DIFFERENTIAL GEOMETRY OF HOLOMORPHIC VECTOR BUNDLES ON A CURVE

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Abstract. These notes are based on a series of five lectures given at the 2009 *Villa de Leyva Summer School on Geometric and Topological Methods for Quantum Field Theory.* The purpose of the lectures was to give an introduction to differential-geometric methods in the study of holomorphic vector bundles on a compact connected Riemann surface, as examplified in the celebrated paper of Atiyah and Bott ([AB83]). In these notes, we take a rather informal point of view and try to paint a global picture of the various notions that come into play in that study, setting Donaldson's theorem on stable holomorphic vector bundles ([Don83]) as a goal for the lectures.

Contents

1. Holomorphic vector bundles on Riemann surfaces

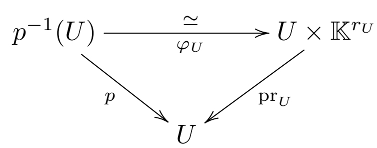
1.1. Vector bundles.

1.1.1. *Definition.* We begin by recalling the definition of a vector bundle. Standard references for the general theory of fibre bundles are the books of Steenrod ([Ste51]) and Husemoller ([Hus93]).

Definition 1.1 (Vector bundle). Let X be a topological space and let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. A **topological $\mathbb{K}$-vector bundle on** X is a continuous map $p : E \longrightarrow X$ such that

(1) $\forall x \in X$, the fibre $E_x := p^{-1}(\{x\})$ is a finite-dimensional $\mathbb{K}$-vector space,

(2) $\forall x \in X$, there exists an open neighbourhood U of x in X, an integer $r_U \geq 0$, and a homeomorphism φ_U such that the diagramme



is commutative (pr_U denotes the projection onto U),

(3) the induced homeomorphism $\varphi_x : p^{-1}(\{x\}) \longrightarrow \{x\} \times \mathbb{K}^{r_U}$ is a $\mathbb{K}$-linear isomorphism.

Most of the times, the map p is understood, and we simply denote E a vector bundle on X. If $\mathbb{K} = \mathbb{R}$, E is called a (topological) *real* vector bundle, and if $\mathbb{K} = \mathbb{C}$, it is called a (topological) *complex* vector bundle. $p^{-1}(\{U\})$ is also denoted $E|_U$ and φ_U is called a *local trivialisation* of E over U. $E|_U$ is itself a vector bundle (on U). The open set $U \subset X$ is said to be trivialising for E, and the pair (U, φ_U) is called a (bundle) *chart*.

Examples 1. The following maps are examples of vector bundles.

(1) The *product bundle* $p : X \times \mathbb{K}^r \longrightarrow X$, where p is the projection onto X.
(2) The tangent bundle $TM \longrightarrow M$ to a differentiable manifold M.
(3) The Möbius bundle on S^1: let $\mathcal{M}$ be the quotient of $[0;1] \times \mathbb{R}$ under the identifications $(0,t) \sim (1,-t)$, with projection map $p : \mathcal{M} \longrightarrow S^1$ induced by the canonical projection $[0;1] \times \mathbb{R} \longrightarrow [0;1]$. Observe that $\mathcal{M}$ is indeed homeomorphic to a Möbius band without its boundary circle.
(4) The canonical line bundle on the n-dimensional projective space $\mathbb{R}\mathbf{P}^n$ (=the space of lines in $\mathbb{R}^{n+1}$):
$$E_{\mathrm{can}} := \{(\ell, v) \in \mathbb{R}\mathbf{P}^n \times \mathbb{R}^{n+1} \mid v \in \ell\}$$
with projection map $p(\ell, v) = l$. The fibre of p above ℓ is canonically identified with ℓ. When $n = 1$, the bundle E_{can} will be shown later to be isomorphic to the Möbius bundle (Exercise 1.2). The same example works with $\mathbb{C}\mathbf{P}^n$ in place of $\mathbb{R}\mathbf{P}^n$.
(5) The Grassmannian of k-dimensional complex sub-spaces of $\mathbb{C}^{n+1}$, denoted $\mathrm{Gr}_k(\mathbb{C}^{n+1})$, has a complex vector bundle structure with k-dimensional fibres:
$$E_{\mathrm{can}} = \{(F, v) \in \mathrm{Gr}_k(\mathbb{C}^{n+1}) \times \mathbb{C}^{n+1} \mid v \in F\}$$
with projection map $p(F, v) = F$. The fibre of p above F is canonically identified with F. The same example works with $\mathbb{R}^{n+1}$ in place of $\mathbb{C}^{n+1}$.

It follows from the definition of a vector bundle that the $\mathbb{Z}_+$-valued map
$$x \longmapsto \mathrm{rk}\, p^{-1}(\{x\})$$
(called the *rank function*) is a locally constant, integer-valued function on X (i.e. an element of $\check{H}^0(X; \underline{\mathbb{Z}})$). In particular, if X is connected, it is a constant map, i.e. an integer.

Definition 1.2 (Rank of a vector bundle)**.** Let X be a connected topological space. The **rank** of a $\mathbb{K}$-vector bundle $p : E \longrightarrow X$ is the dimension of the $\mathbb{K}$-vector space $p^{-1}(\{x\})$, for any $x \in X$. It is denoted $\mathrm{rk}\, E$. A vector bundle of rank 1 is called a **line bundle**.

Definition 1.3 (Homomorphism of vector bundles)**.** A **homomorphism**, or simply morphism, between two $\mathbb{K}$-vector bundles $p : E \longrightarrow X$ and $p' : E' \longrightarrow X'$ is a pair (u, f) of continuous maps $u : E \longrightarrow E'$ and $f : X \longrightarrow X'$ such that

(1) the diagramme

$$\begin{array}{ccc} E & \xrightarrow[u]{} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

is commutative,

(2) for all $x \in X$, the map

$$u_x : p^{-1}(\{x\}) \longrightarrow (p')^{-1}(\{f(x)\})$$

is $\mathbb{K}$-linear.

Topological vector bundles together with their homomorphisms form a category that we denote $\mathrm{Vect}^{\mathrm{top}}$. If X is a fixed topological space, there is a category $\mathrm{Vect}_X^{\mathrm{top}}$ whose objects are topological vector bundles on X and whose morphisms are defined as follows.

Definition 1.4 (Homomorphisms of vector bundles on X)**.** Let X be a fixed topological space and let $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$ be two $\mathbb{K}$-vector bundles on X. A **morphism of $\mathbb{K}$-vector bundles on** X is a continous map $u : E \longrightarrow E'$ such that

(1) the diagramme

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\varphi_U]{\simeq} & U \times \mathbb{K}^{r_U} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

is commutative,

(2) for all $x \in x$, the map

$$u_x : p^{-1}(\{x\}) \longrightarrow (p')^{-1}(\{f(x)\})$$

is $\mathbb{K}$-linear.

As usual, an isomorphism is a homomorphism which admits an inverse homomorphism. A $\mathbb{K}$-vector bundle isomorphic to a product bundle is called a *trivial bundle*.

When the base space X is a smooth manifold, one defines smooth vector bundles on X using smooth maps in place of continuous ones, and when X is a complex analytic manifold manifold, one may accordingly define holomorphic vector bundles on X (this last notion only makes sense, of course, if the field $\mathbb{K}$ in Definition 1.1 is assumed to be the field of complex numbers).

1.1.2. *Transition maps.* We shall henceforth assume that X is connected. Let E be a $\mathbb{K}$-vector bundle on X, and denote $r = \mathrm{rk}\, E$. If (U, φ_U) and (V, φ_V) are two overlapping charts in the sense that $U \cap V \neq \emptyset$, one gets a map

$$\varphi_U \circ \varphi_V^{-1} : \begin{array}{ccc} (U\cap V)\times \mathbb{K}^r & \longrightarrow & (U\cap V)\times\mathbb{K}^r \\ (x,v) & \longmapsto & (x, g_{UV}(x)\cdot v) \end{array}$$

where $g_{UV} : U\cap V \longrightarrow \mathrm{Aut}(\mathbb{K}^r) = \mathbf{GL}(r,\mathbb{K})$. It satisfies, for any triple of open sets (U,V,W),

$$g_{UV}g_{VW} = g_{UW} \tag{1.1}$$

(the product on the left-hand side being the pointwise product of $\mathbf{GL}(r,\mathbb{K})$-valued functions). Setting $U=V=W$, we obtain $(g_{UU})^2 = g_{UU}$, so

$$g_{UU} = I_r$$

(the constant map equal to I_r). This in turn implies that $g_{UV}g_{VU} = g_{UU} = \mathrm{I}_r$, so

$$g_{VU} = g_{UV}^{-1}$$

(the map taking x to $(g_{UV}(x))^{-1} \in \mathbf{GL}(r,\mathbb{K})$). The condition (1.1) is called a *cocycle condition*. If $(U_i)_{i\in I}$ is a covering of X by trivialising open sets, with associated local trivialisations $(\varphi_i)_{i\in I}$, we get a family

$$g_{ij} : U_i \cap U_j \longrightarrow \mathbf{GL}(r,\mathbb{K})$$

of maps satisying condition (1.1): the family $(g_{ij})_{(i,j)\in I\times I}$ is called a $\mathbf{GL}(r,\mathbb{K})$-valued 1-*cocycle* subordinate to the open covering $(U_i)_{i\in I}$. It is completely determined by the transition maps $(\varphi_i\circ\varphi_j^{-1})_{(i,j)\in I\times I}$. Conversely, such a cocycle defines a topological $\mathbb{K}$-vector bundle of rank r

$$E := \left(\bigcup_{i\in I} \{i\}\times U_i \times \mathbb{K}^r\right) \Big/ \sim\ , \tag{1.2}$$

where the equivalence relation $\sim$ identifies (i,x,v) and (j,y,w) if $y=x$ (in particular, $U_i\cap U_j \neq \emptyset$) and $w = g_{ij}(x)\cdot v$, the projection map $p : E \longrightarrow X$ being induced by the projections maps $\{i\}\times U_i\times\mathbb{K}^r \longrightarrow U_i \subset X$. In other words, a $\mathbb{K}$-vector bundle of rank r on X is a fibre bundle with typical fibre $\mathbb{K}^r$ and structure group $\mathbf{GL}(r,\mathbb{K})$, acting on $\mathbb{K}^r$ by linear transformations (see for instance [Ste51]). Two vector bundles E and E' on X, represented by two cocycles $(g_{ij})_{(i,j)}$ and $(g'_{ij})_{(i,j)}$ subordinate to a same open covering $(U_i)_{i\in I}$ are isomorphic if and only if there exists a family

$$u_i\ :\ U_i \longrightarrow \mathbf{GL}(r,\mathbb{K})$$

of maps satisfying

$$g'_{ij} = u_i g_{ij} u_j^{-1}.$$

Indeed, simply define u_i in the following way

$$U_i\times\mathbb{K}^r \underset{\varphi_i^{-1}}{\longrightarrow} E|_{U_i} \underset{u}{\longrightarrow} E'|_{U_i} \underset{\varphi'_i}{\longrightarrow} U_i\times\mathbb{K}^r\ ,$$

the map taking (x,v) to $(x, u_i(x)\cdot v)$, and check that, for all $x\in U_i\cap U_j$, $u_i(x) = g'_{ij}(x)u_j(x)g_{ij}^{-1}(x)$. This defines an equivalence relation on the set of $\mathbf{GL}(r,\mathbb{K})$-valued 1-cocycles subordinate to a given open covering $\mathcal{U} = (U_i)_{i\in I}$ of X. The set of equivalence classes for this relations is usually denoted

$$\check{H}^1_{\mathrm{top}}(\mathcal{U};\mathbf{GL}(r,\mathbb{K})).$$

These sets form a direct system relative to the operation of passing from an open covering of X to a finer one, and the direct limit is denoted

$$\check{H}^1_{\mathrm{top}}(X;\mathbf{GL}(r,\mathbb{K})) := \varinjlim_{\mathcal{U}} \check{H}^1_{\mathrm{top}}(\mathcal{U};\mathbf{GL}(r,\mathbb{K}))$$

(see for instance [Gun66]). *This set is the set of isomorphism classes of topological* $\mathbb{K}$*-vector bundles on* X (if X is not connected, $\check{H}^1(X;\mathbf{GL}(r,\mathbb{K}))$ is the disjoint union $\sqcup_{i=1}^k \check{H}^1(X_i\,;\mathbf{GL}(r,\mathbb{K}))$, where $\sqcup_{i=1}^k X_i$ is the disjoint union of connected components of X). If one considers smooth 1-cocycles instead of continuous ones, one obtains a similar description of smooth vector bundles on X. Likewise, if X is a Riemann surface, a holomorphic vector bundle of rank r on X is represented by a holomorphic 1-cocycle

$$g_{ij} : U_i \cap U_j \longrightarrow \mathbf{GL}(r,\mathbb{C})$$

in the sense that all the components of g_{ij} are holomorphic functions of one variable. An automorphism of a topological (resp. smooth, resp. holomorphic) vector bundle E on X represented by the cocycle $(g_{ij})_{(i,j)}$ may be represented by a family $(u_i : U_i \longrightarrow \mathbf{GL}(r,\mathbb{K}))_i$ of continuous (resp. smooth, resp. holomorphic) maps satisfying $u_i g_{ij} = g_{ij} u_j$ for all $(i,j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

1.1.3. *Sections of a bundle.* Sections of a bundle are a generalisation of mappings between two spaces X and Y in the sense that a map from X to Y is a section of the product bundle $X \times Y \longrightarrow X$.

Definition 1.5 (Sections of a vector bundle)**.** A (global) **section** of a topological $\mathbb{K}$-vector bundle $p : E \longrightarrow X$ is a continuous map $s : X \longrightarrow E$ such that $p \circ s = \mathrm{Id}_X$. The set $\Gamma(E)$ of global sections of E is an infinite-dimensional $\mathbb{K}$-vector space and a module over the ring of $\mathbb{K}$-valued functions on X.

Local sections of E are sections $s_U : U \longrightarrow E|_U \simeq U \times \mathbb{K}^r$ of the vector bundle $E|_U$ where $U \subset X$ is an open subset. They may be seen as maps from U to $\mathbb{K}^r$. If $(g_{UV})_{(U,V)}$ is a 1-cocycle representing the vector bundle E, then a global section $s : X \longrightarrow E$ is the same as a collection $(s_U)_U$ of local sections subject to the condition

$$s_U = g_{UV} s_V$$

for any pair (U,V) of open subsets of X satisfying $U \cap V \neq \emptyset$. Smooth (resp. holomorphic) sections of a smooth (resp. holomorphic) vector bundles are defined accordingly.

Example 1.6. A section of the vector bundle $TM \longrightarrow M$ is a *vector field* on M. A section of $T^*M \longrightarrow M$ is a *differential* 1*-form* on M.

1.2. **Topological classification.** Evidently, if two $\mathbb{K}$-vector bundles on X are isomorphic, they have the same rank (=dimension over $\mathbb{K}$ of the typical fibre of E). If $\mathbb{K} = \mathbb{C}$ and $X = \Sigma_g$ is a compact connected oriented *surface*, isomorphism classes of *topological* or *smooth* vector bundles on X are completely classified by a pair (r,d) of integers, namely the rank and the *degree* of a complex vector bundle. We give the following definition, which makes a free use of the notion of *Chern class* of a complex vector bundle (see for instance [BT82] or [Hat02]).

Definition 1.7 (Degree)**.** Let E be a complex vector bundle on a compact connected oriented surface Σ_g. The *degree* of E is by definition the integral of the first Chern class $c_1(E) \in H^2(\Sigma_g;\mathbb{Z})$ of E:

$$deg(E) := \int_{\Sigma_g} c_1(E) \in \mathbb{Z}.$$

The degree of a vector bundle satisfies the following relations, which are often useful in computations:

$$deg(E^*) = -deg(E) \quad \text{and} \quad deg(E_1 \otimes E_2) = deg(E_1)\,\mathrm{rk}\,(E_2) + \mathrm{rk}\,(E_1)\,deg(E_2).$$

Theorem 1.8 (Topological classification of complex vector bundles on a curve)**.** *Let Σ_g be a compact connected Riemann surface, and let E, E' be two topological (resp. smooth) complex vector bundles on Σ_g. Denote $r = \mathrm{rk}\,(E)$, $r' = \mathrm{rk}\,(E')$, $d = deg(E)$ and $d' = deg(E')$. Then E and E' are isomorphic as topological (resp. smooth) complex vector bundles on Σ_g if and only if $r = r'$ and $d = d'$. Moreover, for any pair $(r,d) \in \mathbb{Z}_+ \times \mathbb{Z}$, there exists a complex vector bundle of rank r and degree d on Σ_g.*

We refer for instance to [Tha97] for a proof of this theorem. In Section 3, we shall be interested in the (much more involved) classification problem for *holomorphic* vector bundles on a compact connected Riemann surface Σ_g, and the preceding result will be used to reduce it to the study of *holomorphic structures* on a given smooth complex vector bundle of *topological type* (r,d).

1.3. **Dolbeault operators and the space of holomorphic structures.** In this subsection, we only consider smooth complex vector bundles over a fixed Riemann surface Σ (although, most of the time, a similar, albeit slightly more complicated, statement holds for complex vector bundles over a higher-dimensional complex analytic manifold, see for instance [Kob87] or [Wel08] for more on this topic).

1.3.1. *Smooth complex vector bundles and their sections.* A holomorphic structure on a topological complex vector bundle is by definition a (maximal) holomorphic atlas on it (local trivialisations with holomorphic transition maps). Such a holomorphic structure defines (up to conjugation by an automorphism of the bundle) a remarkable object on the underlying *smooth* complex vector bundle: a Dolbeault operator.

In what follows, we will denote E a smooth complex vector bundle on Σ. When E is endowed with a holomorphic structure, we will designate by $\mathcal{E}$ the resulting holomorphic vector bundle. We denote $\Omega^0(\Sigma;E) = \Gamma(E)$ the complex vector space of smooth sections of E, and $\Omega^k(\Sigma;E)$ the complex vector space of E-valued, smooth, $\mathbb{R}$-linear k-forms on Σ. For any $k \geq 0$, $\Omega^k(\Sigma;E)$ is also a module over the ring $\Omega^0(\Sigma;\mathbb{C}) = C^\infty(\Sigma;\mathbb{C})$ of $\mathbb{C}$-valued smooth functions on Σ. It is important to stress that, for $k \geq 1$, $\Omega^k(\Sigma;E)$ is the space of smooth sections of the complex vector bundle $\wedge^k(T^*\Sigma) \otimes_{\mathbb{R}} E$. It is a complex vector space because the fibres of E are complex vector spaces, but a single element $\omega \in \Omega^k(\Sigma;E)$, when evaluated at a point $x \in \Sigma$, defines an $\mathbb{R}$-*linear* map

$$\omega_x : T_x\Sigma \wedge \cdots \wedge T_x\Sigma \longrightarrow E_x.$$

In particular, we do not restrict our attention to $\mathbb{C}$-linear such maps. Instead, we write, for instance if $k = 1$,

$$\Omega^1(\Sigma;E) = \Omega^{1,0}(\Sigma;E) \oplus \Omega^{0,1}(\Sigma;E)$$

where $\Omega^{1,0}(\Sigma; E)$ is the complex vector space of $\mathbb{C}$-linear 1-forms

$$\omega : T\Sigma \longrightarrow E$$

and $\Omega^{0,1}(\Sigma; E)$ is the complex vector space of $\mathbb{C}$-antilinear such forms. We recall that $\Omega^k(\Sigma; \mathbb{C}) = \Omega^k(\Sigma; E)$ for $E = \Sigma \times \mathbb{C}$, so the remark above is in particular valid for $\Omega^1(\Sigma; \mathbb{C})$. For $k > 1$ (and over a complex analytic manifold M of arbitrary dimension), we would have a decomposition

$$\Omega^k(M; E) = \Omega^{k,0}(M; E) \oplus \Omega^{k-1,1}(M; E) \oplus \cdots \oplus \Omega^{1,k-1}(M; E) \oplus \Omega^{0,k}(M; E)$$

where $\Omega^{p,q}(M; E)$ is the space of $\mathbb{R}$-linear $(p+q)$-forms

$$\omega : TM \wedge \cdots \wedge TM \longrightarrow E$$

which are $\mathbb{C}$-linear in p arguments and $\mathbb{C}$-antilinear in q arguments. A consequence of these decompositions is that the de Rham operator d on Σ splits into

$$d = d^{1,0} \oplus d^{0,1} : \Omega^0(\Sigma; \mathbb{C}) \longrightarrow \Omega^1(\Sigma; \mathbb{C}) = \Omega^{1,0}(\Sigma; \mathbb{C}) \oplus \Omega^{0,1}(\Sigma; \mathbb{C}).$$

That is, the exterior derivative df of a smooth function $f : \Sigma \longrightarrow \mathbb{C}$ splits into a $\mathbb{C}$-linear part $d^{1,0}f$ (also denoted ∂f) and a $\mathbb{C}$-antilinear part $d^{0,1}f$ (also denoted $\overline{\partial} f$).

Lemma 1.9. *A smooth function $f : \Sigma \longrightarrow \mathbb{C}$ is holomorphic if and only if $\overline{\partial} f = 0$.*

Proof. It is a question of a purely local nature. In a holomorphic chart $z = x + iy$ of Σ, one has $df = \partial f dz + \overline{\partial} f d\overline{z}$ and $\overline{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$. Write now $f = P + iQ$ with P and Q real-valued. Then $\overline{\partial} f = 0$ if and only if $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ and $\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$. These are the Cauchy-Riemann equations: they mean that the Jacobian matrix of f at any given point is a similitude matrix, i.e. a complex number, which amounts to saying that f is holomorphic. □

Definition 1.10 (Cauchy-Riemann operator)**.** The operator

$$\overline{\partial} : \Omega^0(\Sigma, \mathbb{C}) \longrightarrow \Omega^{0,1}(\Sigma; \mathbb{C})$$

taking a smooth function to the $\mathbb{C}$-antilinear part of its derivative is called the **Cauchy-Riemann operator** of the Riemann surface Σ.

Observe that the Cauchy-Riemann operator satisfies the Leibniz rule

$$\overline{\partial}(fg) = (\overline{\partial} f)g + f(\overline{\partial} g).$$

We would like to have a similar characterisation for the holomorphic *sections* of an arbitrary holomorphic bundle $\mathcal{E}$ (not just $\Sigma \times \mathbb{C}$). The problem is that there is no canonically defined operator

$$D : \Omega^0(\Sigma; E) \longrightarrow \Omega^1(\Sigma; E)$$

which would play the role of the de Rham operator, so we first need to define these. Recall that the de Rham operator satisfies the Leibniz rule

$$d(fg) = (df)g + f(dg).$$

The next object, called a (linear) connection, gives a way to differentiate sections of a vector bundle E *covariantly* (in such a way that the resulting object is an E-valued 1-form, thus generalising the de Rham operator). To give a presentation of connections of a broader interest, we temporarily move back to manifolds more general than Riemann surfaces. The next definition is a bit abstract but designed to incorporate the case, for instance, of smooth complex vector bundles over real smooth manifolds.

Definition 1.11 (Linear connection)**.** Let M be a smooth manifold over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A (linear) **connection** on a smooth vector bundle $E \longrightarrow M$ is a $\mathbb{K}$-linear map

$$D : \Omega^0(M; E) \longrightarrow \Omega^1(M; E)$$

satisfying the following Leibniz rule

$$D(fs) = (df)s + f(Ds)$$

for all $f \in C^\infty(M; \mathbb{K})$ and all $s \in \Omega^0(M; E)$, where d is the de Rham operator on M.

$(df)s$ is the element of $\Omega^1(M; E)$ which, when evaluated at $x \in M$, is the $\mathbb{R}$-linear (*even, and this is crucial, if* $\mathbb{K} = \mathbb{C}$) map

$$\begin{array}{ccc} T_xM & \longrightarrow & E_x \\ v & \longmapsto & \underbrace{(df)_x(v)}_{\in \mathbb{K}} \cdot \underbrace{s(x)}_{\in E_x} \end{array} .$$

fDs is the element of $\Omega^1(M; E)$ which, when evaluated at $x \in M$, is the $\mathbb{R}$-linear map $f(x)(Ds)_x$. Given a section $s \in \Omega^0(M; E)$, the E-valued 1-form Ds is called the *covariant derivative* of s. We observe that the product bundle $U \times \mathbb{K}^r$ always admits a distinguished connection, called the product connection: since elements of $\Omega^1(U; U \times \mathbb{K}^r) \simeq \Omega^1(U; \mathbb{K}^r)$ are r-uplets of k-forms on U, one may define $D = d \oplus \cdots \oplus d$, the de Rham operator repeated r times. One can moreover show that any convex combination of linear connections is a linear connection. It is then a simple consequence of the existence of partitions of unity that any smooth vector bundle admits a connection (see Exercise 1.4). As we now show, the space of all connections is an affine space.

Proposition 1.12. *The space of all linear connections on a smooth* $\mathbb{K}$*-vector bundle is an affine space, whose group of translations is the vector space* $\Omega^1(\Sigma; \mathrm{End}\,(E))$.

Proof. It suffices to show that the difference $D_1 - D_2$ of two linear connections defines an element of $\Omega^1(M; E)$. One has, for all $f \in C^\infty(M; \mathbb{K})$ and all $s \in \Omega^0(M; E)$,

$$\begin{aligned} (D_1 - D_2)(fs) &= D_1(fs) - D_2(fs) \\ &= (df)s + f(D_1 s) - (df)s - f(D_2 s) \\ &= f(D_1 - D_2)s. \end{aligned}$$

So $D_1 - D_2$ is a $C^\infty(M; \mathbb{K})$-linear map from $\Omega^0(M; E)$ to $\Omega^1(M; E)$. This is the same as an $\mathrm{End}\,(E)$-valued 1-form on M. ☐

We postpone the exposition of further generalities on linear connections (curvature and the like) to Subsection 2.1, where they will be presented for a special case of linear connections called unitary connections (but seen to hold in greater generality), and we go back to generalised Cauchy-Riemann generators.

A linear connection on a smooth complex vector bundle $E \longrightarrow \Sigma$ over a Riemann surface splits into

$$D = D^{1,0} \oplus D^{0,1} \, : \, \Omega^0(\Sigma; E) \longrightarrow \Omega^1(\Sigma; E) = \Omega^{1,0}(\Sigma; E) \oplus \Omega^{0,1}(\Sigma; E).$$

Lemma 1.13. *Let D be a linear connection on $E \longrightarrow \Sigma$. The operator*

$$D^{0,1} : \Omega^0(\Sigma; E) \longrightarrow \Omega^{0,1}(\Sigma; E)$$

taking a section of E to the $\mathbb{C}$-antilinear part of its covariant derivative is $\mathbb{C}$-linear and satisfies the following Leibniz rule

$$D^{0,1}(fs) = (\overline{\partial} f)s + f(D^{0,1}s),$$

where $\overline{\partial}$ is the Cauchy-Riemann operator on Σ.

Proof. $D^{0,1}$ is obviously additive. Moreover,

$$\begin{aligned} D(fs) &= (df)s + f(Ds) \\ &= (\partial f)s + f(D^{1,0}s) + (\overline{\partial} f)s + f(D^{0,1}s) \end{aligned}$$

so the $\mathbb{C}$-antilinear part of $D(fs)$ is $(\overline{\partial} f)s + f(D^{0,1}s)$. □

This motivates the following definition.

Definition 1.14 (Dolbeault operator)**.** A Dolbeault operator on a smooth complex vector bundle $E \longrightarrow \Sigma$ over a Riemann surface is a $\mathbb{C}$-linear map

$$D'' : \Omega^0(\Sigma; E) \longrightarrow \Omega^{0,1}(\Sigma; E)$$

satisfying the following Leibniz rule: for all $f \in C^\infty(\Sigma; \mathbb{C})$ and all $s \in \Omega^0(\Sigma; E)$,

$$D''(fs) = (\overline{\partial} f)s + f(D''s)$$

where $\overline{\partial}$ is the Cauchy-Riemann operator of Σ.

A Dolbeault operator is also called a $(0,1)$-connection. As in the case of connections, any smooth complex vector bundle over a complex base space admits a Dolbeault operator, and the space $\mathrm{Dol}(E)$ of all Dolbeault operators on E is an affine space, whose group of translations is the vector space $\Omega^{0,1}(\Sigma; \mathrm{End}\,(E))$. We now show that, given a *holomorphic* vector bundle $\mathcal{E}$ on Σ, there is a Dolbeault operator on the underlying smooth vector bundle E, whose kernel consists exactly of the holomorphic sections of $\mathcal{E}$ (much like $\ker \overline{\partial}$ consists of the holomorphic functions on Σ, see Lemma 1.9). We first observe that $\mathcal{G}_E$, the group[1] of all complex linear bundle automorphisms of E, acts on $\mathrm{Dol}(E)$ in the following way: if $u \in \mathcal{G}_E$ and $D'' \in \mathrm{Dol}(E)$,

$$(u \cdot D'')(s) := u\big(D''(u^{-1}s)\big) \tag{1.3}$$

is a Dolbeault operator on E (Exercise 1.5), and, if v is another automorphism of E,

$$(uv) \cdot D'' = u \cdot (v \cdot D'')$$

so we indeed have a group action. Moreover, D'' is a local operator in the following sense: if s_U is a *local* smooth section of E, we can define a local E-valued 1-form $D''s_U$ using bump functions on U (see the proof below). In particular, local solutions to the equation $D''s = 0$ form a sheaf on Σ.

[1] One may observe that there is a group bundle $\mathbf{GL}(E)$ on Σ, whose typical fibre is $\mathbf{GL}(r; \mathbb{C})$ and whose structure group is $\mathrm{Ad}_{\mathbf{GL}(r;\mathbb{C})}$, such that $\mathcal{G}_E = \Gamma(\mathbf{GL}(E))$.

Proposition 1.15. *Let E be a smooth complex vector bundle on Σ. Given a holomorphic structure on E, denote $\mathcal{E}$ the resulting holomorphic vector bundle. Then, there exists a unique $\mathcal{G}_E$-orbit of Dolbeault operators on E such that, for any D'' in that orbit, local holomorphic sections of $\mathcal{E}$ are in bijection with local solutions to the equation $D''s = 0$.*

Proof. Let $(g_{ij})_{(i,j)}$ be a *holomorphic* 1-cocycle of transition maps on E. Let s be a smooth global section of E, and denote s_i the section s read in the local chart (U_i, φ_i). Then $s_i = g_{ij}s_j$ as maps from $U_i \cap U_j \to \mathbb{C}^r$, so

$$\overline{\partial} s_i = \overline{\partial}(g_{ij}s_j) = (\overline{\partial} g_{ij})s_j + g_{ij}(\overline{\partial} s_j) = g_{ij}(\overline{\partial} s_j),$$

since g_{ij} is holomorphic. This defines, for any $s \in \Omega^0(\Sigma; E)$, a global, E-valued $(0,1)$-form $D''s$ on Σ (such that $(D''s)_i = \overline{\partial} s_i$). The Leibniz rule for the operator thus defined follows from the Leibniz rule for the the local operator $\overline{\partial}$. Let us now identify the local solutions to the equation $D''s = 0$. If σ is a smooth local section of E over an open subset U of Σ, let f be a smooth bump function whose support is contained in U. Then, by definition, there exists an open set $V \subset U$ on which f is identically 1. We may assume that V is trivialising for E. Then $\sigma|_V : V \to \mathbb{C}^r$ is a smooth local section of E over V, and it is holomorphic in V if and only if $\overline{\partial}(\sigma|_V) = 0$, or equivalently, $(D''\sigma)_V = 0$ (observe that this last equation makes sense because we can extend σ *smoothly* to Σ using the bump function, and since $f \equiv 1$ in V, $(D''\sigma)_V$ does not depend on the extension). Using different pairs (V, f), we see that σ is holomorphic in U if and only if it is a local solution to the equation $D''s = 0$. Evidently, two isomorphic holomorphic structures on E determine conjugate Dolbeault operators. □

So we have an injective map

$$\{\text{holomorphic structures on } E\} \,/\, \text{isomorphism} \longrightarrow \mathrm{Dol}(E)/\,\mathcal{G}_E\,.$$

It is a remarkable fact that the image of this map can be entirely described, and that it is in fact a surjective map when the base complex analytic manifold has complex dimension 1. The problem of determining whether a Dolbeault operator comes from a holomorphic structure is a typical *integrability* question, similar to knowing whether a linear connection comes from a linear representation of the fundamental group of M at a given basepoint. The integrability conditions, too, are very similar, and we refer to [DK90] (Chapter 2, Section 2) for an illuminating parallel discussion of the two questions, as well as the proof of the following integrability theorem.

Theorem 1.16 (The Newlander-Nirenberg Theorem in complex dimension one)**.** *Let $E \longrightarrow \Sigma$ be a smooth complex vector bundle on a Riemann surface, and let D'' be a Dolbeault operator on E. Then there exists a unique holomorphic structure on E such that such that the local holomorphic sections of $\mathcal{E}$ are in bijection with smooth local solutions to the equatio $D'' = 0$.*

The proof is a question of showing that the sheaf of local solutions to the equation $D''s = 0$ is a locally free sheaf of rank $r = \mathrm{rk}\, E$ over the sheaf of holomorphic functions of Σ. It is then easy to check that two $\mathcal{G}_E$-conjugate Dolbeault operators determine isomorphic holomorphic structures, since their kernels are conjugate in $\Omega^0(\Sigma; E)$. Therefore, over a Riemann surface Σ, there is a bijection between the set of isomorphism classes of holomorphic structures on E and

$$\mathrm{Dol}(E)/\,\mathcal{G}_E\,.$$

We refer to [LPV85], Exposé 1, for an explanation of why any natural topology of this space is not Hausdorff. We conclude the present subsection by one further remark on Dolbeault operators, namely that a Dolbeault operator D'' on E induces a Dolbeault operator D''_E on $\mathrm{End}\,(E)$. First, note that, for all $k \geq 0$,

$$\Omega^k(\Sigma; \mathrm{End}\,(E)) = \Gamma(\wedge^k T^*\Sigma \otimes_{\mathbb{R}} \mathrm{End}\,(E)) = \mathrm{Hom}(E; \wedge^k T^*\Sigma \otimes_{\mathbb{R}} E)\,.$$

So, given $u \in \Omega^0(\Sigma; \mathrm{End}\,(E))$, we need only specify $(D''_E u)(s) \in \Omega^1(\Sigma; E)$ for all $s \in \Omega^0(\Sigma; E)$ in order to completely determine $D''_E u \in \Omega^1(\Sigma; \mathrm{End}\,(E))$. Moreover, because $u(s)$ is locally a product between a matrix and a column vector, we want the would-be operator D''_E on $\mathrm{End}\,(E)$ to satisfy, for all $u \in \Omega^0(\Sigma; \mathrm{End}\,(E))$ and all $s \in \Omega^0(\Sigma; E)$, the generalised Leibniz identity

$$D''\big(u(s)\big) = (D''_E u)(s) + u(D''s) \tag{1.4}$$

so we *define*

$$(D''_E u)(s) := D''\big(u(s)\big) - u(D''s).$$

Evidently, D''_E is $\mathbb{C}$-linear in u as a map from $\Omega^0(\Sigma; \mathrm{End}\,(E))$ to $\Omega^1(\Sigma; \mathrm{End}\,(E))$, and, if $f \in C^\infty(\Sigma; \mathbb{C})$, one has

$$\begin{aligned}\big(D''_E(fu)\big)(s) &= D''\big(fu(s)\big) - (fu)D''s \\ &= (\overline{\partial} f)\, u(s) + fD''\big(u(s)\big) - f\big(u(D''s)\big) \\ &= [(\overline{\partial} f)u + f(D''_E u)](s)\end{aligned}$$

so D''_E is indeed a Dolbeault operator on $\mathrm{End}\,(E)$. In practice, it is simply denoted D'', which, if anything, makes (1.4) more transparent and easy to remember. As a consequence of the Leibniz identity (1.4), one can modify the way the $\mathcal{G}_E$-action on $\mathrm{Dol}(E)$ is written:

$$\begin{aligned}(u \cdot D'')(s) &= u\big(D''(u^{-1}s)\big) \\ &= u\big((D''(u^{-1}))s + u^{-1}(D''s)\big) \\ &= u(-u^{-1}(D''u)u^{-1}s + u^{-1}D''s) \\ &= D''s - (D''u)u^{-1}s\end{aligned}$$

so

$$u \cdot D'' = D'' - (D''u)u^{-1}\,, \tag{1.5}$$

which is the way the $\mathcal{G}_E$-action on $\mathrm{Dol}(E)$ is usually written when performing explicit computations (we refer to Exercise 1.6 for the computation of $D''(u^{-1})$ used in the above).

1.4. **Exercises.**

EXERCISE 1.1. Show that the quotient topological space defined in (1.2) is a vector bundle of rank r on X.

EXERCISE 1.2. Show that the Möbius bundle $\mathcal{M} \longrightarrow S^1$ is isomorphic to the canonical bundle $E_{\mathrm{can}} \longrightarrow \mathbb{R}\mathbf{P}^1$ (see Example 1).

EXERCISE 1.3. Show that a vector bundle of rank r, $p : E \longrightarrow X$, say, is isomorphic to the product bundle $X \times \mathbb{K}^r$

if and only f there exist r global sections $s_1, \cdots, s_r \in \Gamma(X;E)$ such that, for all $x \in X$, $(s_1(x), \cdots, s_r(x))$ is a basis of E_x over $\mathbb{K}$.

EXERCISE 1.4. **a.** Let $(D_i)_{1\leq i\leq n}$ be n linear connections on a vector bundle $E \longrightarrow M$, and let $(\lambda_i)_{1\leq i\leq n}$ be n non-negative real numbers satisfying $\sum_{i=1}^n \lambda_i = 1$. Show that the convex combination $D = \sum_{i=1}^n \lambda_i D_i$ is a linear connection on E.
b. Let $(U_i)_{i\in I}$ be a covering of M by trivialising open sets for E, and let D_i be the product connection on $E|_{U_i}$. Let $(f_i)_{i\in I}$ be a partition of unity subordinate to $(U_i)_{i\in I}$. Show that $D := \sum_{i\in I} D_i$ is a well-defined map from $\Omega^0(M;E)$ to $\Omega^1(M;E)$, and that it is a linear connection on E.

EXERCISE 1.5. Let E be a smooth complex vector bundle and let D'' be a Dolbeault operator on E. Let u be an automorphism of E. Define $u \cdot D''$ by

$$(u \cdot D'')(s) = u\big(D''(u^{-1}s)\big)$$

on sections of E.
a. Show that $u \cdot D''$ is a Dolbeault operator on E.
b. Show that this defines an action of the group of automorphisms of E on the set of Dolbeault operators.

EXERCISE 1.6. Let D'' be a Dolbeault operator on a smooth complex vector bundle E, and let u be an automorphism of E. Show that

$$D''(u^{-1}) = -u^{-1}(D''u)u^{-1}.$$

2. Holomorphic structures and unitary connections

In this section, we study the space of holomorphic structures on a smooth complex vector bundle E over a Riemann surface Σ *in the additional presence of a Hermitian metric h on E*. This has the effect of replacing the space of Dolbeault operators by another space of differential operators: the space of unitary connections on (E,h). This new affine space turns out to have a natural structure of infinite-dimensional Kähler manifold. Moreover, the action of the group of unitary transformations of (E,h) on the space of unitary connections is a Hamiltonian action, and this geometric point of view, initiated by Atiyah and Bott in [AB83], will be key to understanding Donaldson's Theorem in Subsection 3.2.

2.1. Hermitian metrics and unitary connections.

Definition 2.1 (Hermitian metric)**.** Let E be a smooth complex vector bundle on a smooth manifold M. A **Hermitian metric** h on E is a family $(h_x)_{x\in X}$ of maps

$$h_x : E_x \times E_x \longrightarrow \mathbb{C}$$

such that

(1) $\forall (v, w_1, w_2) \in E_x \times E_x \times E_x$,
$$h(v, w_1 + w_2) = h(v, w_1) + h(v, w_2),$$
(2) $\forall (v, w) \in E_x \times E_x$, $\forall \lambda \in \mathbb{C}$,
$$h(v, \lambda w) = \lambda h(v, w),$$
(3) $\forall (v, w) \in E_x \times E_x$,
$$h(w, v) = \overline{h(v, w)},$$
(4) $\forall v \in E_x \setminus \{0\}$,
$$h(v, v) > 0,$$

(5) for any pair (s, s') of smooth sections of E, the function
$$h(s, s') : M \longrightarrow \mathbb{C}$$
is smooth.

In other words, h is a smooth family of Hermitian products on the fibres of E. A smooth complex vector bundle with a Hermitian metric is called a **smooth Hermitian vector bundle**.

Definition 2.2. A **unitary transformation** of (E, h) is an automorphism u of E satisfying, for any pair (s, s') of smooth sections of E,
$$h\big(u(s), u(s')\big) = h(s, s').$$
In other words, a unitary transformation is fibrewise an isometry. The group $\mathcal{G}_h$ of unitary transformations of (E, h) is called the (unitary) **gauge group**. There is a group bundle $\mathbf{U}(E, h)$, whose typical fibre is $\mathbf{U}(r)$ and whose structure group is $\mathrm{Ad}_{\mathbf{U}(r)}$, such that $\mathcal{G}_h = \Gamma(\mathbf{U}(E, h))$.
A **Hermitian transformation** is an endomorphism u of E satisfying
$$h\big(u(s), s'\big) = h\big(s, u(s')\big).$$
An **anti-Hermitian transformation** is an endomorphism u of E satisfying
$$h\big(u(s), s'\big) = -h\big(s, u(s')\big).$$
The Lie algebra bundle whose sections are anti-Hermitian endomorphisms of E is denoted $\mathfrak{u}(E, h)$. Its typical fibre is the Lie algebra $\mathfrak{u}(r) = Lie(\mathbf{U}(r))$ and its structure group is $\mathrm{Ad}_{\mathbf{U}(r)}$. As one might expect, $\Gamma(\mathfrak{u}(E, h)) = \Omega^0(M; \mathfrak{u}(E, h))$ is actually the Lie algebra of $\Gamma(\mathbf{U}(E, h)) = \mathcal{G}_h$.

Proposition 2.3 (Reduction of structure group)**.** *Let $(E \longrightarrow M)$ be a smooth complex vector bundle. Given a Hermitian metric h on E, there exists a $\mathbf{U}(r)$-valued 1-cocycle*
$$g_{ij} : U_i \cap U_j \longrightarrow \mathbf{U}(r) \subset \mathbf{GL}(r, \mathbb{C})$$
representing E. Two such cocycles differ by an $\mathbf{U}(r)$-valued 0-cocycle. Conversely, an atlas of E whose transition maps are given by a unitary 1-cocycle determines a Hermitian metric on E.

More generally, if H is a subgroup of $\mathbf{GL}(r, \mathbb{C})$ and a vector bundle E can be represented by an H-valued 1-cocycle whose class modulo H-valued 0-cocycles is uniquely defined, one says that the structure group of E has been *reduced* to H. The proposition above says that a Hermitian metric is equivalent to a reduction of the structure group $\mathbf{GL}(r, \mathbb{C})$ of a complex rank r vector bundle to the maximal compact subgroup $\mathbf{U}(r)$. In the general theory of fibre bundles, the existence of such a reduction is usually deduced from the fact that the homogeneous space $\mathbf{GL}(r, \mathbb{C})/\mathbf{U}(r)$ (the space of Hermitian inner products on $\mathbb{C}^r$) is contractible (see for instance [Ste51]).

Proof of Proposition 2.3. Let h be a Hermitian metric on E. Using the Gram-Schmidt process, one can obtain an h-unitary local frame out of any given local frame of E, hereby identifying $E|_U$ with $U \times \mathbb{C}^r$ where $\mathbb{C}^r$ is endowed with its canonical Hermitian inner product. The transition functions of such an atlas have an associated 1-cocycle of transition maps which preserves the Hermitian product and is therefore $\mathbf{U}(r)$-valued. A different choice of unitary frames leads to a $\mathbf{U}(r)$-equivalent 1-cocycle. Conversely, given such a 1-cocycle, the Hermitian products obtained on the fibres of $E|_U$ and $E|_V$ respectively via the identifications with $U \times \mathbb{C}^r$ and $V \times \mathbb{C}^r$ coincide over $U \cap V$. □

Since the structure group of the bundle has changed, it makes sense to ask whether there is a notion of connection which is compatible with this smaller structure group. This is usually better expressed in the language of principal bundles, but we shall not need this point of view in these notes (see for instance [KN96]).

Definition 2.4 (Unitary connection)**.** Let (E, h) be a smooth Hermitian vector bundle on a manifold M. A linear connection

$$D : \Omega^0(M; E) \longrightarrow \Omega^1(M; E)$$

on E is called **unitary** if, for any pair (s, s') of smooth sections of E, one has

$$d\big(h(s, s')\big) = h(Ds, s) + h(s, Ds')\,.$$

The same standard arguments as in the case of linear connections and Dolbeault operators show that a Hermitian vector bundle always admits a unitary connection (locally, the product connection satisfies the unitarity condition), and that the space $\mathcal{A}(E, h)$ of all unitary connections on (E, h) is an affine space, whose group of translations is the vector space $\Omega^1(M; \mathfrak{u}(E, h))$ of $\mathfrak{u}(E, h)$-valued 1-forms on M.

Given a k-form $\alpha \in \Omega^k(M; E)$ and a local trivialisation (U, φ_U) of E, let us denote $\alpha_U \in \Omega^k(U; \mathbb{C}^r)$ the k-form obtained from reading $\alpha|_U$ in the local trivialisation $\varphi_U : E|_U \xrightarrow{\simeq} U \times \mathbb{C}^r$. Then, if $(g_{UV})_{(U,V)}$ is a 1-cocycle of transition maps for E, one has $\alpha_U = g_{UV}\alpha_V$. Moreover, any linear connection is locally of the form

$$(Ds)_U = d(s_U) + A_U s_U$$

where $A_U \in \Omega^1(U; \mathfrak{gl}(r, \mathbb{C}))$ is a family of matrix-valued 1-forms defined on trivialising open sets by $A_U s_U = (Ds)_U - d(s_U)$, d being the product de Rham operator on $\Omega^1(U; \mathbb{C}^r)$, and subject to the condition, for all $s \in \Omega^0(M; E)$,

$$\begin{aligned}
(Ds)_U &= g_{UV}(Ds)_V \\
&= g_{UV}(ds_V + A_V s_V) \\
&= g_{UV}\big(d(g_{UV}^{-1}s_U) + A_V(g_{UV}^{-1}s_U)\big) \\
&= g_{UV}\big(d(g_{UV}^{-1})s_U + g_{UV}^{-1}ds_U\big) + g_{UV}A_V g_{UV}^{-1}\, s_U \\
&= ds_U + \big(g_{UV}A_V g_{UV}^{-1} - (dg_{UV})g_{UV}^{-1}\big)s_U
\end{aligned}$$

so

$$A_U = g_{UV}A_V g_{UV}^{-1} - (dg_{UV})g_{UV}^{-1}\,. \tag{2.1}$$

The family $(A_U)_U$ subject to Condition (2.1) above is sometimes called the connection form, even though it is *not* a global differential form on M. The connection determined by such a family is denoted d_A, or even simply A. Let us now analyse what it means for d_A to be unitary. Given a pair (s, s') of smooth sections of E, and a local chart (U, φ_U) of E, one has, on the one hand,

$$d\big(h(s_U, s'_U)\big) = h(ds_U, s'_U) + h(s_U, ds'_U)$$

and, on the other hand,

$$\begin{aligned} & h\big((Ds)_U, s'_U\big) + h\big(s_U, (Ds')_U\big) \\ = \quad & h(ds_U, s'_U) + h(A_U s_U, s'_U) + h(s_U, ds'_U) + h(s_U, A_U s'_U) \end{aligned}$$

so d_A is unitary if and only if

$$h(A_U s_U, s'_U) + h(s_U, A_U s'_U) = 0$$

which means that the 1-form A_U is in fact $\mathfrak{u}(r)$-valued. Conversely, if (E, h) is represented by a unitary cocycle $(g_{UV} : U \cap V \longrightarrow \mathbf{U}(r))_{(U,V)}$ (Proposition 2.3) and $(A_U)_U$ is a family of $\mathfrak{u}(r)$-valued 1-forms satisfying condition (2.1), then there is a unique unitary connection d_A on (E, h) such that, for all $s \in \Omega^0(M; E)$, one has $(d_A s)_U = ds_U + A_U s_U$ on each U.

Just like any linear connection on a smooth complex vector bundle over a complex manifold (Lemma 1.13), a unitary connection

$$d_A : \Omega^0(M; E) \longrightarrow \Omega^1(M; E) = \Omega^{1,0}(M; E) \oplus \Omega^{0,1}(M; E)$$

splits into $d_A = d_A^{1,0} \oplus d_A^{0,1}$, where $d_A^{1,0}$ takes a section s of E to the $\mathbb{C}$-linear part of its covariant derivative, and $d_A^{0,1}$ takes s to the $\mathbb{C}$-antilinear part of $d_A s$. In particular,

$$d_A^{0,1} \; : \; \Omega^1(M; E) \longrightarrow \Omega^{0,1}(M; E)$$

is a Dolbeault operator. So, if $M = \Sigma$ is a Riemann surface, then, by the Newlander-Nirenberg Theorem, $d_A^{0,1}$ determines a holomorphic structure on E. The next proposition shows what we gain by working in the presence of a Hermitian metric on E: a Dolbeault operator D'' on $E \longrightarrow M$ may be the $(0,1)$-part of various, non-equivalent linear connections, but it is the $(0,1)$-part of a *unique* unitary connection.

Proposition 2.5. *Let (E, h) be a smooth Hermitian vector bundle on a complex manifold M, and let*

$$D'' : \Omega^0(M; E) \longrightarrow \Omega^{0,1}(M; E)$$

be a Dolbeault operator on E. Then there exists a unique unitary connection

$$d_A : \Omega^0(M; E) \longrightarrow \Omega^1(M; E) = \Omega^{1,0}(M; E) \oplus \Omega^{0,1}(M; E)$$

such that $d_A^{0,1} = D''$.

Proof. Like for many other results in these notes, the proof essentially boils down to linear algebra. Let $(g_{UV})_{(U,V)}$ be a unitary 1-cocycle representing (E, h). The Dolbeault operator D'' is locally of the form

$$(D''s)_U = \overline{\partial} s_U + B_U s_U$$

where $B_U \in \Omega^{0,1}(U; \mathfrak{gl}(r, \mathbb{C}))$ and $\overline{\partial}$ is the product Cauchy-Riemann operator on $\Omega^0(U; \mathbb{C}^r)$, and where the family $(B_U)_U$ satisfies

$$B_U = g_{UV} B_V g_{UV}^{-1} - (\overline{\partial} g_{UV}) g_{UV}^{-1} \tag{2.2}$$

(this does not require g_{UV} to be unitary). We then have an isomorphism of real vector spaces

$$\begin{array}{ccc} \Omega^{0,1}(U; \mathfrak{gl}(r, \mathbb{C})) & \longrightarrow & \Omega^1(U; \mathfrak{u}(r)) \\ B_U & \longmapsto & A_U := B_U - B_U^* \end{array}$$

where $B_U^* = \overline{B_U}^t$ is the adjoint of B_U, the converse map being

$$A_U \longmapsto A_U^{0,1} = \frac{A_U(\,\cdot\,) + iA_U(i\,\cdot\,)}{2}.$$

One may observe here that

$$A_U^{1,0} = \frac{A_U(\,\cdot\,) - iA_U(i\,\cdot\,)}{2} = -B_U^*\,.$$

Moreover, as g_{UV} is unitary, $g_{UV}^* = g_{UV}^{-1}$ and therefore

$$\begin{aligned}
A_U &= B_U - B_U^* \\
&= g_{UV}B_Vg_{UV}^{-1} - (\overline{\partial}g_{UV})g_{UV}^{-1} - g_{UV}B_V^*g_{UV}^{-1} + g_{UV}(\partial(g_{UV}^*)) \\
&= g_{UV}(B_V - B_V^*)g_{UV}^{-1} - (\partial g_{UV} + \overline{\partial}g_{UV})g_{UV}^{-1} \\
&= g_{UV}A_Vg_{UV}^{-1} - (dg_{UV})g_{UV}^{-1}
\end{aligned}$$

so the family $(A_U)_U$ is a unitary connection on (E,h), and $d_A^{0,1} = (B_U)_U = D''$. Conversely, if $(A_U)_U$ is a unitary connection on (E,h) such that $A_U^{0,1} = B_U$ for all U, then $A_U^{1,0} = -(A_U^{0,1})^* = -B_U^*$, so such a unitary connection is unique: $A_U = A_U^{1,0} + A_U^{0,1} = -B_U^* + B_U$. □

Observe that the family $(B_U)_U$ satisfying Condition (2.2) completely determines the Dolbeault operator D'', which therefore could be denoted $\overline{\partial}_B$, or even simply B.

Corollary 2.6. *Let Σ be a Riemann surface, and let E be a smooth complex vector bundle on Σ. Then the choice of a Hermitian metric h on E determines an isomorphism of affine spaces*

$$\begin{array}{ccc} \mathcal{A}(E,h) & \xrightarrow{\simeq} & \mathrm{Dol}(E) \\ d_A & \longmapsto & d_A^{0,1} \end{array}$$

between the space of unitary connections on (E,h) and the space of Dolbeault operators on E.

Recall that we denote $\mathcal{G}_h = \Gamma(\mathbf{U}(E,h))$ the group of unitary automorphisms of (E,h). It is commonly called the *unitary gauge group*. As for the group $\mathcal{G}_E = \Gamma(\mathbf{GL}(E))$ of all complex linear automorphisms of E, it is commonly called the *complex gauge group*. A good reason for this terminology is that $\mathcal{G}_E$ is actually the complexification of $\mathcal{G}_h$ (indeed $\mathbf{GL}(r,\mathbb{C})$ is the complexification of $\mathbf{U}(r)$, so the typical fibre of $\mathbf{GL}(E)$ is the complexification of the typical fibre of $\mathbf{U}(E,h)$). We saw in Section 1.3 that, over a Riemann surface Σ, the set of isomorphism classes of holomorphic structures on E was in bijection with the orbit space

$$\mathrm{Dol}(E)/\,\mathcal{G}_E\,.$$

Now if E has Hermitian metric h, we can replace, as we have just seen, $\mathrm{Dol}(E)$ with $\mathcal{A}(E,h)$, and then use the bijection between the two to transport the $\mathcal{G}_E$-action from $\mathrm{Dol}(E)$ to $\mathcal{A}(E,h)$. Computing through this procedure gives, for all $u \in \mathcal{G}_E$ and all $d_A \in \mathcal{A}(E,h)$, the relation

$$u \cdot d_A = d_A - \left[(d_A^{0,1}u)u^{-1} - \left((d_A^{0,1}u)u^{-1}\right)^*\right] \tag{2.3}$$

where $d_A^{0,1}u$ denotes the $\mathbb{C}$-antilinear part of the covariant derivative of the *endomorphism* u (this extension of a Dolbeault operator on a bundle to endomorphisms of that bundle was discussed at the end of Section 1.3) and α^* denotes the h-unitary adjoint of an E-valued, or an $\operatorname{End} E$-valued, k-form α (the proof of relation (2.3) is proposed as an exercise in Exercise 2.3). In particular, if u actually lies in $\mathcal{G}_h \subset \mathcal{G}_E$, then $u^* = u^{-1}$, and

$$(d_A^{0,1}u)^* = d_A^{1,0}(u^*) = d_A^{1,0}(u^{-1}) = -u(d_A^{1,0}u)u^{-1}$$

so

$$\begin{aligned} u \cdot d_A &= d_A - (d_A^{0,1}u + d_A^{1,0}u)u^{-1} \\ &= d_A - (d_A u)u^{-1}, \end{aligned}$$

which is no other than the natural action of $\mathcal{G}_h$ on $\mathcal{A}(E,h)$, defined for all $s \in \Omega^0(M;E)$ by

$$(u \cdot d_A)(s) = u\big(d_A(u^{-1}s)\big)$$

(the exact formal analogue of the $\mathcal{G}_E$-action on $\operatorname{Dol}(E)$, see Equations (1.3) and (1.5)). The fact that the action of $\mathcal{G}_h$ on $\mathcal{A}(E,h)$ extends to an action of $\mathcal{G}_E = \mathcal{G}_h^{\mathbb{C}}$ is what eventually explains the relation between the symplectic picture and the Geometric Invariant Theoretic picture for vector bundles on a curve, a relation which plays an important part in Donaldson's Theorem.

To sum up, the choice of a Hermitian metric on a smooth complex vector bundle $E \longrightarrow \Sigma$ over a Riemann surface provides an identification between the set of isomorphism classes of holomorphic structures on E and the orbit space

$$\mathcal{A}(E,h)/\,\mathcal{G}_E\,.$$

This raises the question: what happens if we choose a different metric? If h and h' are two Hermitian metrics on E, then there exists an automorphism $u \in \mathcal{G}_E$ (in fact unique up to mutiplication by an element of $\mathcal{G}_h$) such that

$$h' = u^*h$$

(meaning that, for any pair (s_1, s_2) of smooth sections of E, one has $h'(s_1,s_2) = h(us_1, us_2)$). In particular, a linear connection D on E is h'-unitary if and only if the linear connection $u \cdot D = u(D(u^{-1}\cdot))$ is h-unitary. Indeed,

$$\begin{aligned} d\big(h(s_1,s_2)\big) &= d\big(h'(u^{-1}s_1, u^{-1}s_2)\big) \\ &= h'\big(D(u^{-1}s_1), u^{-1}s_2\big) + h'\big(u^{-1}s_1, D(u^{-1}s_2)\big) \\ &= h\big((u\cdot D)(s_1), s_2\big) + h\big(s_1, (u\cdot D)(s_2)\big)\,. \end{aligned}$$

Therefore, there is a non-canonical bijection $\mathcal{A}(E,h') \simeq \mathcal{A}(E,h)$ *with the key property that it sends $\mathcal{G}_E$-orbits to $\mathcal{G}_E$-orbits*. In particular, there is a *canonical* bijection

$$\mathcal{A}(E,h')/\,\mathcal{G}_E \simeq \mathcal{A}(E,h)/\,\mathcal{G}_E\,.$$

This renders the choice of the metric unimportant in the whole analysis of holomorphic structures on E: the space $\mathcal{A}(E,h)$ depends on that choice, but not the space $\mathcal{A}(E,h)/\,\mathcal{G}_E$, which is the space of isomorphism classes of holomorphic structures on E.

2.2. **The Atiyah-Bott symplectic form.** Only from this point on does it become truly necessary to assume that the base manifold of our holomorphic bundles be a *compact*, *connected* Riemann surface Σ_g (g being the genus). The fact that Σ_g is of complex dimension one has already been used, though, for instance to show that *any* unitary connection on a smooth Hermitian vector bundle (E,h) over Σ_g defines a holomorphic structure on E. We shall now use the compactness of Σ_g to show that $\mathcal{A}(E,h)$ has a natural structure of infinite-dimensional symplectic (in fact, Kähler) manifold. Actually, for this to be true, we would need to amend our presentation of Dolbeault operators and unitary connections to allow non-smooth such operators. Indeed, as the vector space $\Omega^1(\Sigma;\mathfrak{u}(E,h))$ on which the affine space $\mathcal{A}(E,h)$ is modelled is infinite-dimensional, we have to choose a topology on it. In order to turn the resulting topological vector space into a Banach space, we have to work with connections which are not necessarily smooth, but instead lie in a certain Sobolev completion of the space of smooth connections, and the same goes for gauge transformations. We refer to [AB83] (Section 13) and [Don83, DK90] for a discussion of this problem. Atiyah and Bott have in particular shown that gauge orbits of such unitary connections always contain smooth connections, and that two smooth connections lying in a same gauge orbit can also be conjugated by a smooth gauge transformation. These analytic results enable us to ignore the issue of having to specify the correct connection spaces and gauge groups, and focus on the geometric side of the ideas of Atiyah-Bott and Donaldson instead.

Recall that the space $\mathcal{A}(E,h)$ of unitary connections on a smooth Hermitian vector bundle (E,h) is an affine space whose group of translations is the space $\Omega^1(\Sigma_g;\mathfrak{u}(E,h))$ of 1-forms with values in the bundle of anti-Hermitian endomorphisms of (E,h). In particular, the tangent space at A to $\mathcal{A}(E,h)$ is canonically identified with $\Omega^1(\Sigma_g;\mathfrak{u}(E,h))$. We assume throughout that the Riemann surface Σ_g comes equipped with a compatible Riemannian metric of normalised unit volume. Compatibility in the present context means that the complex structure I on each tangent plane to Σ_g is an isometry of the Riemannian metric. This defines in particular a symplectic form, also a volume form since $\dim_{\mathbb{R}}\Sigma_g = 2$, namely $\mathrm{vol}_{\Sigma_g} = g(I\cdot\,|\,\cdot)$. The typical fibre of $\mathfrak{u}(E,h)$ is the Lie algebra $\mathfrak{u}(r)$ of anti-Hermitian matrices of size r, so it has a canonical, positive definite inner product

$$\kappa := -\mathrm{tr} : \begin{array}{ccc} \mathfrak{u}(r)\otimes\mathfrak{u}(r) & \longrightarrow & \mathbb{R} \\ (X;Y) & \longmapsto & -\mathrm{tr}(XY) \end{array}$$

(the restriction to $\mathfrak{u}(r)$ of the canonical Hermitian product $(X,Y)\longmapsto -\mathrm{tr}(\overline{X}^t Y)$ of $\mathfrak{gl}(r,\mathbb{C})$).

Given $A\in\mathcal{A}(E,h)$ and $a,b\in T_A\mathcal{A}(E,h)\simeq\Omega^1(\Sigma_g;\mathfrak{u}(E,h))$, $a\wedge b$ is the $\mathfrak{u}(E,h)\otimes\mathfrak{u}(E,h)$-valued 2-form defined by

$$(a\wedge b)_x(v,w) = (a_x(v)\otimes b_x(w) - b_x(v)\otimes a_x(w)) \in \mathfrak{u}(r)\otimes\mathfrak{u}(r).$$

So

$$\kappa(a\wedge b)_x(v,w) := -\mathrm{tr}(a_x(v)b_x(w)) + \mathrm{tr}(b_x(v)a_x(w))$$

is an $\mathbb{R}$-valued 2-form on Σ_g. Note indeed that $\kappa(b\wedge a) = -\kappa(a\wedge b)$ because $b\wedge a = -a\wedge b$. Since Σ_g is oriented and compact, the integral

$$\omega_A(a,b) := \int_{\Sigma_g}\kappa(a\wedge b)\in\mathbb{R}$$

defines a 2-form on $\mathcal{A}(E,h)$.

Proposition 2.7 (Atiyah-Bott)**.** *The 2-form ω defined on $\mathcal{A}(E,h)$ by*

$$\omega_A(a,b) := \int_{\Sigma_g} \kappa(a \wedge b)$$

is a symplectic form.

Proof. ω is obviously closed, since it is constant with respect to A. To show that it is non-degenerate, we use local coordinates. The tangent vectors a and b become $\mathfrak{u}(r)$-valued 1-forms on an open subset $U \subset \Sigma_g$,

$$\begin{array}{rcl} a & = & \alpha\, dx + \beta\, dy \\ b & = & \gamma\, dx + \delta\, dy \end{array}$$

with $\alpha, \beta, \gamma, \delta : U \longrightarrow \mathfrak{u}(r)$ smooth functions. If $a \in \ker \omega_A$, then, for $b = *a := -\beta\, dx + \alpha\, dy$, one has

$$\kappa(a \wedge b)_{(x,y)}(v,w) = \underbrace{(\kappa(\alpha(x,y)^2 + \beta(x,y)^2))}_{\geq 0}(v_1 w_2 - v_2 w_1),$$

a positive multiple of the volume form (here we need the local coordinates (x,y) to be appropriately chosen), so

$$\int_{\Sigma_g} \kappa(a \wedge *a) \geq 0$$

and it is 0 if and only if $\alpha = \beta = 0$, i.e. $a = 0$. □

Of course, there is some hidden meaning to this proof: the transformation

$$* : \alpha\, dx + \beta\, dy \longmapsto -\beta\, dx + \alpha\, dy \tag{2.4}$$

is the local expression of the *Hodge star* on $\Omega^1(\Sigma_g; \mathfrak{u}(E,h))$. It squares to minus the identity, so it is a complex structure on $\Omega^1(\Sigma_g; \mathfrak{u}(E,h))$. But in fact, the Hodge star may be defined on all non-zero homogeneous forms on Σ_g: it sends 0-forms to 2-forms and vice versa, the two transformations being inverse to one another. Locally, one has $*(f dx) = f dy$, $*(f dy) = -f dx$, $*f = f dx \wedge dy$, and $*(f dx \wedge dy) = f$. More intrisically, since Σ_g has a Riemannian metric and the fibres of $\mathfrak{u}(E,h)$ have a scalar product κ, the bundle $\bigwedge^k T^*\Sigma_g \otimes_{\mathbb{R}} \mathfrak{u}(E,h)$ has a Riemannian metric π, say. If a, are two $\mathfrak{u}(E,h)$-valued k-forms on Σ_g, i.e. two sections of $\bigwedge^k T^*\Sigma_g \otimes_{\mathbb{R}} \mathfrak{u}(E,h)$, then $\pi(a,b)$ is a smooth function on Σ_g. Now, if η is an arbitrary $\mathfrak{u}(E,h)$-valued k-form on Σ_g, $*\eta$ is defined as the unique $\mathfrak{u}(E,h)$-valued $(2-k)$-form such that

$$\kappa(\eta \wedge *\eta) = \pi(\eta,\eta)\, \mathrm{vol}_{\Sigma_g} \tag{2.5}$$

as 2-forms on Σ_g.

Proposition 2.8. *Set, for all $a, b \in T_A\mathcal{A}(E,h) \simeq \Omega^1(\Sigma_g; \mathfrak{u}(E,h))$,*

$$(a\,|\,b)_{L^2} := \int_{\Sigma_g} \kappa(a \wedge *b) = \omega_A(a, *b).$$

Then $(\cdot\,|\,\cdot)_{L^2}$ is a Riemannian metric on $\mathcal{A}(E,h)$, called the L^2 metric. The Atiyah-Bott symplectic form ω, the complex structure $*$, and the metric $(\cdot\,|\,\cdot)_{L^2}$ turn $\mathcal{A}(E,h)$ into a Kähler manifold.

Proof. Note that $(a\,|\,b)_{L^2} = \int_{\Sigma_g} \pi(a,b)\,\mathrm{vol}_{\Sigma_g}$. The equality with the expression in the statement of the Proposition follows from 2.5. The fact that $(\cdot\,|\,\cdot)_{L^2}$ is positive-definite has been proved in Proposition 2.7. Moreover, it is clear from either of expressions 2.4 or 2.5, that $\|*a\|_{L^2} = \|a\|_{L^2}$. The rest is the definition of a Kähler manifold (see for instance [MS98]). □

Recall now that the gauge group $\mathcal{G}_h = \Gamma(\mathbf{U}(E,h))$ of unitary transformations of (E,h) acts on $\mathcal{A}(E,h)$ via

$$u \cdot A = A - (d_A u)u^{-1}.$$

Proposition 2.9 (Infinitesimal gauge action)**.** *The fundamental vector field*

$$\xi^{\#}_A = \frac{d}{dt}|_{t=0}\,(\exp(t\xi)\cdot A)$$

associated to the element ξ *of the Lie algebra* $\Omega^0(\Sigma_g;\mathfrak{u}(E,h)) \simeq Lie(\mathcal{G}_h)$ *is*

$$\xi^{\#}_A = -d_A\xi \in \Omega^1(\Sigma_g;\mathfrak{u}(E,h))\,.$$

Proof. In local coordinates, A is of the form $d+a$, where a is a $\mathfrak{u}(r)$-valued 1-form defined on an open subset $U \subset \Sigma_g$, u is a smooth map $U \longrightarrow \mathbf{U}(r)$, and $d_A u$ acts on endomorphism of $E|_U$ by $d + [a,\cdot]$ (see Exercise 2.6). So $u\cdot A$ is of the form

$$\begin{aligned}(d+a) - (du + [a,u])u^{-1} &= d + a - (du)u^{-1} - (au - ua)u^{-1}\\ &= d - (du)u^{-1} + uau^{-1}.\end{aligned}$$

Setting $u = \exp(t\xi)$ and taking the derivative at $t=0$ of $-(du)u^{-1} + uau^{-1}$, we obtain

$$-d\xi + \xi a - a\xi = -d\xi - [a,\xi],$$

which is the local expression of $-d_A\xi$. □

Proposition 2.10. *The action of* $\mathcal{G}_h$ *on* $\mathcal{A}(E,h)$ *preserves the Atiyah-Bott symplectic form and the* L^2 *metric on* $\mathcal{A}(E,h)$.

Proof. The tangent map to the action of $u \in \mathcal{G}_h$ on $\mathcal{A}(E,h)$ is the map

$$\begin{array}{ccc}\Omega^1(\Sigma_g;\mathfrak{u}(E,h)) & \longrightarrow & \Omega^1(\Sigma_g;\mathfrak{u}(E,h))\\ a & \longmapsto & uau^{-1}\end{array}$$

so, since $\kappa = -\mathrm{tr}$ is Ad-invariant on $\mathfrak{u}(r)\otimes\mathfrak{u}(r)$,

$$\kappa\big((uau^{-1})\wedge(ubu^{-1})\big) = \kappa\big(a\wedge b\big)$$

and therefore $u^*\omega = \omega$. Since the action is also $\mathbb{C}$-linear (see Exercise 2.5), it is an isometry of the L^2 metric. □

Since we have a symplectic action of a Lie group $\mathcal{G}_h$ on a symplectic manifold $(\mathcal{A}(E,h),\omega)$ (albeit both infinite-dimensional), it makes sense to ask whether this action is Hamiltonian and, more importantly, find the momentum map. To identify a possible momentum map, we need to make $(Lie(\mathcal{G}_H))$ more explicit.

Proposition 2.11. *The map*

$$\begin{array}{ccc}\Omega^2(\Sigma_g;\mathfrak{u}(E,h)) & \longrightarrow & (Lie(\mathcal{G}_h))^*\\ R & \longmapsto & (\xi \longmapsto \int_{\Sigma_g}\kappa(\xi\otimes R)\end{array}$$

is an isomorphism of vector spaces which is $\mathcal{G}_h$*-equivariant with respect to the action* $u \cdot R := \mathrm{Ad}_u \circ R$ *on* $\Omega^2(\Sigma_g; \mathfrak{u}(E,h))$ *and the co-adjoint action on* $(Lie(\mathcal{G}_h))^*$.

Proof. The Lie algebra of $\mathcal{G}_h$ is $\Omega^0(\Sigma_g; \mathfrak{u}(E,h))$. It carries a Riemannian metric

$$(\lambda, \mu) \longmapsto \int_{\Sigma_g} \kappa(\lambda \wedge *\mu)$$

which canonically identifies it with its dual. Then, the Hodge star establishes an isomorphism

$$* : \Omega^2(\Sigma_g; \mathfrak{u}(E,h)) \longrightarrow \Omega^0(\Sigma_g; \mathfrak{u}(E,h)) .$$

The statement on the action follows from the Ad-invariance of κ. □

Now, there is a natural map from $\mathcal{A}(E,h)$ to $\Omega^2(\Sigma_g; \mathfrak{u}(E,h))$, namely the map taking a unitary connection A to its curvature F_A, which we now define.

Proposition 2.12. *A unitary connection*

$$d_A : \Omega^0(\Sigma_g; E) \longrightarrow \Omega^1(\Sigma_g; E)$$

on (E,h) *uniquely extends to an operator*

$$d_A : \Omega^k(\Sigma_g; E) \longrightarrow \Omega^{k+1}(\Sigma_g; E)$$

satisfying the generalised Leibniz rule

$$d_A(\beta \wedge \sigma) = (d\beta) \wedge \sigma + (-1)^{deg\, \beta} \beta \wedge d_A \sigma$$

for all $\beta \in \Omega^j(\Sigma_g; \mathbb{C})$ *and all* $\sigma \in \Omega^k(\Sigma_g; E)$*. The operator*

$$d_A \circ d_A : \Omega^0(\Sigma_g; E) \longrightarrow \Omega^2(\Sigma_g; E)$$

is $C^\infty(\Sigma_g; \mathbb{C})$*-linear, so it defines an element* $F_A \in \Omega^2(\Sigma_g; \mathfrak{u}(E,h))$ *called the* ***curvature*** *of* A*. It satisfies*

$$F_{u \cdot A} = \mathrm{Ad}_u \circ F_A = u F_A u^{-1}$$

for all $u \in \mathcal{G}_h$*. Moreover, if the local expression of* A *is* $d + a$*, the local expression of* F_A *is* $da + \frac{1}{2}[a,a]$*.*

The following theorem is the main result of this subsection.

Theorem 2.13 (Atiyah-Bott, [AB83])**.** *The curvature map*

$$F : \mathcal{A}(E,h) \longrightarrow \Omega^2(\Sigma_g; \mathfrak{u}(E,h))$$

is an equivariant momentum map for the gauge action of $\mathcal{G}_h$ *on* $\mathcal{A}(E,h)$*.*

We shall need the following lemma to prove Theorem 2.13.

Lemma 2.14. *Let* $A \in \mathcal{A}(E,h)$ *be a unitary connection and let* $b \in \Omega^1(\Sigma_g; \mathfrak{u}(E,h))$*. Then* $A + b$ *is a unitary connection and*

$$F_{A+b} = F_A + d_A b + \frac{1}{2}[b,b].$$

Proof. Since $\mathcal{A}(E,h)$ is an affine space on $\Omega^1(\Sigma_g; \mathfrak{u}(E,h))$, $A + b$ is a unitary connection. Let $d + a$ be the local expression of A, where $a \in \Omega^1(U; \mathfrak{u}(r))$. Then the local expression of F_A is $da + \frac{1}{2}[a,a]$, and the local expression of $d_A b$ is $db + [a,b]$

(as in Exercise 2.6). Moreover, the local expression of $A+b$ is $d+(a+b)$, so the local expression of F_{A+b} is

$$\begin{aligned} d(a+b)+\frac{1}{2}[a+b,a+b] &= da+db+\frac{1}{2}[a,a]+[a,b]+\frac{1}{2}[b,b] \\ &= (da+\frac{1}{2}[a,a])+(db+[a,b])+\frac{1}{2}[b,b] \end{aligned}$$

so indeed

$$F_{A+b}=F_A+d_Ab+\frac{1}{2}[b,b]\,.$$

□

Proof of Theorem 2.13. The equivariance of F follows from Proposition 2.12. It remains to show that F is a momentum map for the gauge action, that is, for all $\xi\in Lie(\mathcal{G}_h)=\Omega^0(\Sigma_g;\mathfrak{u}(E,h))$ and all $A\in\mathcal{A}(E,h)$,

$$\omega_A(\xi^{\#}_A,\cdot)=\big(d<F\,,\,\xi>\big)_A\big(\ \cdot\ \big)$$

as linear forms on $T_A\mathcal{A}(E,h)\simeq\Omega^1(\Sigma_g;\mathfrak{u}(E,h))$. By Proposition 2.9, this is equivalent to the fact that, for all $\eta\in\Omega^1(\Sigma_g:\mathfrak{u}(E,h))$,

$$\int_{\Sigma_g}\kappa(-d_A\xi\wedge\eta)=<(dF)_A\cdot\eta\,,\,\xi>.$$

But, by Proposition 2.14,

$$F_{A+t\eta}=F_A+td_A\eta+\frac{1}{2}t^2[\eta,\eta]\,,$$

so

$$(dF)_A\cdot\eta=\frac{d}{dt}|_{t=0}F_{A+t\eta}=d_A\eta\,.$$

In other words, by Proposition 2.11, we want to show that

$$-\int_{\Sigma_g}\kappa(d_A\xi\wedge\eta)=\int_{\Sigma_g}\kappa(\xi\otimes d_A\eta)\,. \tag{2.6}$$

But, since $\partial\Sigma_g=\emptyset$, one has

$$\int_{\Sigma_g}d\big(\kappa(\xi\otimes\eta)\big)=0$$

on the one hand, and on the other hand,

$$d\big(\kappa(\xi\otimes\eta)\big)=\kappa(d_A\xi\wedge\eta)+\kappa(\xi\otimes d_A\eta)$$

whence relation (2.6). □

2.3. **Exercises.**

Exercise 2.1. Show that any complex vector bundle over a smooth manifold admits a Hermitian metric (as usual, use local trivialisations and a partition of unity).

Exercise 2.2. Let u be an endomorphism of a smooth Hermitian vector bundle (E,h). Show that there exists a unique endomorphism u^* of E such that, for all $(s,s')\in\Gamma(E)\times\Gamma(E)$,

$$h\big(u(s),s'\big)=h\big(s,u^*(s')\big).$$

u^* is called the *adjoint* of u. A Hermitian endomorphism is self-adjoint, and an anti-Hermitian one is anti-self-adjoint.

Exercise 2.3. Show that, if $d_A \in \mathcal{A}(E,h)$ and $g \in \mathcal{G}_E$, then the quantity $g \cdot d_A$ defined by

$$d_A - \left[(d_A^{0,1} g)g^{-1} - \left((d_A^{0,1} g)g^{-1}\right)^*\right]$$

is a unitary connection, and that this defines an action of $\mathcal{G}_E$ on $\mathcal{A}(E,h)$ making the isomorphism

$$\mathcal{A}(E,h) \simeq \mathrm{Dol}(E)$$

$\mathcal{G}_E$-equivariant.

Exercise 2.4. Check that relation (2.5) gives a well-defined $\mathbb{R}$-linear map

$$* : \Omega^k(\Sigma_g; \mathfrak{u}(E,h)) \longmapsto \Omega^2(\Sigma_g; \mathfrak{u}(E,h))$$

satisfying $*^2 = (-1)^{k(2-k)}\mathrm{Id}$. Check that, in local coordinates, the map $*$ satisfies

$$*(\alpha\, dx + \beta\, dy) = -\beta\, dx + \alpha\, dy\,.$$

How about $*(\lambda\, dz + \mu\, d\overline{z})$?

Exercise 2.5. Show that the tangent map to the self-diffeomorphism of $\mathcal{A}(E,h)$ defined by the action of an element $u \in \mathcal{G}_h$ is $\mathbb{C}$-linear with respect to the complex structure of $\mathcal{A}(E,h)$ given on each tangent space $T_A\mathcal{A}(E,h) \simeq \Omega^1(\Sigma_g; \mathfrak{u}(E,h))$ by the Hodge star.

Exercise 2.6. Let A be a linear connection on a vector bundle E, and let s be a section of E. Show that if A is locally of the form

$$s \longmapsto ds + as$$

then the covariant derivative $d_A u$ of an endomorphism of E, defined at the end of Section 1.3 by

$$(d_A u)s = d_A(u(s)) - u(d_A s)\,,$$

is locally of the form

$$u \longmapsto du + [a,u].$$

3. Moduli spaces of semi-stable vector bundles

It is sometimes important, while thinking about mathematics, to have a guiding problem to help one organise one's thoughts. For us in these notes, it is the problem of classifying holomorphic vector bundles on a smooth, irreducible complex projective curve Σ_g (=a compact connected Riemann surface of genus g). When the genus is 0 or 1, there are complete classification results for holomorphic vector bundles on Σ_g, due to Grothendieck for the case of the Riemann sphere ([Gro57]), and to Atiyah for the case of elliptic curves ([Ati57]). There are no such classification results available for holomorphic vector bundles on a curve of genus $g > 1$. In such a situation, one generally hopes to replace the classification theorem by the construction of what is called a *moduli space*, the geometry of which can subsequently be studied. Roughly speaking, a moduli space of holomorphic vector bundles is a complex quasi-projective variety which has isomorphism classes of vector bundles over a fixed base for points, and satisfies a universal property controlling the notion of holomorphic or algebraic family of such vector bundles. We shall not get into the formal aspects of the notion of a moduli space and we refer the interested reader to [GÓ1] instead. There are a few situations in which we know how to construct a moduli variety of vector bundles (that is, give a structure of complex quasi-projective variety to *a certain set* of equivalence classes of vector bundles) and vector bundles on a smooth complex projective curve is one of those situations. The difficulty of a *moduli problem* is to understand *which set* one should try to endow with a structure of complex quasi-projective variety.

Common features of many moduli problems include:

(1) Starting with a topological (or smooth) classification of the objects under study. This is typically obtained via *discrete invariants* (for vector bundles on curves: the rank and the degree) and has the virtue of dividing the

moduli problem into various, more tractable moduli problems for objects of a fixed topological type.

(2) Getting rid of certain objects in order to get a moduli space that admits a structure of projective algebraic variety (=a closed subspace of a projective space), or at least quasi-projective (=an open subset of a projective variety). This is where continuous invariants, called moduli, enter the picture (moduli may be thought of as some sort of local coordinates on the would-be moduli space). It is usually a difficult problem to find moduli for a class of objects, and one solution has been to use Mumford's Geometric Invariant Theory (GIT, [MFK93]) to decide which objects one should consider in order to get a nice moduli space (these objects are called *semi-stable* objects).

In fact, *stable objects* exhibit even better properties in the sense that the moduli space is then typically an orbit space (also called a geometric quotient, as opposed to a categorical quotient in the semi-stable case, see for instance [Tho06, New09]) admitting a structure of quasi-projective variety. GIT really is a way of defining quotients in algebraic geometry, and it has been applied very successfully to the study of moduli problems (Mumford's original motivation indeed). We shall not say anything else about GIT in these notes, and focus on *slope stability* for vector bundles on a curve only (it can be shown that this is in fact a GIT type of stability condition, see for instance [New09]). Nor shall we say anything about moduli functors and their coarse/fine moduli spaces (the interested reader might consult, for instance, [Muk03]).

3.1. **Stable and semi-stable vector bundles.** A basic property of holomorphic line bundles on a compact connected Riemann surface Σ_g states that they do not admit non-zero global holomorphic sections if their degree is negative (see for instance [For91], Theorem 16.5). Since a homomorphism between the holomorphic line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ is a section of the *line bundle* $\mathcal{L}_1^* \otimes \mathcal{L}_2$, a non-zero such homomorphism may only exist if $deg\,(\mathcal{L}_1^* \otimes \mathcal{L}_2) \geq 0$, which is equivalent to $deg\,\mathcal{L}_1 \leq deg\,\mathcal{L}_2$. Semi-stable vector bundles of rank $r \geq 2$ provide a class of higher rank vector bundles for which the statement above remains true (see Proposition 3.5). Note that, for higher rank vector bundles, the degree of $\mathcal{E}_1^* \otimes \mathcal{E}_2$ is

$$deg\,(\mathcal{E}_1^* \otimes \mathcal{E}_2) = \mathrm{rk}\,(\mathcal{E}_1)\, deg\,(\mathcal{E}_2) - deg\,(\mathcal{E}_1)\mathrm{rk}\,\mathcal{E}_2$$

so the non-negativity condition is equivalent to

$$\frac{deg\,\mathcal{E}_1}{\mathrm{rk}\,\mathcal{E}_1} \leq \frac{deg\,\mathcal{E}_2}{\mathrm{rk}\,\mathcal{E}_2}.$$

This motivates the following definition.

Definition 3.1 (Slope)**.** The slope of a non-zero complex vector bundle $E \longrightarrow \Sigma_g$ on an orientable, compact, connected surface Σ_g is the rational number

$$\mu(E) := \frac{deg\,E}{\mathrm{rk}\,E} \in \mathbb{Q}\,.$$

We point out that no use is made of the holomorphic structures in the definition of the slope. It is a purely topological quantity, that will, nonetheless, have strong holomorphic properties (another example of a topological invariant with strong holomorphic properties is the genus: on a compact, connected, orientable surface of genus g, the dimension of the space of holomorphic 1-forms is equal to g for *any* complex analytic structure on the surface).

In what follows, we call a sub-bundle $\mathcal{F} \subset \mathcal{E}$ non-trivial if it is distinct from 0 and $\mathcal{E}$. We emphasise that the definition that we give here is that of *slope* stability. However, since this is the only notion of stability that we shall consider in these notes, we will only say stable and semi-stable afterwards.

Definition 3.2 (Slope stability)**.** A (non-zero) holomorphic vector bundle $\mathcal{E} \longrightarrow \Sigma_g$ on a compact, connected Riemann surface Σ_g is called

(1) **slope stable**, or simply stable, if for any non-trivial holomorphic sub-bundle $\mathcal{F}$, one has
$$\mu(\mathcal{F}) < \mu(\mathcal{E})\,.$$
(2) **slope semi-stable**, or simply semi-stable, if for any non-trivial holomorphic sub-bundle $\mathcal{F}$, one has
$$\mu(\mathcal{F}) \leq \mu(\mathcal{E})\,.$$

A couple of remarks are in order. First, all holomorphic line bundles are stable (since they do not even have non-trivial sub-bundles), and all stable bundles are semi-stable. Second, a semi-stable vector bundle with coprime rank and degree is actually stable (this only uses the definition of slope stability and the properties of Euclidean division in $\mathbb{Z}$). Next, we have the following equivalent characterisation of stability and semi-stability, which is sometimes useful in practice.

Proposition 3.3. *A holomorphic vector bundle $\mathcal{E}$ on Σ_g is stable if and only if, for any non-trivial sub-bundle $\mathcal{F} \subset \mathcal{E}$, one has $\mu(\mathcal{E}/\mathcal{F}) > \mu(\mathcal{E})$. It is semi-stable if and only if $\mu(\mathcal{E}/\mathcal{F}) \geq \mu(\mathcal{F})$ for all such $\mathcal{F}$.*

Proof. Denote r, r', r'' the respective ranks of $\mathcal{E}, \mathcal{F}$ and $\mathcal{E}/\mathcal{F}$, and d, d', d'' their respective degrees. One has an exact sequence
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F} \longrightarrow 0$$
so $r = r' + r''$ and $d = d' + d''$. Therefore,
$$\frac{d'}{r'} < \frac{d'+d''}{r'+r''} \Leftrightarrow \frac{d'}{r'} < \frac{d''}{r''} \Leftrightarrow \frac{d'+d''}{r'+r''} < \frac{d''}{r''}$$
and likewise with large inequalities or with equalities. This readily implies the Proposition. □

In a way, semi-stable holomorphic vector bundles are holomorphic vector bundles that do not admit *too many* sub-bundles, since any sub-bundle they may have is of slope no greater than their own. This turns out to have a number of interesting consequences that we now study. We recall that the category of vector bundles on a curve is a typical example of an additive category which is not Abelian: even though it admits kernels and co-kernels (hence also images and co-images), the canonical map $\mathcal{E}/\ker u \longrightarrow \operatorname{im} u$ is in general not an isomorphism. We can, however, always compare the slopes of these two bundles.

Lemma 3.4. *If $u : \mathcal{E} \longrightarrow \mathcal{E}'$ is a non-zero homomorphism of vector bundles over Σ_g, then*
$$\mu(\mathcal{E}/\ker u) \leq \mu(\operatorname{im} u)$$
with equality if and only if the canonical map $\mathcal{E}/\ker u \longrightarrow \operatorname{im} u$ is an isomorphism.

One says that u is *strict* if the canonical homomorphism $\mathcal{E}/\ker u \longrightarrow \operatorname{im} u$ is an isomorphism. In this case, u is injective if and only if $\ker u = 0$ and u is surjective if and only $\operatorname{im} u = \mathcal{E}'$. The proof we give below, of Lemma 3.4, requires notions on coherent modules over the sheaf O_{Σ_g}; it may be skipped upon firt reading of these notes. Recall that the category of vector bundles on Σ_g is equivalent to the category of locally free O_{Σ_g}-modules (=torsion-free coherent O_{Σ_g}-modules). Let $\mathcal{E}$ be a vector bundle on Σ_g and let $\underline{\mathcal{E}}$ be the corresponding torsion-free coherent module. Even though a coherent sub-module $\underline{\mathcal{F}}$ of $\underline{\mathcal{E}}$ is torsion-free, it only corresponds to a sub-bundle $\mathcal{F}$ of $\mathcal{E}$ if the coherent module $\underline{\mathcal{E}}/\underline{\mathcal{F}}$ is also torsion-free (and the latter then corresponds to the vector bundle $\mathcal{E}/\mathcal{F}$). This is equivalent to saying that $\underline{\mathcal{F}}$ is locally a direct summand of $\underline{\mathcal{E}}$. Given a coherent sub-module $\underline{\mathcal{F}}$ of $\underline{\mathcal{E}}$, there exists a smallest coherent sub-module $\underline{\widetilde{\mathcal{F}}}$ containing $\underline{\mathcal{F}}$ and such that $\underline{\mathcal{E}}/\underline{\widetilde{\mathcal{F}}}$ is torsion-free, namely the pre-image of the torsion sub-module of $\underline{\mathcal{E}}/\underline{\mathcal{F}}$. Then $\underline{\widetilde{\mathcal{F}}}/\underline{\mathcal{F}}$ has finite support and $\underline{\mathcal{F}}$ and $\underline{\widetilde{\mathcal{F}}}$ have same rank. Moreover, $deg\,\underline{\mathcal{F}}$ (which is well-defined since $\mathcal{F}$ is locally free) satisfies $deg\,\underline{\mathcal{F}} \leq deg\,\underline{\widetilde{\mathcal{F}}}$, with equality if and only if $\underline{\widetilde{\mathcal{F}}} = \underline{\mathcal{F}}$. It is convenient to call the *sub-bundle* $\widetilde{\mathcal{F}}$ corresponding to $\underline{\widetilde{\mathcal{F}}}$ the sub-bundle of $\mathcal{E}$ *generated* by $\underline{\mathcal{F}} \subset \underline{\mathcal{E}}$.

Proof of Lemma 3.4. Recall that the category of coherent O_{Σ_g}-modules is Abelian. In particular, the isomorphism theorem between co-images and images holds in that category. Let $\underline{u} : \underline{\mathcal{E}} \longrightarrow \underline{\mathcal{E}'}$ be the homomorphism of coherent O_{Σ_g}-modules associated to $u : \mathcal{E} \longrightarrow \mathcal{E}'$. Then, on the one hand, the locally free O_{Σ_g}-module associated to $\ker u$ is $\ker \underline{u}$ and the locally free O_{Σ_g}-module associated to $\mathcal{E}/\ker u$ is $\underline{\mathcal{E}}/\ker \underline{u}$. On the other hand, $\operatorname{im} u$ is the vector bundle *generated* by $\operatorname{im} \underline{u} \simeq \underline{\mathcal{E}}/\underline{\ker u}$. So $\operatorname{rk}(\operatorname{im} u) = \operatorname{rk}(\operatorname{im} \underline{u})$ and $deg\,(\operatorname{im} u) \geq deg\,(\operatorname{im} \underline{u})$, with equality if and only if u is strict. Therefore $\mu(\operatorname{im} u) \geq \mu(\operatorname{im} \underline{u})$, with equality if and only if u is strict. So

$$\mu(\mathcal{E}/\ker u) = \mu(\underline{\mathcal{E}}/\ker \underline{u}) = \mu(\operatorname{im} \underline{u}) \leq \mu(\operatorname{im} u)$$

with equality if and only if $\mathcal{E}/\ker u \simeq \operatorname{im} u$. □

This immediately implies the result alluded to in the introduction to the present subsection.

Proposition 3.5. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two semi-stable vector bundles such that $\mu(\mathcal{E}) > \mu(\mathcal{E}')$. Then any homomorphism $u : \mathcal{E} \longrightarrow \mathcal{E}'$ is zero.*

Proof. If u is non-zero, then, since $\mathcal{E}$ is semi-stable, Proposition 3.3 and Lemma 3.4 imply that

$$\mu(\operatorname{im} u) \geq \mu(\mathcal{E}/\ker u) \geq \mu(\mathcal{E}) > \mu(\mathcal{E}'),$$

which contradicts the semi-stability of $\mathcal{E}'$. □

We now focus on the category of semi-stable vector bundles of fixed slope $\mu \in \mathbb{Q}$. Unlike the category of all vector bundles on Σ_g, this is an Abelian category: it is additive, and we prove below that it admits kernels and co-kernels and that the isomorphism theorem holds.

Proposition 3.6. *Let $u : \mathcal{E} \longrightarrow \mathcal{E}'$ be a non-zero homomorphism of semi-stable vector bundles of slope μ. Then $\ker u$ and $\operatorname{im} u$ are semi-stable vector bundles of slope μ, and the natural map $\mathcal{E}/\ker u \longrightarrow \operatorname{im} u$ is an isomorphism. In particular, the category of semi-stable vector bundles of slope μ is Abelian.*

Proof. Since $u \neq 0$, $\operatorname{im} u$ is a non-zero sub-bundle of $\mathcal{E}'$, so $\mu(\operatorname{im} u) \leq \mu(\mathcal{E}') = \mu$. But, by Lemma 3.4,

$$\mu(\operatorname{im} u) \geq \mu(\mathcal{E}/\ker u) \geq \mu(\mathcal{E}) = \mu\,.$$

So $\mu(\operatorname{im} u) = \mu$ and $\mu(\mathcal{E}/\ker u) = \mu$. In particular, by Lemma 3.4, $\mathcal{E}/\ker u \simeq \operatorname{im} u$. Consider now the exact sequence

$$0 \longrightarrow \ker u \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\ker u \longrightarrow 0\,.$$

Since $\mu(\mathcal{E}) = \mu(\mathcal{E}/\ker u) = \mu$, one also has $\mu(\ker u) = \mu$. Finally, since a sub-bundle of $\ker u$ (resp. $\operatorname{im} u$) is also a sub-bundle of $\mathcal{E}$ (resp. $\mathcal{E}'$), its slope is no greater than $\mu(\mathcal{E}) = \mu = \mu(\ker u)$ (resp. $\mu(\mathcal{E}') = \mu = \mu(\operatorname{im} u)$), so $\ker u$ (resp. $\operatorname{im} u$) is semi-stable. □

As an easy consequence of the above, the following result shows that, by considering only stable bundles of the same slope, we can better control the homomorphisms between them.

Proposition 3.7. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two stable vector bundles on Σ_g such that $\mu(\mathcal{E}) = \mu(\mathcal{E}')$, and let $u : \mathcal{E} \longrightarrow \mathcal{E}'$ be a non-zero homomorphism. Then u is an isomorphism.*

Proof. Recall that $\ker u \neq \mathcal{E}$ by assumption. Since $u : \mathcal{E} \longrightarrow \mathcal{E}'$ is a non-zero homomorphism between semi-stable bundles of the same slope, Proposition 3.6 implies that u is strict and that $\ker u$ is either 0 or has slope equal to $\mu(\mathcal{E})$. Since $\mathcal{E}$ is actually stable, $\ker u$ must be 0. Since u is strict, this implies that u is injective. Likewise, $\operatorname{im} u \neq 0$ by assumption, and has slope equal to $\mu(\mathcal{E}')$ by Proposition 3.6. Since $\mathcal{E}'$ is actually stable, this forces $\operatorname{im} u$ to be equal to $\mathcal{E}'$. Then, again since u is strict, $\operatorname{im} u = \mathcal{E}'$ implies that u is surjective. Therefore, u is an isomorphism. □

Note that a vector bundle always has non-trivial automorphisms (multiplication by a non-zero scalar on the fibres). When these are all the automorphisms of a given bundle, it is called a *simple* bundle. We now show that stable implies simple.

Proposition 3.8. *If $\mathcal{E}$ is a stable vector bundle on Σ_g, then* End $\mathcal{E}$ *is a field, isomorphic to $\mathbb{C}$. In particular,* $\operatorname{Aut}\mathcal{E} \simeq \mathbb{C}^*$.

Proof. Let u be a non-zero endomorphism of $\mathcal{E}$. By Proposition 3.7, u is an automorphism of $\mathcal{E}$, so End $\mathcal{E}$ is a field, which contains $\mathbb{C}$ as its sub-field of scalar endomorphisms. Then, for any $u \in \operatorname{End}\mathcal{E}$, the sub-field $\mathbb{C}(u) \subset \operatorname{End}\mathcal{E}$ is a commutative field, and the Cayley-Hamilton Theorem shows that u is algebraic over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, this shows that $u \in \mathbb{C}$. So $\operatorname{End}\mathcal{E} \simeq \mathbb{C}$ (in particular, the field End $\mathcal{E}$ is commutative) and therefore $\operatorname{Aut}\mathcal{E} \simeq \mathbb{C}^*$. □

Corollary 3.9. *A stable vector bundle is indecomposable: it is not isomorphic to a direct sum of non-trivial sub-bundles.*

Proof. The automorphism group of a direct sum $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ contains $\mathbb{C}^* \times \mathbb{C}^*$, so $\mathcal{E}$ cannot be simple. Then, by Proposition 3.8, it cannot be stable. □

The following result is key to understanding semi-stable bundles: these are extensions of stable bundles of the same slope.

Theorem 3.10 (Seshadri, [Ses67]). *The simple objects in the category of semi-stable bundles of slope μ are the stable bundles of slope μ. Any semi-stable holomorphic vector bundle of slope μ on Σ_g admits a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

by holomorphic sub-bundles such that, for all $i \in \{1; \cdots ; k\}$,

(1) *$\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable,*
(2) *$\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$.*

Such a filtration is called a ***Jordan-Hölder filtration of length*** *k of $\mathcal{E}$.*

Proof. Recall that a simple object in an Abelian category is an object with no non-trivial sub-object. In particular, a stable bundle $\mathcal{E}$ is simple in that sense (it contains no non-trivial sub-bundle *of slope equal to* $\mu(\mathcal{E})$). Conversely, if a semi-stable bundle $\mathcal{E}$ is simple in that sense, then any non-trivial sub-bundle $\mathcal{F} \subset \mathcal{E}$ satisfies $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ because $\mathcal{E}$ is semi-stable, and $\mu(\mathcal{F}) \neq \mu(\mathcal{E})$ because $\mathcal{E}$ has no non-trivial sub-objects in the category of semi-stable bundles with slope μ.
To prove the existence of a Jordan-Hölder filtration for a semi-stable bundle $\mathcal{E}$, observe that increasing and decreasing sequences of sub-bundles of $\mathcal{E}$ are stationary because of the bounds on the rank. If $\mathcal{E}$ is not a simple object, there exists a non-trivial sub-bundle $\mathcal{E}'$ of $\mathcal{E}$ which is semi-stable and of slope μ. If $\mathcal{E}'$ is not a simple object, we can go on and find a decreasing sequence of non-trivial (semi-stable) sub-bundles (of slope μ) in $\mathcal{E}$. Such a sequence is stationary, and we call $\mathcal{E}_1$ the final term: it is a simple sub-object of $\mathcal{E}$, so it is a stable bundle of slope μ. In particular, $\mathcal{E}/\mathcal{E}_1$ is semi-stable and also has slope μ (see Exercise 3.3). So there is a sub-bundle $\mathcal{E}_2/\mathcal{E}_1$ which is stable and of slope μ. This gives an increasing sequence

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$$

of (semi-stable) sub-bundles of $\mathcal{E}$ (of slope μ) whose successive quotients are stable bundles of slope μ. Such a sequence is stationary, so there is a k such that $\mathcal{E}_k = \mathcal{E}$, and the resulting filtration of $\mathcal{E}$ is a Jordan-Hölder filtration. □

One may observe that, to show the existence of a filtration whose successive quotients are simple objects in the category of semi-stable bundles of slope μ, the proof only used that decreasing and increasing sequences *of such bundles* were stationary. An Abelian category satisfying these properties is called *Artinian* (decreasing sequences of sub-objects are stationary) and *Noetherian* (increasing sequences of sub-objects are stationary).

Observe that if a bundle is stable, it admits a Jordan-Hölder filtration of length 1, namely $0 = \mathcal{E}_0 \subset \mathcal{E}_1 = \mathcal{E}$. In general, there is no unicity of the Jordan-Hölder filtration, but the isomorphism class of the graded object associated to a filtration is unique, as shown by the next result. In particular, the lengths of any two Jordan-Hölder filtrations of $\mathcal{E}$ are equal and a semi-stable bundle is stable if and only if its Jordan-Hölder filtrations have length 1.

Proposition 3.11 (Seshadri, [Ses67]). *Any two Jordan-Hölder filtrations*

$$(S) : 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

and

$$(S') : 0 = \mathcal{E}'_0 \subset \mathcal{E}'_1 \subset \cdots \subset \mathcal{E}'_l = \mathcal{E}$$

of a semi-stable vector bundle $\mathcal{E}$ have same length $k = l$, and the associated graded objects

$$\mathrm{gr}(S) := \mathcal{E}_1/\mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k/\mathcal{E}_{k-1}$$

and

$$\mathrm{gr}(S') := \mathcal{E}'_1/\mathcal{E}'_0 \oplus \cdots \oplus \mathcal{E}'_k/\mathcal{E}'_{k-1}$$

satisfy

$$\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{E}'_i/\mathcal{E}'_{i-1}$$

for all $i \in \{1; \cdots ; k\}$.

Proof. Assume for instance that $l < k$. Then there exists an $i \in \{1; \cdots ; k\}$ such that $\mathcal{E}'_1 \subset \mathcal{E}_i$ and $\mathcal{E}'_1 \not\subset \mathcal{E}_{i-1}$. So the map $\mathcal{E}_1 \hookrightarrow \mathcal{E}_i \longrightarrow \mathcal{E}_i/\mathcal{E}_{i-1}$ is a non-zero morphism between stable bundles of slope μ. By Proposition 3.7, it is an isomorphism. So $\mathcal{E}'_1 \cap \mathcal{E}_{i-1} = 0$ and $\mathcal{E}_i = \mathcal{E}_{i-1} \oplus \mathcal{E}'_1$. Then,

$$(S_1) : 0 \subset \mathcal{E}'_1 \subset \mathcal{E}'_1 \oplus \mathcal{E}_1 \subset \cdots \subset \mathcal{E}'_1 \oplus \mathcal{E}_{i-1} \subset \mathcal{E}_{i+1} \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

is a Jordan-Hölder filtration of length k of $\mathcal{E}$. Since (S') and (S_1) have the same first term, they induce Jordan-Hölder filtrations of $\mathcal{E}/\mathcal{E}'_1$, of respective lengths $l-1$ and $k-1$, with $l-1 < k-1$. Repeating this process $l-1$ more times, we eventually reach $\mathcal{E}/\mathcal{E}'_{l-1}$ with a Jordan-Hölder filtration of length $k-l > 0$. In particular, if the inclusions $\mathcal{E}'_{l-1} \subset \mathcal{E}_{k-1} \subset \mathcal{E}_k = \mathcal{E}$ are strict, there is a sub-bundle of $\mathcal{E}_{k-1}/\mathcal{E}'_{l-1}$ contradicting the stability of $\mathcal{E}/\mathcal{E}'_{l-1}$. So $l = k$.
Then we prove the second assertion by induction on the length k of Jordan-Hölder filtrations of $\mathcal{E}$. If $k = 1$, it is obvious. If $k > 1$, consider again the filtration (S_1). It satisfies $\mathrm{gr}(S_1) \simeq \mathrm{gr}(S)$. Moreover, (S_1) and (S') have the same first term, so they induce Jordan-Hölder filtrations $(\overline{S_1})$ and $(\overline{S'})$ of length $k-1$ of $\mathcal{E}/\mathcal{E}'_1$. By the induction hypothesis $\mathrm{gr}(\overline{S_1}) \simeq \mathrm{gr}(\overline{S'})$. So

$$\mathrm{gr}(S) \simeq \mathrm{gr}(S_1) \simeq \mathrm{gr}(\overline{S_1}) \oplus \mathcal{E}'_1 \simeq \mathrm{gr}(\overline{S'}) \oplus \mathcal{E}'_1 \simeq \mathrm{gr}(S')\,.$$

□

This motivates the following definition.

Definition 3.12 (Poly-stable bundles)**.** A holomorphic vector bundle $\mathcal{E}$ on Σ_g is called poly-stable if it is isomorphic to a direct sum

$$\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$$

of stable vector bundles of the same slope.

Evidently, a stable bundle is poly-stable. We point out that a poly-stable vector bundle of rank r admits a reduction of its structure group $\mathbf{GL}(r, \mathbb{C})$ to a sub-group of the form $\mathbf{GL}(r_1, \mathbb{C}) \times \cdots \times \mathbf{GL}(r_k, \mathbb{C})$, with $r_1 + \cdots + r_k = r$.

Proposition 3.13. *Let $\mathcal{F}_1, \cdots , \mathcal{F}_k$ be stable vector bundles of slope μ. Then $\mathcal{E} := \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$ is a semi-stable vector bundle of slope μ.*

Proof. $\mathcal{E}$ is a (trivial) extension of semi-stable vector bundles of slope μ, so it is semi-stable of slope μ (see Exercise 3.3). □

The graded object associated to any Jordan-Hölder filtration of a semi-stable vector bundle $\mathcal{E}$ is a poly-stable vector bundle (since it is a direct sum of simple objects in the category of semi-stable bundles of slope $\mu(\mathcal{E})$, it is a semi-simple object in that category). By Proposition 3.11, its isomorphism class is uniquely defined; it is usually denoted $\mathrm{gr}(\mathcal{E})$ and it is a graded isomorphism class of poly-stable vector bundles. The following notion is due to Seshadri (he used it to compactify the quasi-projective moduli variety of stable bundles constructed by Mumford).

Definition 3.14 (S-equivalence class, [Ses67])**.** The graded isomorphism class $\mathrm{gr}(\mathcal{E})$ associated to a semi-stable vector bundle $\mathcal{E}$ is called the **S-equivalence class** of $\mathcal{E}$. If $\mathrm{gr}(\mathcal{E}) \simeq \mathrm{gr}(\mathcal{E}')$, we say that $\mathcal{E}$ and $\mathcal{E}'$ are S-equivalent, and we write $\mathcal{E} \sim_S \mathcal{E}'$.

This defines an equivalence relation between semi-stable bundles of a given fixed slope. If two bundles of slope μ are S-equivalent, they have the same rank and the same degree (because the rank and degree of $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$ are equal to those of $\mathcal{E}$, see Exercise 3.4). The important point is that *two non-isomorphic semi-stable vector bundles may be S-equivalent.* Two S-equivalent stable bundles, however, are isomorphic, by definition of the S-equivalence class.

Definition 3.15 (Moduli set of semi-stable vector bundles)**.** The set $\mathcal{M}_{\Sigma_g}(r,d)$ of S-equivalence classes of semi-stable holomorphic vector bundles of rank r and degree d on Σ_g is called the **moduli set** of semi-stable vector bundles of rank r and degree d. It contains the set $\mathcal{N}_{\Sigma_g}(r,d)$ of isomorphism classes of stable vector bundles of rank r and degree d. When r and d are coprime, every semi-stable bundle is in fact stable and these two sets coincide.

Equivalently, $\mathcal{M}_{\Sigma_g}(r,d)$ is the set of isomorphism classes of poly-stable holomorphic vector bundles of rank r and degree d. This will be important in Subsection 3.2, where Donaldson's Theorem will be presented. The following theorem is the main result of the basic theory of vector bundles on a curve. It is due to Mumford for the first part ([Mum63]) and Seshadri for the second part ([Ses67]).

Theorem 3.16 (Mumford-Seshadri, [Mum63, Ses67])**.** *Let $g \geq 2$, $r \geq 1$ and $d \in \mathbb{Z}$.*

1. *The set $\mathcal{N}_{\Sigma_g}(r,d)$ of isomorphism classes of stable holomorphic vector bundles of rank r and degree d admits a structure of smooth, complex quasi-projective variety of dimension $r^2(g-1)+1$.*
2. *The set $\mathcal{M}_{\Sigma_g}(r,d)$ of S-equivalence classes of semi-stable holomorphic vector bundles of rank r and degree d admits a structure of complex projective variety of dimension $r^2(g-1)+1$. $\mathcal{N}_{\Sigma_g}(r,d)$ is an open dense sub-variety of $\mathcal{M}_{\Sigma_g}(r,d)$.*

In particular, when $r \wedge d = 1$, $\mathcal{M}_{\Sigma_g}(r,d) = \mathcal{N}_{\Sigma_g}(r,d)$ is a smooth complex projective variety.

For general r and d, it can in fact be shown that the set of *isomorphism* classes of semi-stable vector bundles of rank r and degree d does not admit such an algebraic structure ([Ses82]). In other words, to obtain a moduli variety, we *have to* identify S-equivalent, possibly non-isomorphic, objects.

3.2. **Donaldson's Theorem.** In [Don83], Donaldson proposed a differential-geometric proof of the celebrated Narasimhan-Seshadri theorem ([NS65]) which magnificiently complemented the symplectic approach to holomorphic vector bundles on a curve of Atiyah and Bott. Donaldson's theorem echoes, in an infinite-dimensional setting, a result by Kempf and Ness, relating semi-stable closed orbits of the action of a complex reductive group to the action of a maximal compact sub-group of that group. Thanks to a differential-geometric characterisation of stability, Donaldson's theorem establishes a homeomorphism between the moduli space $\mathcal{M}_{\Sigma_g}(r,d)$ and the symplectic quotient $F^{-1}(\{*i2\pi \frac{d}{r\mathrm{Id}_E}\})/\mathcal{G}_h$.

Theorem 3.17 (Donaldson, [Don83]). *Fix a smooth Hermitian vector bundle (E,h) of rank r and degree d. Let $\mathcal{E}$ be a holomorphic vector bundle of rank r and degree d, and let $O(\mathcal{E})$ be the corresponding orbit of unitary connections on (E,h). Then $\mathcal{E}$ is stable if and only if $O(\mathcal{E})$ contains a unitary connection A satisfying:*

(1) $\mathrm{Stab}_{\mathcal{G}_E}(A) \simeq \mathbb{C}^*$.
(2) $F_A = *i2\pi \frac{d}{r}\mathrm{Id}_E$.

Moreover, such a connection, if it exists, is unique up to an element of the unitary *gauge group $\mathcal{G}_h$.*

Indeed, since we know that isomorphism classes of holomorphic vector bundles of rank r and degree d are in one-to-one correspondence with *complex* gauge group orbits of unitary connections on (E,h), it seems natural to look for which unitary connections or more accurately, which *orbits* of unitary connections, correspond to isomorphism classes of *stable* holomorphic vector bundles of rank r and degree d. Donaldson's theorem states that these orbits are precisely the complex gauge orbits of unitary connections which are both *irreducible* (condition (1): $\mathcal{E} = (E,A)$ is an indecomposable holomorphic vector bundle) and *minimal Yang-Mills connections* (condition (2): A is an absolute minimum of the Yang-Mills functional $A \longmapsto \int_{\Sigma_g} \|F_A\|^2 \mathrm{vol}_{\Sigma_g}$, see [AB83, Don83]). Moreover, any such complex gauge orbit contains a unique *unitary* gauge orbit.

Corollary 3.18 (The Narasimhan-Seshadri theorem, [NS65]). *Graded isomorphism classes of poly-stable vector bundles of rank r and degree d are in one-to-one correspondence with unitary gauge orbits of minimal Yang-Mills connections:*

$$\mathcal{M}_{\Sigma_g}(r,d) \simeq F^{-1}\left(\{*i2\pi \frac{d}{r}\mathrm{Id}_E\}\right)/\mathcal{G}_h\,.$$

3.3. **Exercises.**

Exercise 3.1. Show that a semi-stable holomorphic which has coprime rank and degree is in fact stable.

Exercise 3.2. Show that $\mu(\mathcal{E}^*) = -\mu(\mathcal{E})$ and $\mu(\mathcal{E}\otimes\mathcal{E}') = \mu(\mathcal{E}) + \mu(\mathcal{E}')$. Compute $\mu(\mathrm{Hom}(\mathcal{E},\mathcal{E}'))$.

Exercise 3.3. Consider the extension (short exact sequence)

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

of $\mathcal{E}''$ by $\mathcal{E}'$.
a. Assume that $\mathcal{E}'$ and $\mathcal{E}''$ are semi-stable and both have slope μ. Show that $\mu(\mathcal{E}) = \mu$ and that $\mathcal{E}$ is semi-stable.
b. Show that if $\mathcal{E}'$ and $\mathcal{E}''$ are stable and have the same slope, $\mathcal{E}$ is not stable. *Hint*: By **a**, $\mu(\mathcal{E}) = \mu(\mathcal{E}')$ and $\mathcal{E}'$ is a sub-bundle of $\mathcal{E}$.
c. Let μ, μ', μ'' be the respective slopes of the bundles $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}''$. Show that

$$\mu' < \mu \Leftrightarrow \mu' < \mu'' \Leftrightarrow \mu < \mu''\,,$$
$$\mu' = \mu \Leftrightarrow \mu' = \mu'' \Leftrightarrow \mu = \mu''\,,$$
$$\mu' > \mu \Leftrightarrow \mu' > \mu'' \Leftrightarrow \mu > \mu''\,.$$

c. Suppose that the three bundles $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ have the same slope. Show that $\mathcal{E}$ is semi-stable if and only if $\mathcal{E}'$ and $\mathcal{E}''$ are semi-stable.

Exercise 3.4. Let $\mathcal{E}$ and $\mathcal{E}'$ be two semi-stable bundles of slope μ and assume that $\mathcal{E}$ and $\mathcal{E}'$ are S-equivalent. Show that $\mathrm{rk}\,\mathcal{E} = \mathrm{rk}\,\mathcal{E}'$ and $deg\,\mathcal{E} = deg\,\mathcal{E}'$. *Hint*: Consider the poly-stable object $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$ associated to an arbitrary Jordan-Hölder filtration of $\mathcal{E}$, and show that $deg\,(\mathcal{F}_1) + \cdots + deg\,(\mathcal{F}_k) = deg\,(\mathcal{E})$. Beware that the direct sum $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$ is not isomorphic to $\mathcal{E}$ in general.

EXERCISE 3.5. Let $\mathcal{E}$ be a holomorphic vector bundle and let $\mathcal{L}$ be a holomorphic line bundle.

a. Show that $\mu(\mathcal{E} \otimes \mathcal{L}) = \mu(\mathcal{E}) + \mu(\mathcal{L})$.
b. Show that $\mathcal{E}$ is stable (resp. semi-stable) if and only if $\mathcal{E} \otimes \mathcal{L}$ is stable (resp. semi-stable). *Hint*: Sub-bundles of $\mathcal{E} \otimes \mathcal{L}$ are of the form $\mathcal{F} \otimes \mathcal{L}$, where $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$.

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