}$$

but $(\varphi_j^{-1}u_{j+1} \dots u_l\varphi_j^{-1})^t = \varphi_j^{-1}u_{j+1}\varphi_j^{-1}$ since $(\varphi_j^{-1})^t = (\varphi_j^t)^{-1} = \varphi_j^{-1}$ and $(u_{j+1} \dots u_l)^t = u_{j+1} \dots u_l$. Therefore, by proposition 2.3, we have $\sigma'_{j+1}\sigma'_j = \varphi_j^{-1}u_{j+1} \dots u_l\varphi_j^{-1}$. Since $\varphi_j^2 = u_j \dots u_l$, we then have $\varphi_j^{-1}u_{j+1} \dots u_l\varphi_j^{-1} = \varphi_j^{-1}(u_j^{-1}\varphi_j^2)\varphi_j^{-1} = \varphi_j^{-1}u_j^{-1}\varphi_j$, therefore $u_j^{-1} = \varphi_j\sigma'_{j+1}\sigma'_j\varphi_j^{-1} = \sigma_{j+1}\sigma_j$ since $L_j = \varphi_j(L'_j)$, $L_{j+1} = \varphi_j(L'_{j+1})$ and $\sigma_{\varphi(L)} = \varphi\sigma_L\varphi_{-1}$. Hence $u_j = \sigma_j\sigma_{j+1}$ and the representation of π corresponding to $(u_1, \dots, u_l)$ is σ_0-Lagrangian.
Conversely, assume that a given representation $(u_1, \dots, u_l)$ is σ_0-Lagrangian. Then $u_l = \sigma_l\sigma_0$. Now observe that for any unitary map u, one has $\overline{u} = \sigma_0 u \sigma_0$, therefore here $u_l^t = \overline{u}_l^{-1} = \sigma_0 u_l^{-1}\sigma_0 = \sigma_0(\sigma_l\sigma_0)^{-1}\sigma_0 = \sigma_0(\sigma_0\sigma_l)\sigma_0 = \sigma_l\sigma_0 = u_l$. Likewise :

$$\begin{array}{rcl} \overline{u}_l^{-1}u_{l-1}^t\overline{u}_l & = & (\sigma_0 u_l^{-1}\sigma_0)(\sigma_0 u_{l-1}^{-1}\sigma_0)(\sigma_0 u_l \sigma_0) \\ & = & \sigma_0(u_l^{-1}u_{l-1}^{-1}u_l)\sigma_0 \\ & = & \sigma_0(\sigma_0\sigma_l)(\sigma_l\sigma_{l-1})(\sigma_l\sigma_0)\sigma_0 \\ & = & \sigma_{l-1}\sigma_l \\ & = & u_{l-1} \end{array}$$

and so on, until :

$$\begin{array}{rcl} \overline{u}_l^{-1} \dots \overline{u}_2^{-1}u_1^t\overline{u}_2 \dots \overline{u}_l & = & \sigma_0(\sigma_0\sigma_l) \dots (\sigma_3\sigma_2)(\sigma_2\sigma_1)(\sigma_2\sigma_3) \dots (\sigma_3\sigma_0)\sigma_0 \\ & = & \sigma_1\sigma_2 \\ & = & u_1 \end{array}$$

so that $\beta(u_1, \dots, u_l) = (u_1, \dots, u_l)$. □

We can then characterize those among representations of $\pi_1(S^2\backslash\{s_1, \dots, s_l\})$ which are Lagrangian in the following way :

Corollary 6.11 (Characterization of Lagrangian representations)**.** *Suppose that one of the $\mathcal{C}_j$ is defined by pairwise distinct eigenvalues and let $u_1, \dots, u_l$ be l unitary matrices such that $u_j \in \mathcal{C}_j$ and $u_1 \dots u_l = 1$. Then there exist l Lagrangian subspaces $L_1, \dots, L_l$ of $\mathbb{C}^n$ such that $u_1 = \sigma_1\sigma_2, u_2 = \sigma_2\sigma_3, \dots, u_l = \sigma_l\sigma_1$ (where σ_j is the Lagrangian involution associated to L_j) if and only if $\beta(u_1, \dots, u_l)$ is equivalent to $(u_1, \dots, u_l)$ as representations of π. In this case, if ψ is any unitary map such that $\beta(u_1, \dots, u_l) = (\psi u_1 \psi^{-1}, \dots, \psi u_l \psi^{-1})$ then $\psi^t = \psi$ and if φ is any unitary map such that $\varphi^t\varphi = \psi$ then the representation of π corresponding to $(\varphi u_1 \varphi^{-1}, \dots, \varphi u_l \varphi^{-1})$ is σ_0-Lagrangian.*

Proof. Suppose first that $u_1 = \sigma_1\sigma_2, \dots, u_l = \sigma_l\sigma_1$. Take $\varphi \in U(n) \mid \varphi(L_1) = L_0$. Then $\varphi.(u_1, \dots, u_l)$ is σ_0-Lagrangian, hence $\beta(\varphi.(u_1, \dots, u_l)) = \varphi.(u_1, \dots, u_l)$, hence $\overline{\varphi}.\beta(u_1, \dots, u_l) = \varphi.(u_1, \dots, u_l)$ hence $\beta(u_1, \dots, u_l) = (\overline{\varphi}^{-1}\varphi).(u_1, \dots, u_l) \sim_{U(n)} (u_1, \dots, u_l)$. Observe that $\overline{\varphi}^{-1}\varphi = \varphi^t\varphi$ is symmetric.
Conversely, suppose that $\exists\ \psi \in U(n) \mid \beta(u_1, \dots, u_l) = \psi.(u_1, \dots, u_l)$ and assume first that the conjugacy class $\mathcal{C}_l$ is defined by pairwise distinct eigenvalues. Write $u_l = vdv^{-1}$ where d is diagonal. Then, since $\beta(u) = \psi.u$, we have in particular $\psi u_l \psi^{-1} = u_l^t$, from which we obtain $\psi v d v^{-1} \psi^{-1} = (v^{-1})^t d^t v^t = (v^t)^{-1} d v^t$, so that $(v^t\psi v) d (v^t \psi v)^{-1} = d$. Since d is diagonal with pairwise distinct elements, $v^t\psi v$ is itself diagonal and therefore symmetric, so that ψ is symmetric. If now it is a different $\mathcal{C}_j$ which is defined by pairwise distinct eigenvalues, say $\mathcal{C}_{l-1}$, then consider the representation $(u_l, u_1, \dots, u_{l-1})$: it is indeed a representation of π since the relation $u_1 \dots u_l = 1$ is invariant by circular permutation (as can be seen by conjugating by u_l) and via this transformation $\psi.(u_1, \dots, u_l)$ is sent to $\psi.(u_l, u_1, \dots, u_{l-1})$. The representation $(u_l, u_1, \dots, u_{l-1})$ is Lagrangian iff $(u_1, \dots, u_l)$ is Lagrangian. We can define a corresponding β accordingly and proceed as above to show that ψ is indeed symmetric.
Now, to conclude, let φ be any unitary map such that $\varphi^t\varphi = \psi$ (such a map always exists by proposition 2.3). Starting from $\beta(u_1, \dots, u_l) = \psi.(u_1, \dots, u_l)$, we obtain $(\varphi^t)^{-1}.\beta(u) = \varphi.u$, hence $\beta(\varphi.u) = \varphi.u$ so that, by theorem 6.10, $\varphi.(u_1, \dots, u_l)$ is σ_0-Lagrangian. Hence $(u_1, \dots, u_l)$ is Lagrangian, with $L_1 = \varphi^{-1}(L_0)$. □

Before passing on to studying Lagrangian representations in the moduli space, we would like to point out that if a representation u is irreducible then so is $\beta(u)$ and, more interestingly maybe, that it is possible to characterize Lagrangian representations with arbitrarily fixed first Lagrangian L_1 in a way similar to theorem 6.10. In order to do so, we define, for a given Lagrangian subspace L_1, the involution $\beta_{L_1}(u_1, u_2, u_3) := (\sigma_1 u_3^{-1} u_2^{-1} u_1^{-1} u_2 u_3 \sigma_1, \sigma_1 u_3^{-1} u_2^{-1} u_3 \sigma_1, \sigma_1 u_3^{-1} \sigma_1)$ (remember that when $L_1 = L_0$, $\sigma_0 u \sigma_0 = \overline{u}$). If we write $L_1 = \varphi(L_0)$ for some $\varphi \in U(n)$, we obtain $\beta_{L_1}(u) = (\varphi\varphi^t).\beta(u)$ (this does not depend on the choice of φ such that $\varphi(L_0) = L_1$ as seen from the argument used in proposition 2.3). Finally, it was proved in [FMS04] that when $n = 2$ and $l = 3$, every (two-dimensional) unitary representation of $\pi_1(S^2 \backslash \{s_1, s_2, s_3\})$ is Lagrangian : this is because in this case the moduli space is a single point (it is zero-dimensional and connected), so that the submanifold consisting of Lagrangian representations is the point itself (see [FW06] for dimensions of moduli spaces of representations). As a matter of fact, we believe that the characterization of Lagrangian representations as representations u satisfying $\beta(u) \sim_{U(n)} u$ is true even without the (generic) assumption made on the $\mathcal{C}_j$ but we have been unable to prove it so far. One would only need to show that if $\beta(u) = \psi.u$ for some $\psi \in U(n)$ then there exists such a ψ which is symmetric. In the remainder of this paper, we will assume that one of the $\mathcal{C}_j$ is defined by pairwise distinct eigenvalues, so that corollary 6.11 holds.

Remark (Addendum - 26.07.06)**.** As a matter of fact, corollary 6.11 does hold without any assumption on the conjugacy classes $\mathcal{C}_j$ and a proof of this is availablle in [Sch05].

6.7 Lagrangian representations in the moduli space

Recall from section 4 that the moduli space of unitary representations of $\pi = \pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ is the quasi-Hamiltonian quotient $\mathcal{M}_{\mathcal{C}} = \mu^{-1}(\{1\})/U(n)$ where $\mu : \mathcal{C}_1 \times \dots \times \mathcal{C}_l \to U(n)$ is the product map. Since the involution β we constructed on $\mathcal{C}_1 \times \dots \times \mathcal{C}_l$ in 6.6 satisfies $\beta \circ \mu = \tau^- \circ \mu$ (where $\tau(u) = \overline{u}$ on $U(n)$), β preserves $\mu^{-1}(\{1\})$ and since $\beta(\varphi.u) = \tau(\varphi).\beta(u)$, β induces an involution $\hat{\beta}$ on $\mathcal{M}_{\mathcal{C}} = \mu^{-1}(\{1\})/U(n)$ given by $\hat{\beta}([u]) = [\beta(u)]$. Observe that if $\beta^{(l)}$ is defined as in the end of the previous subsection by $\beta_L = (\varphi\varphi^t).\beta$ (where $\varphi \in U(n)$ satisfies $\varphi(L_0) = L$) then $\widehat{\beta^{(l)}} = \hat{\beta}$. Furthermore, if $[u] \in \mathcal{M}_{\mathcal{C}}$ is the equivalence class of a unitary representation of π, then it is Lagrangian if and only if any of its representatives is Lagrangian (for, if $u_j = \sigma_j\sigma_{j+1}$, then $\varphi u_j \varphi^{-1} = \sigma'_j\sigma'_{j+1}$ where $L'_j = \varphi(L_j)$, for any $\varphi \in U(n)$). Corollary 6.11 then shows that a given $[u] \in \mathcal{M}_{\mathcal{C}}$ is Lagrangian if and only if $\hat{\beta}([u]) = [u]$. We then have the following result, which is a direct consequence of theorem 5.2.

Theorem 6.12. *The set of equivalence classes of Lagrangian representations of $\pi = \pi_1(S^2\backslash\{s_1, \dots, s_l\})$ is exactly $Fix(\hat{\beta})$. It is a Lagrangian submanifold of the moduli space $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U(n))/U(n)$ of unitary representations of π (in particular it is always non-empty).*

To apply theorem 5.2, the only condition left to check is that $\beta^*\omega = -\omega$, where ω is the 2-form defining the quasi-Hamiltonian structure on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ described in section 4. Actually, we also need to check that $Fix(\hat{\beta}) \neq \emptyset$. As indicated before theorem 5.3, this is always true for an involution β which satisfies the hypotheses of theorem 5.2 and which has fixed points itself, but since this paper does not contain a proof of this fact, we instead refer to theorem 1 of [FW06], which we state here.

Theorem 6.13. [FW06] *Let $\mathcal{C}_1, \dots, \mathcal{C}_l$ be $l \geq 1$ conjugacy classes in $U(n)$ such that there exist $(u_1, \dots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ satisfying $u_1 \dots u_l = 1$. Then there exist l Lagrangian subspaces $L_1, \dots, L_l$ of $\mathbb{C}^n$ such that $\sigma_j\sigma_{j+1} \in \mathcal{C}_j$ for all $j \in \{1, \dots, l\}$, where σ_j is the Lagrangian involution associated with L_j and where $\sigma_{l+1} = \sigma_1$.*

This shows that $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ as one can construct, from the Lagrangian representation $(\sigma_j\sigma_{j+1})_j$ whose existence is guaranteed by the theorem, a σ_0-Lagrangian representation $(\sigma'_j\sigma'_{j+1})_j \in Fix(\beta)$ by applying $\varphi \in U(n)$ such that $\varphi(L_1) = L_0$.

Proof of theorem 6.12. As observed, we only have to check that $\beta^*\omega = -\omega$. We prove it by induction on l. For $l = 1$, we have, for any $X, Y \in \mathfrak{u}$ (denoting $[X]_u = X.u - u.X \in T_u\mathcal{C}_1$),

$$\omega_u([X]_u, [Y]_u) = \frac{1}{2}\big((Adu.X \,|\, Y) - (Adu.Y \,|\, X)\big)$$

as well as $\beta(u) = \tau(u^{-1})$ and $T_u\beta.[X]_u = [\tau(X)]_{\tau(u^{-1})}$. Therefore :

$$\begin{aligned}
(\beta^*\omega)_u\big([X]_u, [Y]_u\big) &= \omega_{\beta(u)}\big(T_u\beta.[X]_u, T_u\beta.[Y]_u\big) \\
&= \frac{1}{2}\Big(\big(Ad\,\tau(u^{-1}).\tau(X) \,|\, \tau(Y)\big) - \big(Ad\,\tau(u^{-1}).\tau(Y) \,|\, \tau(X)\big)\Big) \\
&= \frac{1}{2}\Big(\big(\tau(Ad\,u^{-1}.X) \,|\, \tau(Y)\big) - \big(\tau(Ad\,u^{-1}.Y) \,|\, \tau(X)\big)\Big)
\end{aligned}$$

Since τ is an isometry for $(.\,|\,.)$, we then have :

$$\begin{aligned}
(\beta^*\omega)_u\big([X]_u, [Y]_u\big) &= \frac{1}{2}\big((Ad\,u^{-1}.X \,|\, Y) - (Ad\,u^{-1}.Y \,|\, X)\big) \\
&= \frac{1}{2}\big((X \,|\, Ad\,u.Y) - (Y \,|\, Ad\,u.X)\big) \\
&= -\omega_u\big([X]_u, [Y]_u\big)
\end{aligned}$$

To complete the induction, we will use the following lemma, which is general in nature and can be used to construct form-reversing involutions on quasi-Hamiltonian spaces.

Lemma 6.14. *Let $(M_1, \omega_1, \mu_1 : M_1 \to U)$ and $(M_2, \omega_2, \mu_2 : M_2 \to U)$ be two quasi-Hamiltonian U-spaces. Let τ be an involutive automorphism of $(U, (.\,|\,.))$ and let β_i be an involution on M_i satisfying :*

(i) $\beta_i^*\omega_i = -\omega_i$

(ii) $\beta_i(u.x_i) = \tau(u).\beta_i(x_i)$ *for all* $u \in U$ *and all* $x_i \in M_i$

(iii) $\mu_i \circ \beta_i = \tau^- \circ \mu_i$

Consider the quasi-Hamiltonian U-space $(M := M_1 \times M_2, \omega := \omega_1 \oplus \omega_2 + (\mu_1^*\theta^L \wedge \mu_2^*\theta^R), \mu := \mu_1 \cdot \mu_2)$ (with respect to the diagonal action of U) and the map :

$$\beta := \big((\mu_2 \circ \beta_2).\beta_1, \beta_2\big) : \begin{array}{ccc} M & \longrightarrow & M \\ (x_1, x_2) & \longmapsto & \big((\mu_2 \circ \beta_2(x_2)).\beta_1(x_1), \beta_2(x_2)\big) \end{array}$$

Then β is an involution on M satisfying :

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $u \in U$ *and all* $x \in M$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

We postpone the proof of the lemma and give the end of the proof of theorem 6.12. To complete the induction, all one has to do is check that our involution $\beta = \beta^{(l)}$ (see (1)) on the product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ of l conjugacy classes is indeed obtained like in the lemma starting from the form-reversing involution $\beta^{(1)} := \tau^- : u \to u^t$ on each single conjugacy class. This is easily checked since on $\mathcal{C}_1 \times \mathcal{C}_2$:

$$\begin{aligned} \beta^{(2)}(u_1, u_2) &= (\overline{u_2}^{-1}\, u_1^t\, \overline{u_2}\,, u_2^t) \\ &= (u_2^t.u_1^t\,, u_2^t) \\ &= \big((\mu_2 \circ \beta^{(1)}(u_2)).\beta^{(1)}(u_1), \beta^{(1)}(u_2)\big) \end{aligned}$$

and on $\mathcal{C}_1 \times (\mathcal{C}_2 \times \mathcal{C}_3)$:

$$\begin{aligned} \beta^{(3)}(u_1, u_2, u_3) &= (\overline{u_3}^{-1}\,\overline{u_2}^{-1}\, u_1^t\, \overline{u_2}\,\overline{u_3}\,, \overline{u_3}^{-1}\, u_2^t\, \overline{u_3}\,, u_3^t) \\ &= \big((u_2u_3)^t.u_1^t\,, u_3^t.u_2^t\,, u_3^t\big) \\ &= \Big(\big((\mu_2 \cdot \mu_3) \circ \beta^{(2)}(u_2, u_3)\big).\beta^{(1)}(u_1), \beta^{(2)}(u_2, u_3)\Big) \end{aligned}$$

and so on. It is of course the very form of the involution β which inspired the formulation of the lemma. □

Proof of lemma 6.14. First, we have :

$$\begin{aligned} \beta(\beta(x_1, x_2)) &= \bigg(\Big(\mu_2 \circ \beta_2\big(\beta_2(x_2)\big)\Big).\beta_1\Big(\big(\mu_2 \circ \beta_2(x_2)\big).\beta_1(x_1)\Big), \beta_2\big(\beta_2(x_2)\big)\bigg) \\ &= \bigg(\big(\mu_2(x_2)\big).\Big(\tau\big(\underbrace{\mu_2 \circ \beta_2}_{=\tau^- \circ \mu_2}(x_2)\big).\beta_1\big(\beta_1(x_1)\big)\Big), x_2\bigg) \\ &= \Big(\big(\mu_2(x_2)\big)\big(\mu_2(x_2)\big)^{-1}.x_1, x_2\Big) \\ &= (x_1, x_2) \end{aligned}$$

so that β is indeed an involution. Second :

$$\begin{aligned} \beta(u.x_1, u.x_2) &= \big(\mu_2 \circ \beta_2(u.x_2).\beta_1(u.x_1), \beta_2(u.x_2)\big) \\ &= \Big(\underbrace{\mu_2\big(\tau(u).\beta_2(x_2)\big)}_{=\tau(u)\mu_2\left(\beta_2(x_2)\right)\tau(u)^{-1}}.\big(\tau(u).\beta_1(x_1)\big), \tau(u)\beta_2(x_2)\Big) \\ &= \bigg(\tau(u).\Big(\big(\mu_2 \circ \beta_2(x_2)\big).\beta_1(x_1)\Big), \tau(u).\beta_2(x_2)\bigg) \\ &= \tau(u).\beta(x_1, x_2) \end{aligned}$$

and :

$$\begin{aligned}
\mu\circ\beta(x_1,x_2) &= \mu_1\Big(\big(\mu_2\circ\beta_2(x_2)\big).\beta_1(x_1)\Big)\mu_2\big(\beta_2(x_2)\big)\\
&= \Big(\mu_2\circ\beta_2(x_2)\mu_1\circ\beta_1(x_1)\big(\mu_2\circ\beta_2(x_2)\big)^{-1}\Big)\Big(\mu_2\circ\beta_2(x_2)\Big)\\
&= \tau^-\circ\mu_2(x_2)\tau^-\circ\mu_1(x_1)\\
&= \tau^-\circ(\mu_1\cdot\mu_2)(x_1,x_2)\\
&= \tau^-\circ\mu(x_1,x_2)
\end{aligned}$$

So the only thing left to prove is that $\beta^*\omega=-\omega$. Let us start by computing $T\beta$. For all $(x_1,x_2)\in M$, and all $(v_1,v_2):=\frac{d}{dt}|_{t=0}(x_1(t),x_2(t))$ (where $x_i(0)=x_i$), one has :

$$\begin{aligned}
T_{(x_1,x_2)}\beta.(v_1,v_2) &= \frac{d}{dt}|_{t=0}\Big(\big(\mu_2\circ\beta_2(x_2)\big).\beta_1\big(x_1(t)\big),\beta_2\big(x_2(t)\big)\Big)\\
&= \bigg(\big(\mu_2\circ\beta_2(x_2)\big).\bigg\{\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)}+T_{x_1}\beta_1.v_1\bigg\},T_{x_2}\beta_2.v_2\bigg)
\end{aligned}$$

Recall indeed that if a Lie group U acts on a manifold M then :

$$\frac{d}{dt}|_{t=0}\ (u_t.x_t)=u_0.X^\sharp_{x_0}+u_0.\Big(\frac{d}{dt}|_{t=0}\ x_t\Big)$$

where $X\in\mathfrak{u}=Lie(U)$ is such that $u_t=u_0\exp(tX)$ for all t, that is :

$$X=u_0^{-1}.\Big(\frac{d}{dt}|_{t=0}\ u_t\Big)=\theta^L_{u_0}\Big(\frac{d}{dt}|_{t=0}\ u_t\Big)$$

Let us now compute $\beta^*(\omega_1\oplus\omega_2)$. We obtain, for all $(x_1,x_2)\in M$ and all $(v_1,v_2),(w_1,w_2)\in T_{(x_1,x_2)}M$:

$$\big(\beta^*(\omega_1\oplus\omega_2)\big)_{(x_1,x_2)}\big((v_1,v_2),(w_1,w_2)\big)$$

$$= (\omega_1)_{\left(\mu_2\circ\beta_2(x_2)\right).\beta_1(x_1)}\Bigg(\big(\mu_2\circ\beta_2(x_2)\big).\bigg\{\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)}+\underbrace{T_{x_1}\beta_1.v_1}_{(A)}\bigg\}, \tag{4}$$

$$\big(\mu_2\circ\beta_2(x_2)\big).\bigg\{\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).w_2\big)\Big)^\sharp_{\beta_1(x_1)}+\underbrace{T_{x_1}\beta_1.w_1}_{(A)}\bigg\}\Bigg) \tag{5}$$

$$+\underbrace{(\omega_2)_{\beta_2(x_2)}(T_{x_2}\beta_2.v_2,T_{x_2}\beta_2.w_2)}_{(B)}$$

Since ω_1 is U-invariant, we can drop the terms $\mu_2\circ\beta_2(x_2)\in U$ appearing on lines (4) and (5). Further, since $\beta^*\omega_1=-\omega_1$ and $\beta^*\omega_2=-\omega_2$, we have, by the $l=1$ case :

$$(A)+(B) = -(w_1)_{x_1}(v_1,w_1)-(w_2)_{x_2}(v_2,w_2) \tag{6}$$

$$= -(\omega_1\oplus\omega_2)_{(x_1,x_2)}\big((v_1,v_2),(w_1,w_2)\big) \tag{7}$$

The remaining terms on lines (4) and (5) then are :

$$(w_1)_{\beta_1(x_1)}\bigg(\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)},T_{x_1}\beta_1.w_1\bigg) \tag{8}$$

$$+\quad (w_1)_{\beta_1(x_1)}\bigg(T_{x_1}\beta_1.v_1\,,\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).w_2\big)\Big)^\sharp_{\beta_1(x_1)}\bigg) \tag{9}$$

$$+\quad (w_1)_{\beta_1(x_1)}\bigg(\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)},\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2\circ\beta_2).w_2\big)\Big)^\sharp_{\beta_1(x_1)}\bigg) \tag{10}$$

and we notice that each of these three terms is of the form $\iota_{X^\sharp}\omega_1 = \frac{1}{2}\mu_1^*(\theta^L + \theta^R \,|\, X)$ for some $X \in \mathfrak{u}$. To facilitate the computations, we set, for $i = 1, 2$:

$$\begin{aligned} g_i &:= \mu_i \circ \beta_i(x_i) \in U \\ \zeta_i &:= T_{x_i}(\mu_i \circ \beta_i).v_i \in T_{\mu_i\circ\beta_i(x_i)}U = T_{g_i}U \\ \eta_i &:= T_{x_i}(\mu_i \circ \beta_i).w_i \in T_{\mu_i\circ\beta_i(x_i)}U = T_{g_i}U \end{aligned}$$

We can then rewrite lines (8), (9) and (10) under the form :

$$\frac{1}{2}\big(\underbrace{\theta^L_{g_1}(\eta_1)}_{(1)} + \underbrace{\theta^R_{g_1}(\eta_1)}_{C} \,|\, \theta^L_{g_2}(\zeta_2)\big) \tag{11}$$

$$-\frac{1}{2}\big(\underbrace{\theta^L_{g_1}(\zeta_1)}_{(2)} + \underbrace{\theta^R_{g_1}(\zeta_1)}_{D} \,|\, \theta^L_{g_2}(\eta_2)\big) \tag{12}$$

$$+\frac{1}{2}\Big(\theta^L_{g_1}\big(\theta^L_{g_2}(\eta_2).g_1 - g_1.\theta^L_{g_2}(\eta_2)\big) + \theta^R_{g_1}\big(\theta^L_{g_2}(\eta_2).g_1 - g_1.\theta^L_{g_2}(\eta_2)\big) \,|\, \theta^L_{g_2}(\zeta_2)\Big) \tag{13}$$

where the expression for the last term follows from the equivariance of μ_1 :

$$T_{\beta_1(x_1)}\mu_1.\big(\theta^L_{\mu_2\circ\beta_2(x_2)}(T_{x_2}(\mu_2\circ\beta_2).w_2)\big)^\sharp_{\beta_1(x_1)} = \big(\theta^L_{\mu_2\circ\beta_2(x_2)}(T_{x_2}(\mu_2\circ\beta_2).w_2)\big)^\sim_{\mu_1\circ\beta_1(x_1)} = \big(\theta^L_{g_2}(\eta_2)\big)^\sim_{g_2}$$

(where $X^\sim_u = X.u - u.X$ is the value at u of the fundamental vector field associated to $X \in \mathfrak{u}$ by the action of U on itself by conjugation). We can simplify the expression in (13) further by using the definition of θ^L and θ^R and the Ad-invariance of $(\,.\,|\,.\,)$:

$$\begin{aligned} (13) \quad &= \quad \frac{1}{2}\big(Ad\,g_1^{-1}.\theta^L_{g_2}(\eta_2) - Ad\,g_1.\theta^L_{g_2}(\eta_2) \,|\, \theta^L_{g_2}(\zeta_2)\big) && (14) \\ &= \quad \frac{1}{2}\big(\theta^L_{g_2}(\eta_2) \,|\, Ad\,g_1.\theta^L_{g_2}(\zeta_2)\big) - \frac{1}{2}\big(Ad\,g_1.\theta^L_{g_2}(\eta_2) \,|\, \theta^L_{g_2}(\zeta_2)\big) && (15) \end{aligned}$$

Let us now compute $\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$.

$$\begin{aligned} & \big(\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)\big)_{(x_1,x_2)}\big((v_1,v_2),(w_1,w_2)\big) && (16) \\ = \quad & (\mu_1^*\theta^L \wedge \mu_2^*\theta^R)_{\big(\mu_2\circ\beta_2(x_2).\beta_1(x_1),\beta_2(x_2)\big)}\big(T_{(x_1,x_2)}\beta.(v_1,v_2), T_{(x_1,x_2)}\beta.(w_1,w_2)\big) && (17) \\ = \quad & \frac{1}{2}\Big(\theta^L_{\underbrace{\mu_1(g_2.\beta_1(x_1)}_{g_2\mu_1(\beta_1(x_1))g_2^{-1}}}\Big(T_{g_2.\beta_1(x_1)}\mu_1.\big(\mu_2\circ\beta_2(x_2)\big).\Big\{\big(\theta^L_{g_2}(\zeta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.v_1\Big\}\Big) \,|\, \theta^R_{g_2}(\eta_2)\Big) && (18) \\ & -\frac{1}{2}\Big(\theta^L_{\mu_1(g_2.\beta_1(x_1)}\Big(T_{g_2.\beta_1(x_1)}\mu_1.\big(\mu_2\circ\beta_2(x_2)\big).\Big\{\big(\theta^L_{g_2}(\eta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.w_1\Big\}\Big) \,|\, \theta^R_{g_2}(\zeta_2)\Big) \end{aligned}$$

Since μ_1 is equivariant, we have, for any $v \in T_{\beta_1(x_1)}M_1$:

$$T_{g_2.\beta_1(x_1)}\mu_1.\big(\mu_2\circ\beta_2(x_2)\big).v = \big(\mu_2\circ\beta_2(x_2)\big).\Big(T_{\beta_1(x_1)}\mu_1.v\Big)$$

where the action in the right side term is conjugation. We then have :

$$\begin{aligned} (18) \quad = \quad & \frac{1}{2}\Bigg(\theta^L_{g_2g_1g_2^{-1}}\Big(g_2.\Big(T_{\beta_1(x_1)}\mu_1.\Big(\big(\theta^L_{g_2}(\zeta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.v_1\Big)\Big).g_2^{-1}\Big) \,|\, \theta^R_{g_2}(\eta_2)\Bigg) && (19) \\ & -\frac{1}{2}\Bigg(\theta^L_{g_2g_1g_2^{-1}}\Big(g_2.\Big(T_{\beta_1(x_1)}\mu_1.\Big(\big(\theta^L_{g_2}(\eta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.w_1\Big)\Big).g_2^{-1}\Big) \,|\, \theta^R_{g_2}(\zeta_2)\Bigg) \end{aligned}$$

$$= \frac{1}{2}\Big(g_2 g_1^{-1} g_2^{-1} g_2.\big(\theta^L_{g_2}(\zeta_2).g_1 - g_1.\theta^L_{g_2}(\zeta_2)\big).g_2^{-1} \,|\, \theta^R_{g_2}(\eta_2)\Big) + \frac{1}{2}\Big(g_2 \underbrace{g_1^{-1} g_2^{-1} g_2.\zeta_1}_{=\theta^L_{g_1}(\zeta_1)}.g_2^{-1} \,|\, \theta^R_{g_2}(\eta_2)\Big) \quad (20)$$

$$-\frac{1}{2}\Big(g_2 g_1^{-1} g_2^{-1} g_2.\big(\theta^L_{g_2}(\eta_2).g_1 - g_1.\theta^L_{g_2}(\eta_2)\big).g_2^{-1} \,|\, \theta^R_{g_2}(\zeta_2)\Big) + \frac{1}{2}\Big(g_2 \underbrace{g_1^{-1} g_2^{-1} g_2.\eta_1}_{=\theta^L_{g_1}(\eta_1)}.g_2^{-1} \,|\, \theta^R_{g_2}(\zeta_2)\Big)$$

$$= \frac{1}{2}\Big(Ad\, g_2 Ad\, g_1^{-1}.\theta^L_{g_2}(\zeta_2) \,|\, \theta^R_{g_2}(\eta_2)\Big) - \frac{1}{2}\Big(Ad\, g_2.\theta^L_{g_2}(\zeta_2) \,|\, \theta^R_{g_2}(\eta_2)\Big) + \frac{1}{2}\Big(Ad\, g_2.\theta^L_{g_1}(\zeta_1) \,|\, \theta^R_{g_2}(\eta_2)\Big) \quad (21)$$

$$-\frac{1}{2}\Big(Ad\, g_2 Ad\, g_1^{-1}.\theta^L_{g_2}(\eta_2) \,|\, \theta^R_{g_2}(\zeta_2)\Big) + \frac{1}{2}\Big(Ad\, g_2.\theta^L_{g_2}(\eta_2) \,|\, \theta^R_{g_2}(\zeta_2)\Big) - \frac{1}{2}\Big(Ad\, g_2.\theta^L_{g_1}(\eta_1) \,|\, \theta^R_{g_2}(\zeta_2)\Big)$$

$$= \underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\zeta_2) \,|\, Ad\, g_1.\theta^L_{g_2}(\eta_2)\Big)}_{(3')} - \underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\zeta_2) \,|\, \theta^L_{g_2}(\eta_2)\Big)}_{(4)} + \underbrace{\frac{1}{2}\Big(\theta^L_{g_1}(\zeta_1) \,|\, \theta^L_{g_2}(\eta_2)\Big)}_{(2')} \quad (22)$$

$$-\underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\eta_2) \,|\, Ad\, g_1.\theta^L_{g_2}(\zeta_2)\Big)}_{(3')} + \underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\eta_2) \,|\, \theta^L_{g_2}(\zeta_2)\Big)}_{(4')} - \underbrace{\frac{1}{2}\Big(\theta^L_{g_1}(\eta_1) \,|\, \theta^L_{g_2}(\zeta_2)\Big)}_{(1')}$$

(to obtain this last expression, one uses the Ad-invariance of $(.\,|\,.)$ and the fact that $Ad\, g_2^{-1} \circ \theta^R_{g_2} = \theta^L_{g_2}$). Observe that (4) and (4′) cancel in the above expression. Likewise, (1′), (2′) and (3′) in (22) cancel respectively with (1), (2) in (11) and (12) and with (15) when computing the sum $\beta^*(\omega_1 \oplus \omega_2) + \beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$. The non-vanishing terms in this sum are therefore (A) and (B) from (6) and (C) and (D) from (11) and (12), so that :

$$\begin{aligned}
& (\beta^*\omega)_x(v,w) && (23)\\
= \;& \big(\beta^*(\omega_1 \oplus \omega_2)\big)_x(v,w) + \big(\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)\big)_x(v,w) && (24)\\
= \;& (A) + (B) + (C) + (D) && (25)\\
= \;& -(\omega_1 \oplus \omega_2)_x(v,w) - \frac{1}{2}\Big(\big(\theta^R_{g_1}(\zeta_1) \,|\, \theta^L_{g_2}(\eta_2)\big) - \big(\theta^R_{g_1}(\eta_1) \,|\, \theta^L_{g_2}(\zeta_2)\big)\Big) && (26)
\end{aligned}$$

But $\mu_i \circ \beta_i = \tau^- \circ \mu_i$, so that :

$$\begin{aligned}
\big(\theta^R_{g_1}(\zeta_1) \,|\, \theta^L_{g_2}(\eta_2)\big) &= \big(\theta^R_{\mu_1\circ\beta_1(x_1)}(T_{x_1}(\mu_1 \circ \beta_1).v_1) \,|\, \theta^L_{\mu_2\circ\beta_2(x_2)}(T_{x_2}(\mu_2 \circ \beta_2).w_2)\big) && (27)\\
&= \big(\theta^R_{\tau^-\circ\mu_1(x_1)}(T_{x_1}(\tau^- \circ \mu_1).v_1) \,|\, \theta^L_{\tau^-\circ\mu_2(x_2)}(T_{x_2}(\tau^- \circ \mu_2).w_2)\big) && (28)
\end{aligned}$$

and $\tau^- = Inv \circ \tau$, where $Inv : u \mapsto u^{-1}$ is inversion on U, so $T_u\tau^-.\xi = -\tau^-(u).(T_u\tau.\xi).\tau^-(u)$. Hence :

$$\begin{aligned}
\theta^R_{\tau^-(u)}(T_u\tau^-.\xi) &= \theta^R_{\tau^-(u)}\big(-\tau^-(u).(T_u\tau.\xi).\tau^-(u)\big)\\
&= -\tau^-(u).(T_u\tau.\xi)\\
&= -\theta^L_{\tau(u)}(T_u\tau.\xi)
\end{aligned}$$

(and likewise θ^L changes into θ^R). Since in addition to that τ is a group automorphism and an isometry for $(.\,|\,.)$, the expression (28) becomes :

$$\begin{aligned}
(28) \quad &= \Big(\theta^L_{\tau\left(\mu_1(x_1)\right)}\big(T_{\mu_1(x_1)}\tau.(T_{x_1}\mu_1.v_1)\big) \,|\, \theta^R_{\tau\left(\mu_2(x_2)\right)}\big(T_{\mu_2(x_2)}\tau.(T_{x_2}\mu_2.w_2)\big)\Big)\\
&= \Big(T_1\tau.\big(\theta^L_{\mu_1(x_1)}(T_{x_1}\mu_1.v_1)\big) \,|\, T_1\tau.\big(\theta^R_{\mu_2(x_2)}(T_{x_2}\mu_2.w_2)\big)\Big)\\
&= \big(\theta^L_{\mu_1(x_1)}(T_{x_1}\mu_1.v_1) \,|\, \theta^R_{\mu_2(x_2)}(T_{x_2}\mu_2.w_2)\big)\\
&= \big((\mu_1^*\theta^L)_{x_1}(v_1) \,|\, (\mu_2^*\theta^R)_{x_2}(w_2)\big)
\end{aligned}$$

so that we have :

$$\begin{aligned}
(\beta^*\omega)_x(v,w) &= (26)\\
&= -(\omega_1\oplus\omega_2)_x(v,w)\\
&\quad -\frac{1}{2}\Big(\big((\mu_1^*\theta^L)_{x_1}(v_1)\,|\,(\mu_2^*\theta^R)_{x_2}(w_2)\big)-\big((\mu_1^*\theta^L)_{x_1}(w_1)\,|\,(\mu_2^*\theta^R)_{x_2}(v_2)\big)\Big)\\
&= -(\omega_1\oplus\omega_2)_x(v,w)-(\mu_1^*\theta^L\wedge\mu_2^*\theta^R)_x(v,w)\\
&= -\omega_x(v,w)
\end{aligned}$$

which completes the proof of lemma 6.14. □

Remark. As a last comment on theorem 6.12, we would like to say that even if we drop the assumption on the conjugacy classes, the set of equivalence clases of Lagrangian representations is still a Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}}$. Indeed, it is always contained in $Fix(\hat{\beta})$, and therefore it is isotropic, and its dimension is half the dimension of $\mathcal{M}_{\mathcal{C}}$ (see [FW06]). With our hypothesis on the $\mathcal{C}_j$, the upshot is that we are able to show that this Lagrangian submanifold is exactly $Fix(\hat{\beta})$.

The main tool to obtain theorem 6.12 was theorem 5.2, which is very general. It may for instance help to find Lagrangian submanifolds in the moduli space of polygons in S^3, which also admits a quasi-Hamiltonian description (see [Tre02]). In fact, in [Tre02], the symplectic structure of the moduli space of polygons with fixed sidelengths in $S^3 \simeq SU(2)$ is obtained by reduction from the quasi-Hamiltonian space $\mathcal{C}_1\times\dots\times\mathcal{C}_l$ where $\mathcal{C}_j$ is a conjugacy class in $SU(2)$, so that our involution β can be defined in this context. In analogy with results in [FH05], the fixed-point set of this involution should consist of polygons in S^3 which are contained in the equatorial $S^2\subset S^3$ (I would like to thank Philip Foth for suggesting this to me).

6.8 The case of an arbitrary compact connected Lie group

To conclude, we wish to explain, using the description of $\widehat{U(n)}$ given in section 2, how to make the notion of Lagrangian representation make sense when the compact connected Lie group U at hand is not necessarily the unitary group $U(n)$. We suppose that such a group U is endowed with an involution τ leaving a *maximal* torus of U pointwise fixed (for instance the Cartan involution defining its non-compact dual), and we define an action of $\mathbb{Z}/2\mathbb{Z}=\{1,\sigma_0\}$ on U by $\sigma_0.u=\tau(u)$. We then consider the semi-direct product $U\rtimes\mathbb{Z}/2\mathbb{Z}$ for this action. Recall that if $U=U(n)$ and $\tau(u)=\overline{u}$ then $U(n)\rtimes\mathbb{Z}/2\mathbb{Z}=\widehat{U(n)}=U(n)\sqcup U(n)\sigma_{L_0}$. Under this identification, $\sigma_0.u=\tau(u)=\overline{u}=\sigma_{L_0}u\sigma_{L_0}$ and the Lagrangian involutions are the elements $\sigma_L=\sigma_{\varphi(L_0)}=\varphi\sigma_{L_0}\varphi^{-1}=(\varphi\sigma_{L_0}\varphi^{-1}\sigma_{L_0})\sigma_{L_0}=(\varphi\overline{\varphi}^{-1})\sigma_{L_0}=(\varphi\varphi^t)\sigma_{L_0}\leftrightarrow(\varphi\varphi^t,\sigma_{L_0})\in U(n)\rtimes\mathbb{Z}/2\mathbb{Z}$. Observe that the element $\varphi\varphi^t$ does not depend on the choice of $\varphi\in U(n)$ such that $L=\varphi(L_0)$, as was shown in proposition 2.3. Thus, we see again that the elements of order 2 that we are interested in are in one-to-one correspondence with the symmetric elements of $U(n)$. In the general case, the elements of order 2 that we are interested in are the elements $(w,\sigma_0)\in U\rtimes\mathbb{Z}/2\mathbb{Z}$ where $w\in U$ satisfies $\tau(w)=w^{-1}$. The product of two such elements is then of the form $(w_1,\sigma_0).(w_2,\sigma_0)=(w_1(\sigma_0.w_2),\sigma_0^2)=(w_1\tau(w_2),1)\in U\subset U\rtimes\mathbb{Z}/2\mathbb{Z}$ (observe that when $w_2=w_1$, we indeed obtain 1 because $\tau(w_1)=w_1^{-1}$). One can then say that a U-representation $(u_1,\dots,u_l)$ of $\pi=\pi_1(S^2\backslash\{s_1,\dots,s_l\})$ is *decomposable* (or *Lagrangian*) if there exist $w_1,\dots,w_l\in U$ such that $\tau(w_j)=w_j^{-1}$ for all j and $u_1=(w_1,\sigma_0).(w_2,\sigma_0), u_2=(w_2,\sigma_0).(w_3,\sigma_0),\dots,u_l=(w_l,\sigma_0).(w_1,\sigma_0)$. Observe that we then have indeed $u_1\dots u_l=1$, for $u_1\dots u_l=(w_1\tau(w_2),1).(w_2\tau(w_3),1)\dots(w_l\tau(w_1),1)=(w_1\tau(w_2)w_2\tau(w_3)\dots w_l\tau(w_1),1)=1$ since $\tau(w_j)=w_j^{-1}$. A representation will be called σ_0-*decomposable* if it is decomposable with $w_1=Id$. Then, theorem 6.10 and corollary 6.11, along with theorem 6.12 are still true in this setting (the condition on the eigenvalues of some $\mathcal{C}_j$ to be pairwise distinct is to be replaced by the condition that the centralizer Z_u of any $u\in\mathcal{C}_j$ is a maximal torus of U, and therefore conjugate to a maximal torus fixed pointwise by τ^-). All one has to do is then define β as in (1) in subsection 6.5 (that is, replace u^t by $\tau(u^{-1})$ in the definition of β given in subsection 6.6) : the σ_0-decomposable representations are exactly the elements of the fixed-point set of β, a given representation u is decomposable if and only if $\beta(u)$ is equivalent to u, and the set of equivalence classes of decomposable representations is a Lagrangian submanifold of $\mathrm{Hom}_{\mathcal{C}}(\pi,U)/U$, obtained as the fixed-point set of an antisymplectic involution $\hat{\beta}$. We refer to [Sch05] for further details in that direction.

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