A NOTE ON QUASI-HAMILTONIAN GEOMETRY AND REPRESENTATION SPACES OF SURFACES GROUPS

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ABSTRACT. In this note, we gather known applications of quasi-Hamiltonian geometry to the study of representations spaces of surface groups. We consider three apects of the geometry of representation spaces of surface groups: the symplectic structure that they carry, the number of connected components of representation spaces and the construction of Lagrangian submanifolds. The present survey is based on the work of Alekseev, Malkin and Meinrenken in [3], of Ho and Liu in [12, 13, 11] and of the author in [21, 20, 19]. This note grew out of a talk given at Keio University in September 2006.

1. INTRODUCTION

Given a compact Riemann surface Σ_g of genus $g \geq 0$, we consider the surface

$$\Sigma_{g,l} := \Sigma_g \backslash \{s_1, \dots, s_l\}$$

obtained from Σ_g by removing l pairwise disctinct points $s_1, \dots, s_l \in \Sigma_g$. The fundamental group of the surface $\Sigma_{g,l}$ thus obtained has the following finite presentation:

$$\pi_{g,l} := \pi_1(\Sigma_g \backslash \{s_1, \dots, s_l\}) =< \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_l \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{l} \gamma_j = 1 >$$

The object of study in this note will be the space (of equivalence classes) of representations of this group $\pi_{g,l}$ into a *compact connected* Lie group U. More precisely, given a compact connected Lie group U and l conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_l$of U, we fix a set of generators of $\pi_{g,l}$ and consider the set

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U) := \{(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l) \in (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{l} c_j = 1\} \subset U^{2g+l}$$

which is identified, via the choice of generators of $\pi_{g,l}$, to the set of group morphisms ϱ from $\pi_{g,l}$ to U satisfying $\varrho(\gamma_j) \in \mathcal{C}_j$ for all $j \in \{1, \dots, l\}$. This set is called the set of *representations of* $\pi_{g,l}$ *into* U. A first remark here is that depending on the choice of conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_l$ in U, this set might very well be empty. In fact, it is proved in [13] that when $g \geq 1$ and the compact connected group U is semi-simple, the representation space is always non-empty. Conditions for the representation space to be non-empty when U is an arbitrary compact connected Lie group can be found in [13]. For the $g = 0$ case, we refer to [1, 23], where the situation is seen to be much more complicated.
Observe then that the group U acting diagonally by conjugation on $(U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ preserves the relation $\prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{l} c_j = 1$, hence the set $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ of representations of $\pi_{g,l}$ into U. We may the consider the orbit space

$$\mathcal{M}_{g,l} := \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$$

of this action, usually called the *representation space* or *representation variety*, or yet, to avoid confusion, the *moduli space associated to* $\pi_{g,l}$.
The moduli spaces $\mathcal{M}_{g,l}$ are an important object of study and are connected to various areas of mathematics (see for instance [17, 5, 6]). In this note, we will consider three aspects of the geometry of these spaces:

1. the symplectic structure that they carry

Supported by the Japanese Society for Promotion of Science (JSPS).

2. the number of conected components of these moduli spaces
3. the construction of Lagrangian submanifolds of these moduli spaces

The symplectic structure of the moduli spaces $\mathcal{M}_{g,l}$ was first studied in [5] and [6], and extensively after that. In this note, we will present in section 2 the quasi-Hamiltonian description of this symplectic structure. This description was obtained by Alekseev, Malkin and Meinrenken in [3]: the representation space $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$ carries a symplectic structure because it is the quasi-Hamiltonian quotient associated to the space $(U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\mathcal{M}_{g,l} = \mu^{-1}(\{1\})/U$$

where μ is the momentum map

$$\begin{array}{rcl} \mu : (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ (a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l) & \longmapsto & [a_1, b_1] \dots [a_g, b_g] c_1 \dots c_l \end{array}$$

As a further remark on the sympectic structure of representation spaces, we would like to recall here that it is necessary to prescribe the conjugacy classes of generators c_j of $\pi_{g,l}$ correponding to loops around removed points of Σ_g, otherwise one only obtains Poisson structures (see [2]). Let us now come back to the three themes above. Although the second one is a question of a purely topological nature, the method used to compute the number of connected components of $\mathcal{M}_{g,l}$, which is due to Ho and Liu that we will present in section 3, uses the description of $\mathcal{M}_{g,l}$ as a quasi-Hamiltonian quotient. So will the construction of a Lagrangian submanifold of $\mathcal{M}_{g,l}$ presented in section 4.

2. Quasi-Hamiltonian geometry

In this section, we recall the notions of quasi-Hamiltonian geometry that we will need to study the moduli spaces $\mathcal{M}_{g,l} = \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$.

2.1. Quasi-Hamiltonian quotients.

We have seen that we have the following set-theoretic description:

$$\mathcal{M}_{g,l} = \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U = \mu^{-1}(\{1\})/U$$

with

$$\begin{array}{rcl} \mu : (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ (a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l) & \longmapsto & [a_1, b_1] \dots [a_g, b_g] c_1 \dots c_l \end{array}$$

The work of Alekseev, Malkin and Meinrenken in [3] shows that the map μ above is a *momentum map* for the diagonal conjugacy action of U on $(U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ in the following sense: there exists a 2-form ω on $(U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $((U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l, \omega, \mu)$ satisfies the axioms of definition 2.1 below.

Definition 2.1 (Quasi-Hamiltonian space, [3])**.** *Let (M, ω) be a manifold endowed with a 2-form ω and an action of the Lie group $(U, (.\,|\,.))$ leaving the 2-form ω invariant. We denote by $(.\,|\,.)$ an Ad-invariant non-degenerate symmetric bilinear form on $\mathfrak{u} = Lie(U)$, by $\theta^L = g^{-1}.dg$ and $\theta^R = dg.g^{-1}$ the Maurer-Cartan 1-forms of U, and by $\chi = \frac{1}{2}([\theta^L, \theta^L] \,|\, \theta^L)$ the Cartan 3-form of U. Finally, we denote by $X^\sharp$ the fundamental vector field on M associated to $X \in \mathfrak{u}$. Its value at $x \in M$ is : $X^\sharp_x := \frac{d}{dt}|_{t=0}(\exp(tX).x)$. Let $\mu : M \to U$ be a U-equivariant map (for the conjugacy action of U on itself).*
Then $(M, \omega, \mu : M \to U)$ is said to be a quasi-Hamiltonian space *(with respect to the action of U) if the map $\mu : M \to U$ satisfies the following three conditions:*

(i) $d\omega = -\mu^* \chi$
(ii) *for all* $x \in M$, $\ker \omega_x = \{X^\sharp_x \;:\; X \in \mathfrak{u} \,|\, (Ad\,\mu(x) + Id).X = 0\}$
(iii) *for all* $X \in \mathfrak{u}$, $\iota_{X^\sharp}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)$

where $(\theta^L + \theta^R \,|\, X)$ is the real-valued 1-form defined on U for any $X \in \mathfrak{u}$ by $(\theta^L + \theta^R \,|\, X)_u(\xi) := (\theta^L_u(\xi) + \theta^R_u(\xi) \,|\, X)$ (where $u \in U$ and $\xi \in T_u U$).
The map μ is called the momentum map.

We then have the following theorem, due to Alekseev, Malkin and Meinrenken.

Theorem 2.2 (Reduction of quasi-Hamiltonian spaces,[3]). *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space. If the compact connected Lie group U acts freely on the fiber $\mu^{-1}(\{1\})$, then:*

(i) *$1 \in U$ is a regular value of the momentum map μ (consequently, the set $\mu^{-1}(\{1\})$ is a submanifold of M).*
(ii) *the set $\mu^{-1}(\{1\})/U$ is a manifold.*
(iii) *if we denote by $i : \mu^{-1}(\{1\}) \hookrightarrow M$ the inclusion of the level submanifold $\mu^{-1}(\{1\})$ in M and by p the principal fibration $p : \mu^{-1}(\{1\}) \to \mu^{-1}(\{1\})/U$, the 2-form $i^*\omega$ is basis with respect to p: there exists a (unique) 2-form ω^{red} on $\mu^{-1}(\{1\})/U$ such that $i^*\omega = p^*\omega^{red}$.*
(iv) *the 2-form ω^{red} is symplectic.*

The symplectic manifold $\mu^{-1}(\{1\})/U$ is often denoted $M//U$ and called the quasi-Hamiltonian quotient *associated to the quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$.*

Dropping the assumption on the freeness of the action of U on $\mu^{-1}(\{1\})/U$, we also have the following generalization of theorem 2.2, saying that the set $M//U = \mu^{-1}(\{1\})/U$ is a disjoint union of symplectic manifolds, each of which is in fact obtained (see [20]) by the reduction procedure above from a quasi-Hamiltonian space endowed with a free action of a compact Lie group.

Theorem 2.3 (Structure of a quasi-Hamiltonian quotient, [20]). *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space. For any closed subgroup $K \subset U$, denote by M_K the isotropy manifold of type K in M:*

$$M_K = \{x \in M \mid U_x = K\}.$$

Denote by $\mathcal{N}(K)$ the normalizer of K in U and by L_K the quotient group $L_K := \mathcal{N}(K)/K$. Then the orbit space

$$(\mu^{-1}(\{1_U\}) \cap M_K)/L_K$$

is a smooth symplectic manifold.
Denote by $(K_j)_{j \in J}$ a system of representatives of closed subgroups of U. Then the orbit space $M^{red} := \mu^{-1}(\{1_U\})/U$ is the disjoint union of the following symplectic manifolds:

$$\mu^{-1}(\{1_U\})/U = \bigsqcup_{j \in J} (\mu^{-1}(\{1_U\}) \cap M_{K_j})/L_{K_j}.$$

Theorems 2.2 and 2.3 show that the representation spaces $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U = \mu^{-1}(\{1\})/U$ indeed carry a symplectic structure. We refer to [3] for a proof that this symplectic structure is the same as the one previously obtained in [5], [6], [8] and [16].

2.2. **Convexity theorems.** When the compact connected Lie group U is in addition simply connected, there exists a subset $\mathcal{W} \subset \mathfrak{u} = Lie(U)$ called a *Weyl alcove* such that:

(i) $\mathcal{W}$ is convex.
(ii) the exponential map restricts to an homeomorphism $\exp|_{\overline{\mathcal{W}}} : \overline{\mathcal{W}} \to \exp(\overline{\mathcal{W}}) \subset U$.
(iii) the set $\exp(\overline{\mathcal{W}})$ is a fundamental domain for the conjugacy action of U on itself: it contains exactly one point of each conjugacy class.

As a consequence of the existence of such a set $\mathcal{W} \subset \mathfrak{u}$, we see that it makes sense to ask whether a given subset $A \subset \exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{u}$ is convex. We then have the following theorem, due to Meinrenken and Woodward in [15].

Theorem 2.4 (Convexity theorem for group-valued momentum maps, [15, 3]). *Let U be a compact connected simply connected Lie group and let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian space with proper momentum map μ. Then, for any choice of a maximal torus $T \subset U$ and any choice of a closed Weyl alcove $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$, the set $\mu(M) \cap \exp(\overline{\mathcal{W}}) \subset \exp(\overline{\mathcal{W}})$ is a convex subpolytope of $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}}$, called the* momentum polytope. *Moreover, the fibres of μ are connected. In particular, the set $\mu^{-1}(\{1\})$ is a connected subset of M.*

In the presence of an involution β on the quasi-Hamiltonian space M, we have the following result:

Theorem 2.5 (A real convexity theorem for group-valued momentum maps, [22]). *Let (U,τ) be a compact connected simply connected Lie group endowed with an involutive automorphism τ such that the involution $\tau^-: u \in U \mapsto \tau(u^{-1})$ leaves a maximal torus T of U pointwise fixed, and let $\mathcal{W} \subset \mathfrak{t} := Lie(T)$ be a Weyl alcove. Let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian U-space with proper momentum map $\mu : M \to U$ and let $\beta : M \to M$ be an involution on M satisfying:*

(i) $\beta^*\omega = -\omega$
(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $x \in M$ *and all* $u \in U$
(iii) $\mu \circ \beta = \tau^- \circ \mu$
(iv) $M^\beta := Fix(\beta) \neq \emptyset$
(v) $\mu(M^\beta)$ *has a non-empty intersection with the fixed-point set* Q_0 *of* 1 *in* $Fix(\tau^-) \subset U$

Then:

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \mu(M) \cap \exp(\overline{\mathcal{W}})$$

In particular, $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ *is a convex subpolytope of* $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$*, equal to the full momentum polytope* $\mu(M) \cap \exp(\overline{\mathcal{W}})$.

Observe that an involutive automorphism τ of U such that τ^- leaves a maximal torus of U pointwise fixed always exists (it is a consequence of the existence of a *split* real form of the complexified Lie group $U^{\mathbb{C}}$). Theorems 2.4 and 2.5 are quasi-Hamiltonian analogues of convexity theorems for momentum maps in the usual Hamiltonian setting. We will see in sections 3 and 4 how they imply results on the number of connected components of representation spaces and construction of Lagrangian submanifolds of these spaces.

2.3. **Coverings of quasi-Hamiltonian spaces.** Before ending this section, we observe that theorems 2.4 and 2.5 only apply to quasi-Hamiltonian U-spaces with U compact connected and simply connected. To be able, in sections 3 and 4, to study connected components and construct Lagrangian submanifolds of $\mathcal{M}_{g,l} = \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$ when the compact connected Lie group is not simply connected, we will need the following result, due to Alekseev, Meinrenken and Woodward:

Proposition 2.6 ([4]). *Let U be a compact connected Lie group and let $\pi : \widetilde{U} \to U$ be a covering map. Set*

$$\widetilde{M} := M \times_U \widetilde{U} = \{(x, \widetilde{u}) \mid \mu(x) = \pi(\widetilde{u})\}$$

and

$$\begin{array}{ll} p : \widetilde{M} \longrightarrow M & \widetilde{\mu} : \widetilde{M} \longrightarrow \widetilde{U} \\ (x, \widetilde{u}) \longmapsto x & (x, \widetilde{u}) \longmapsto \widetilde{u} \end{array}$$

so that we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{M} = M \times_U \widetilde{U} & \xrightarrow{\widetilde{\mu}} & \widetilde{U} \\ p\downarrow & & \downarrow\pi \\ M & \xrightarrow{\mu} & U \end{array}$$

Finally, let us set $\widetilde{\omega} := p^*\omega$ and observe that $\widetilde{U}$ acts on $\widetilde{M}$ via

$$\widetilde{u}_0.(x, \widetilde{u}) := (\pi(\widetilde{u}_0).x, \widetilde{u}_0\widetilde{u}\widetilde{u}_0^{-1})$$

and that $\widetilde{\mu}$ is equivariant for this action. Then we have: $(\widetilde{M}, \widetilde{\omega}, \widetilde{\mu} : \widetilde{M} \to \widetilde{U})$ is a quasi-Hamiltonian $\widetilde{U}$-space.

The proof shows that this works because $\pi : \widetilde{U} \to \widetilde{U}$ is a *covering* homomorphism. In particular, $\widetilde{U}$ and U have isomorphic Lie algebras. Further, we have:

Proposition 2.7 ([4]). *The quasi-Hamiltonian quotients associated to M and $\widetilde{M}$ are isomorphic: the map $p : \widetilde{M} \to M$ sends $\widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\})$ to $\mu^{-1}(\{1_U\})$ and induces an isomorphism*

$$\widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\})/\widetilde{U} \simeq \mu^{-1}(\{1_U\})/U$$

In particular, if $\mu^{-1}(\{1_U\}) \neq \emptyset$ *then* $\widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\}) \neq \emptyset$.

3. Connected components of representation spaces

In this section, we will outline the proof, due to Ho and Liu in [12] of the following theorem, first proved by Goldman in [7] for $U = SU(2)$ and $U = SO(3)$, and by Li in [14] for an arbitrary compact connected semisimple Lie group.

Theorem 3.1 (Connected components of representation spaces, [7, 14])**.** *Let* Σ_g *be a* compact *Riemann surface of genus* $g \geq 1$ *and let* U *be a compact connected* semi-simple *Lie group. Denote by* $\pi_0(\mathrm{Hom}(\pi_1(\Sigma_g), U)/U)$ *the set of connected components of the representation space*

$$\mathcal{M}_{g,0} = \mathrm{Hom}(\pi_1(\Sigma_g), U)/U$$

and by $\pi_1(U)$ *the fundamental group of* U*, which, since the compact connected Lie group* U *is semi-simple, is a* finite *abelian group.*
Then, we have a bijection:

$$\pi_0(\mathrm{Hom}(\pi_1(\Sigma_g), U)/U) \stackrel{\simeq}{\longrightarrow} \pi_1(U)$$

When $g = 0$, the group $\pi_1(\Sigma_g)$ is $\pi_1(S^2) = \{1\}$, so that $Hom(\pi_1(\Sigma_g), U)$ is a single point, hence the moduli space $\mathcal{M}_{g,0}$ is always connected and the above theorem is no longer true.

Remark 3.2. *Recall that, in this note, our purpose is to show how one can use quasi-Hamiltonian geometry to study the geometry and the topology of representation spaces. To be able to illustrate this with simple examples, we limit ourselves, in this section, to* compact *surfaces and* semi-simple *Lie groups. We refer to* [13] *for the computation of the number connected components for surfaces with* removed points *and* arbitrary *compact connected Lie groups.*

It is remarkable in the above theorem that the number of connected components of the moduli space $\mathcal{M}_{g,0} = \mathrm{Hom}(\pi_{g,0}, U)/U$ depends only on the Lie group U and not on the genus $g \geq 1$. Such a phenomenon also occurs (as a matter of fact, the exact statement of theorem 3.1 still holds) for *complex* semi-simple Lie groups, as shown in [7] for $U = SL(2,\mathbb{C})$ and in [14] for arbitrary complex semi-simple Lie groups. This is no longer true for non-compact real semi-simple Lie groups. For instance, Goldman showed in [7] that if $U = PSL(2,\mathbb{R})$ then $\mathcal{M}_{g,0}$ has $4g-3$ connected components. Likewise, if $U = SL(2,\mathbb{R})$, the number of connected components of $\mathcal{M}_{g,0}$ is shown in [7] to be equal to $2^{2g+1}+2g-3$. Similar results for non-compact real Lie groups such as $PU(n,1)$ can be found in [25, 26] (see also [9, 10, 24]). It would be interesting to know if one can write a quasi-Hamiltonian proof of these results. As we shall soon see, this would require an analogue of theorem 2.4. Finally, Goldman also showed that if U is an algebraic semi-simple group then $\mathcal{M}_{g,0}$ has finitely many connected components, but that this is no longer true for non-simply connected nilpotent Lie groups (such as the Heisenberg group for instance).
We can now come back to giving a proof of theorem 3.1:

$$\pi_0(\mathrm{Hom}(\pi_1(\Sigma_g), U)/U) \stackrel{\simeq}{\longrightarrow} \pi_1(U)$$

This proof is due to Ho and Liu in [12]. Observe that theorem 3.1 says that if U is simply connected then the moduli space $\mathcal{M}_{g,0}$ is connected. This is a direct consequence of the convexity theorem 2.4: the moduli space $\mathcal{M}_{g,0}$ is the quasi-Hamiltonian quotient $\mathcal{M}_{g,0} = \mu^{-1}(\{1\})/U$ and since U is simply connected the fiber $\mu^{-1}(\{1\})$ of the momentum map μ is connected. To be able to reduce the general case to the case where U is simply connected, we will use proposition 2.6 when $\rho : \widetilde{U} \to U$ is the universal cover of U. Since U is semi-simple, the simply connected Lie group $\widetilde{U}$ is still compact. Further, we have an identification $\pi_1(U) \simeq \ker\rho \subset \mathcal{Z}(\widetilde{U})$:=center of $\widetilde{U}$. To prove that we have a bijection between $\pi_0(\mathcal{M}_{g,0})$ and $\pi_1(U) \simeq \ker\rho$, the strategy of Ho and Liu consists, following Goldman in [7], in constructing a continuous map

$$\sigma : \mathrm{Hom}(\pi_1(\Sigma_g), U) \longrightarrow \ker\rho$$

(this map σ is called the *obstruction map* in [7]) and showing, by methods of quasi-Hamiltonian geometry, that this continuous map σ is *surjective with connected fibres*, which will eventually imply

theorem 3.1.
Recall that the moduli space $\mathcal{M}_{g,0}$ is the quasi-Hamiltonian quotient

$$\mathcal{M}_{g,0} = \mathrm{Hom}(\pi_{g,0}, U)/U = \mu_U^{-1}(\{1\})/U$$

where μ_U is the momentum map

$$\begin{array}{rcl} \mu_U : M = (U\times U)\times\cdots\times(U\times U) & \longrightarrow & U \\ (a_1, b_1, \dots, a_g, b_g) & \longmapsto & \prod_{i=1}^{g}[a_i, b_i] \end{array}$$

Applying proposition 2.6 to the universal cover $\rho : \widetilde{U} \to U$ of U, the situation is as follows:

$$\begin{array}{ccc} (\widetilde{U}\times\widetilde{U})\times\cdots\times(\widetilde{U}\times\widetilde{U}) & \xrightarrow{\mu_{\widetilde{U}}} & \widetilde{U} \\ {\scriptstyle \rho^{2g}}\downarrow & & \downarrow{\scriptstyle \rho} \\ (U\times U)\times\cdots\times(U\times U) & \xrightarrow{\mu_U} & U \end{array}$$

Following Goldman, Ho and Liu define the obstruction map

$$\sigma : (U\times U)\times\cdots\times(U\times U) \longrightarrow \widetilde{U}$$

in the following way:

Definition 3.3. *Let $\sigma : (U\times U)\times\cdots\times(U\times U) \to \widetilde{U}$ be the map defined by*

$$\sigma(a_1, b_1, \dots, a_g, b_g) := \prod_{i=1}^{g}[\widetilde{a}_i, \widetilde{b}_i]$$

where $\rho(\widetilde{a}_i) = a_i$ and $\rho(\widetilde{b}_i) = b_i$ for all $i \in \{1, \dots, g\}$.

Lemma 3.4. *The map σ is well-defined and satisfies $\sigma\circ\rho^{2g} = \mu_{\widetilde{U}}$. In particular, since the covering map ρ^{2g} is an open surjective map, the obstruction map σ is continuous.*

Proof. If $\rho(\widetilde{a}_i) = \rho(\widetilde{a}'_i)$ and $\rho(\widetilde{b}_i) = \rho(\widetilde{b}'_i)$, then $\widetilde{a}'_i = x_i\widetilde{a}_i$ and $\widetilde{b}'_i = y_i\widetilde{b}_i$ with $x_i, y_i \in \ker\rho \subset \mathcal{Z}(\widetilde{U})$. It follows that $[\widetilde{a}'_i, \widetilde{b}'_i] = [a_i, b_i]$ for all i, hence that σ is well-defined and satisfies $\sigma\circ\rho^{2g} = \mu_{\widetilde{U}}$. □

To sum up, we have:

$$\begin{array}{ccc} (\widetilde{U}\times\widetilde{U})^g & \xrightarrow{\mu_{\widetilde{U}}} & \widetilde{U} \\ {\scriptstyle \rho^{2g}}\downarrow & \nearrow{\scriptstyle \sigma} & \downarrow{\scriptstyle \rho} \\ (U\times U)^g & \xrightarrow{\mu_U} & U \end{array}$$

Further:

Lemma 3.5. *We have $\sigma(\mu_U^{-1}(\{1\})) \subset \ker\rho$.*

Proof. If $\prod_{i=1}^{g}[a_i, b_i] = 1$, then:

$$\begin{array}{rcl} \rho\circ(a_1, b_1, \dots, a_g, b_g) & = & \rho\circ\mu_{\widetilde{U}}((\widetilde{a}_1, \widetilde{b}_1, \dots, \widetilde{a}_g, \widetilde{b}_g)) \\ & = & \mu_U\circ\rho^{2g}((\widetilde{a}_1, \widetilde{b}_1, \dots, \widetilde{a}_g, \widetilde{b}_g)) \\ & = & \mu(a_1, b_1, \dots, a_g, b_g) \\ & = & \prod_{i=1}^{g}[a_i, b_i] \\ & = & 1 \end{array}$$

□

We now begin the study of the fibres of the obstruction map σ.

Lemma 3.6. *For any* $z \in \ker\rho \subset \widetilde{U}$, *the fiber* $\mu_{\widetilde{U}}(\{z\})$ *is non-empty and connected. The map* $\rho^{2g} : (\widetilde{U}\times\widetilde{U})^g \to (U\times U)^g$ *restricts to a continuous surjective map*

$$\alpha_z : \mu_{\widetilde{U}}^{-1}(\{z\}) \longrightarrow \sigma^{-1}(\{z\}) \subset (U\times U)^g$$

Proof. The fibres of $\mu_{\widetilde{U}}$ are non-empty because $\widetilde{U}$ is a compact connected semi-simple Lie group, hence $[\widetilde{U},\widetilde{U}] = \widetilde{U}$ and $z = 1\times\cdots\times 1\times[\widetilde{a}_g,\widetilde{b}_g]$ for some $\widetilde{a}_g,\widetilde{b}_g \in \widetilde{U}$. Since $\widetilde{U}$ is in addition simply connected, theorem 2.4 shows that the fiber $\mu_{\widetilde{U}}^{-1}(\{z\})$ is connected.
Consider now $(\widetilde{a}_1,\widetilde{b}_1,\dots,\widetilde{a}_g,\widetilde{b}_g) \in \mu_{\widetilde{U}}^{-1}(\{z\})$ (that is: $\prod_{i=1}^g[\widetilde{a}_i,\widetilde{b}_i] = z$) and set $a_i := \rho(\widetilde{a}_i)$ and $b_i := \rho(\widetilde{b}_i)$ for all i. Then $\sigma(a_1,b_1,\dots,a_g,b_g) = \prod_{i=1}^g[\widetilde{a}_i,\widetilde{b}_i] = z$ so that ρ^{2g} indeed restricts to a continuous map $\alpha_z : \mu_{\widetilde{U}}^{-1}(\{z\}) \to \sigma^{-1}(\{z\})$. Surjectivity of α_z follows from the construction of σ. □

From this we deduce immediately:

Proposition 3.7. *The fibres of the continuous map* $\sigma|_{\mu_U^{-1}(\{1_U\})} : \mu_U^{-1}(\{1_U\}) \to \ker\rho$ *are non-empty and connected. Since* $\ker\rho$ *is a finite set, the connected components of* $\mu_U^{-1}(\{1_U\})$ *are precisely the fibres of* σ *above* $\ker\rho$. *Consequently the number of connected components of* $\mu^{-1}(\{1_U\})$ *and therefore of* $\mathcal{M}_{g,0} = \mathrm{Hom}(\pi_{g,0},U)/U$ *is equal to the cardinal of* $\ker\rho \simeq \pi_1(U)$. *More precisely, the map*

$$\sigma|_{\mu_U^{-1}(\{1_U\})} : \mu_U^{-1}(\{1_U\}) \longrightarrow \ker\rho$$

induces a map

$$\overline{\sigma} : \mathcal{M}_{g,0} = \mu_U^{-1}(\{1\})/U \longrightarrow \ker\rho \simeq \pi_1(U)$$

whose fibres are the connected components of $\mathcal{M}_{g,0}$, *thereby proving theorem 3.1.*

Proof. The fact that the map $\sigma|_{\mu^{-1}(\{1_U\})}$ has non-empty connected fibres follows from lemma 3.6: the continuous image $\alpha_z(\mu_{\widetilde{U}}^{-1}(\{z\})$ of a connected set is connected. The fact that $\sigma : \mu_U^{-1}(\{1_U\}) \to \ker\rho$ induces a map $\overline{\sigma} : \mu_U^{-1}(\{1_U\}) \to \ker\rho$ follows from the fact that $\ker\rho \subset \mathcal{Z}(\widetilde{U})$. □

As an application, we state the following result, first proved by Goldman in [7]: if $U = SO(3)$ (so that $\pi_1(U) = \mathbb{Z}/2\mathbb{Z}$) the moduli space $\mathrm{Hom}(\pi_{g,0},U)/U$ has 2 connected components.

4. Lagrangian submanifolds of representation spaces

In this section, we outline a general strategy for constructing Lagrangian submanifolds of quasi-Hamiltonian submanifolds of a quasi-Hamiltonian quotient $M//U = \mu^{-1}(\{1\})/U$ starting from a quasi-Hamiltonian space $(M,\omega,\mu : M \to U)$ and provide an example by applying this strategy to moduli spaces associated to surface groups.
Henceforth *we shall assume that* U *acts freely on* $\mu^{-1}(\{1\})$, so that theorem 2.2 applies and $\mu^{-1}(\{1\})/U$ is a symplectic manifold. Our strategy consists in obtaining a Lagrangian submanifold of the quasi-Hamiltonian quotient $M//U = \mu^{-1}(\{1\})/U$ by constructing an anti-symplectic involution ν on the symplectic space $M//U$. Then, if the fixed-point set of ν is non-empty, it is a Lagrangian submanifold of $M//U$. More precisely, we give sufficient conditions on an involution β on the quasi-Hamiltonian space $(M,\omega,\mu : M \to U)$ for it to induce an anti-symplectic involution $\nu := \hat{\beta}$ on the associated quasi-Hamiltonian quotient $M//U = \mu^{-1}(\{1\})/U$. To state such a result, we draw on the usual Hamiltonian case considered in [18] and assume that the compact connected Lie group U is endowed with an involutive automorphism τ. We then have:

Proposition 4.1 ([21])**.** *Let* $(M,\omega,\mu : M \to U)$ *be a quasi-Hamiltonian space and let* τ *be an involutive automorphism of* U. *Denote by* τ^- *the involution on* U *defined by* $\tau^-(u) = \tau(u^{-1})$ *and let* β *be an involution on* M *such that:*

(i) $\forall u \in U,\ \forall x \in M,\ \beta(u.x) = \tau(u).\beta(x)$
(ii) $\forall x \in M,\ \mu\circ\beta(x) = \tau^-\circ\mu(x)$
(iii) $\beta^*\omega = -\omega$

then β induces an anti-symplectic involution $\hat{\beta}$ on the quasi-Hamiltonian quotient

$$M//U := \mu^{-1}(\{1\})/U$$

defined by $\hat{\beta}([x]) = [\beta(x)]$. If $\hat{\beta}$ has fixed points, then $Fix(\hat{\beta})$ is a Lagrangian submanifold of $M//U$.

From now on, we assume additionally that the involution τ^- leaves a *maximal* torus $T \subset U$ *pointwise* fixed, so that the assumptions (U, τ) appearing in theorem 2.5 are satisfied. Recall that such an involutive automorphism τ of U always exists, as was recalled earlier. The rest of this section will be devoted to proving that the assumption $Fix(\hat{\beta}) \neq \emptyset$ is in fact *always* satisfied if U is a compact connected *semi-simple* Lie group, provided that the involution β on M has fixed points whose image lies in the connected component of $Fix(\tau^-) \subset U$ containing 1 (so that we can apply the real convexity theorem for group-valued momentum maps -theorem 2.5- stated in subsection 2.2). In fact, we will prove the following stronger result:

$$Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset \tag{1}$$

which immediately implies:

$$Fix(\hat{\beta}) \neq \emptyset$$

by definition of $\hat{\beta}$.
In order to prove that $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$, we will distinguish two cases. We begin with the case where the compact connected Lie group U is in addition simply connected and then deal with the case of a compact connected semi-simple Lie group. In this last case, we will reduce the situation to the case of simply connected groups by using proposition 2.6, much like what was done in section 3 in order to compute the number of connected components of the representation spaces.

4.1. **The case where U is simply connected.** When U is a compact connected simply connected Lie group, theorem 2.5 holds. We then have the following corollary, which is exacly the result we set out to prove (see (1)).

Proposition 4.2 ($Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$)**.** *If β satisfies the assumptions of theorem 2.5 and $\hat{\beta}$ designates the induced involution $\hat{\beta}([x]) := [\beta(x)]$ on the quasi-Hamiltonian quotient $M//U = \mu^{-1}(\{1\})/U$, we have: $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ and therefore $Fix(\hat{\beta}) \neq \emptyset$.*

Proof. Since $\mu^{-1}(\{1\}) \neq \emptyset$ and since we always have $1 \in \exp(\overline{\mathcal{W}})$, we obtain, using theorem 2.5:

$$1 \in \mu(M) \cap \exp(\overline{\mathcal{W}}) = \mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$$

that is:

$$Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$$

If $x \in Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$, then by definition $\hat{\beta}([x]) = [\beta(x)] = [x]$. □

Observe that, as in section 3, to prove the proposed statement (1) for simply connected compact connected Lie groups, one applies directly a theorem from quasi-Hamiltonian geometry.

4.2. **The case where U is semi-simple.** To prove that the statement $Fix(\beta) \cap \mu^{-1}(\{1\})$ still holds when U is assumed to be semi-simple but not necessarily simply connected, we use proposition 2.6 to construct a quasi-Hamiltonian $\widetilde{U}$-space $(\widetilde{M} = M \times_U \widetilde{U}, \widetilde{\omega}, \mu : \widetilde{M} \to \widetilde{U})$, where $\widetilde{U}$ is the universal cover of U. Since U is semi-simple, the simply connected group $\widetilde{U}$ is still compact and we can therefore apply proposition 4.2 to the quasi-Hamiltonian space $(\widetilde{M}, \widetilde{\omega}, \mu : \widetilde{M} \to \widetilde{U})$. This will turn out to be sufficient.
First, we need to observe that if β is a form-reversing involution on M, it induces a form-reversing involution $\widetilde{\beta}$ on $\widetilde{M}$. As a first step, observe that since the compact connected groups U and $\widetilde{U}$ have isomorphic Lie algebras, the involutive automorphism τ of U induces an involutive automorphism of $\widetilde{U}$, that we denote by $\widetilde{\tau}$. In particular, we have $\pi \circ \widetilde{\tau} = \tau \circ \pi$, where π is the covering map $\pi : \widetilde{U} \to U$. We will denote by $\widetilde{\tau}^-$ the involution $\widetilde{\tau}^-(\widetilde{u}) := \widetilde{\tau}(\widetilde{u}^{-1})$. If τ is of maximal rank, so is $\widetilde{\tau}$. If we denote by Q_0 the connected component of 1_U in $Fix(\tau^-) \subset U$ and by $\widetilde{Q}_0$ the connected component of $1_{\widetilde{U}}$ in $Fix(\widetilde{\tau}^-) \subset \widetilde{U}$, the covering map $\pi : \widetilde{U} \to U$ restricts to a covering map $\pi|_{\widetilde{Q}_0} : \widetilde{Q}_0 \to Q_0$. Then:

Proposition 4.3. *Let β be a form-reversing involution on the quasi-Hamiltonian space $(M,\omega,\mu : M \to U)$, compatible with the action of (U,τ) and the momentum map μ. Then the map*

$$\begin{array}{rcl} \widetilde{\beta} : \widetilde{M} & \longrightarrow & \widetilde{M} \\ (x,\widetilde{u}) & \longmapsto & (\beta(x), \widetilde{\tau}^-(\widetilde{u})) \end{array}$$

is a form-reversing involution on the quasi-Hamiltonian space $(\widetilde{M},\widetilde{\omega},\widetilde{\mu} : \widetilde{M} \to \widetilde{U})$, satisfying $\widetilde{\beta}(\widetilde{u}.x) = \widetilde{\tau}(u).\widetilde{\beta}(x)$ and $\widetilde{\mu} \circ \widetilde{\beta} = \widetilde{\tau}^- \circ \widetilde{\mu}$.

We then have:

Theorem 4.4. *Let (U,τ) be a compact connected semi-simple Lie group endowed with an involutive automorphism τ of maximal rank, and let $(M,\omega,\mu : M \to U)$ be a connected quasi-Hamiltonian U-space such that $\mu^{-1}(\{1_U\}) \neq \emptyset$. Let β be a form-reversing compatible involution β on M, whose fixed-point set $Fix(\beta)$ is not empty and has an image under μ that intersects the connected component of 1_U in $Fix(\tau^-) \subset U$. Then:*

$$Fix(\beta) \cap \mu^{-1}(\{1_U\}) \neq \emptyset$$

Proof. We will show that there exists a connected component of $\widetilde{M} = M \times_U \widetilde{U}$ which contains points of $\widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\})$ *and* fixed points of $\widetilde{\beta}$, and apply the corollary of the convexity theorem (corollary 4.2) to this connected component, which is a quasi-Hamiltonian space. From this we will deduce the statement of the theorem.
Since $\mu^{-1}(\{1\}) \neq \emptyset$ and $\mu(Fix(\beta)) \cap Q_0 \neq \emptyset$, there exist $x_0 \in M$ such that $\mu(x_0) = 1_U$ and $x_1 \in M$ such that $\beta(x_1) = x_1$ and $\mu(x_1) \in Q_0$. Since M is connected, there is a path $(x_t)_{t\in[0,1]}$ from x_0 to x_1. Set $u_t := \mu(x_t) \in U$ for all $t \in [0,1]$. Since $\pi : \widetilde{U} \to U$ is a covering map, we can lift the path $(u_t)_{t\in[0,1]}$ to a path $(\widetilde{u}_t)_{t\in[0,1]}$ on $\widetilde{U}$ such that $\pi(\widetilde{u}_t) = u_t = \mu(x_t)$ and $\widetilde{u}_0 = 1_{\widetilde{U}}$. Then $(x_t, \widetilde{u}_t) \in \widetilde{M} = M \times_U \widetilde{U}$ and it is a path going from $(x_0, \widetilde{u}_0) = (x_0, 1_{\widetilde{U}})$ to $(x_1, \widetilde{u}_1)$, which are therefore contained in a same connected component $\widetilde{M_0}$ of $\widetilde{M}$. Then, we have $\widetilde{\mu}(x_0, 1_{\widetilde{U}}) = 1_{\widetilde{U}}$ and, since $\pi(\widetilde{u}_1) = u_1 = \mu(x_1) \in Q_0 \subset Fix(\tau^-)$, we have $\widetilde{u}_1 \in \widetilde{Q}_0 \subset Fix(\widetilde{\tau}^-)$, hence

$$\widetilde{\beta}(x_1, \widetilde{u}_1) = (\beta(x_1), \widetilde{\tau}^-(\widetilde{u}_1)) = (x_1, \widetilde{u}_1)$$

and $\widetilde{\mu}(x_1, \widetilde{u}_1) = \widetilde{u}_1 \in \widetilde{Q}_0$. Therefore, the connected component $\widetilde{M_0}$ of $\widetilde{M}$, which is a quasi-Hamiltonian $\widetilde{U}$-space, contains points of $\widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\})$ and points of $Fix(\widetilde{\beta})$ whose image is contained in $\widetilde{Q}_0$. Since $\widetilde{U}$ is simply connected, we can apply corollary 4.2 and conclude that $Fix(\widetilde{\beta}) \cap \widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\}) \neq \emptyset$. Take now $(x,\widetilde{u}) \in Fix(\widetilde{\beta}) \cap \widetilde{\mu}^{-1}(\{1_{\widetilde{U}}\})$. In particular, $\widetilde{u} = 1_{\widetilde{U}}$. Since $\widetilde{\beta}(x,\widetilde{u}) = (x,\widetilde{u})$, we have $\beta(x) = x$ and $\mu(x) = \mu \circ p(x,\widetilde{u}) = \pi \circ \widetilde{\mu}(x,\widetilde{u}) = \pi(\widetilde{u}) = \pi(1_{\widetilde{U}}) = 1_U$. That is: $x \in Fix(\beta) \cap \mu^{-1}(\{1_U\})$, which is therefore non-empty. □

This completes the program announced at the beginning of this section (see (1)). We refer to [19] for a proof of the fact that when $M = (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ and U is an *arbitrary* compact connected Lie group, we still have $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ (see also subsection 4.3 below).

4.3. **An example of form-reversing involution β.** We end this note with an example of a form-reversing involution $\beta : M \to M$ on the quasi-Hamiltonian space

$$M = (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$$

This involution satisfies the assumptions of theorem 2.5, as is shown in [19], which, as explained in the above subsections, provides an example of Lagrangian submanifold of the representation space

$$\mathcal{M}_{g,l} = \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U = M//U$$

for any compact connected semi-simple Lie group U. As a matter of fact, it is shown in [19] that the condition $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ is also satisfied for an *arbitrary* compact semi-simple Lie group (in the case where M is $(U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$). Indeed, in this case, the situation is reduced to the case of a simply connected Lie group by using proposition 2.6 for the covering $\rho : S \times G \to U$ where

$S \subset \mathcal{Z}(U)$ is a torus and G is a compact connected simply connected Lie group. The same technique also works for computing the number of connected components of the moduli spaces $\mathcal{M}_{g,l}$ (see [13]). In [19], the involution β is otained by introducing the following notion of *decomposable representation* of the fundamental group $\pi_{g,l} = \pi_1(\Sigma_g \backslash \{s_1, \dots, s_l\})$ into U:

Definition 4.5 (Decomposable representations of $\pi_1(\Sigma_g \backslash \{s_1, \dots, s_l\})$, [19])**.** *Let (U, τ) be a compact connected Lie group endowed with an involutive automorphism τ of maximal rank. A representation $(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l)$ of $\pi_{g,l} = \pi_1(\Sigma_g \backslash \{s_1, \dots, s_l\})$ into U is called* decomposable *if there exist $(g + l)$ elements $v_1, \dots, v_g, w_1, \dots, w_l \in U$ satisfying:*

(i) $\tau(v_i) = v_i^{-1}$ *for all i and* $\tau(w_j) = w_j^{-1}$ *for all j.*
(ii) $[a_1, b_1] = v_1 v_2^{-1}$, $[a_2, b_2] = v_2 v_3^{-1}$, $\dots$, $[a_g, b_g] = v_g w_1^{-1}$, $c_1 = w_1 w_2^{-1}$, $c_2 = w_2 w_3^{-1}$, $\dots$, $c_l = w_l v_1^{-1}$.
(iii) $\tau(a_i) = v_{i+1}^{-1} b_i v_{i+1}$ *for all* $i \in \{1, \dots, g\}$ *(with $v_{g+1} = w_1$).*

We then show that these decomposable representations are characterized in terms of an involution β on $M = (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ satisfying the assumptions of theorem 2.5, from which we can deduce (see proposition 4.1 and theorem 4.4) that $Fix(\hat{\beta}) \neq \emptyset$ and is therefore a Lagrangian submanifold of the moduli space $\mathcal{M}_{g,l}$. Namely, we have:

Theorem 4.6 (A Lagrangian submanifold of the representation space [19])**.** *There exists a form-reversing involution β on the quasi-Hamiltonian space $M = (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that a representation $(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l)$ of $\pi_{g,l}$ into U is decomposable in the sense of definition 4.5 if and only if there exists an element $u \in U$ such that*

$$\beta(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l) = u.(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_l) \quad \text{and} \quad u \in Fix(\tau^-).$$

This involution β satisfies the assumptions of theorem 2.5, hence, if U is semi-simple, by theorem 4.4 we have

$$Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$$

which proves by proposition 4.1 that β induces an anti-symplectic involution $\hat{\beta}$ on the quasi-Hamiltonian quotient

$$\mathcal{M}_{g,l} = \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$$

whose fixed-point set $Fix(\hat{\beta})$ is non-empty and consists of equivalence classes of decomposable representations of $\pi_{g,l}$ into U: it is a Lagrangian submanifold of the moduli space $\mathcal{M}_{g,l}$.

In fact, we cannot immediately apply the results of subsections 4.1 and 4.2 because in general we do not have U acting freely on $\mu^{-1}(\{1\})$ in the above example where $M = (U \times U)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. We refer to [19] to see how to circumvent this difficulty. We also refer to [19] for a general expression of β. When $g = 0$ and $l = 3$, we have the following expression:

$$\beta(c_1, c_2, c_3) = (\tau^-(c_2 c_3)\tau^-(c_1)\tau(c_2 c_3), \tau^-(c_3)\tau^-(c_2)\tau(c_3), \tau^-(c_3))$$

When $g = 1$ and $l = 0$, we have:

$$\beta(a, b) = \big(\tau(b), \tau(a)\big)$$

Finally, we refer to [11] for another example of an anti-symplectic involution σ on the representation space $\mathcal{M}_{g,0} = \mathrm{Hom}(\pi_{g,0}, U)/U$ of the fundamental group of a *compact* surface.

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