Summer School

3 – 6 June, 2025

Aspects of Spectral Theory for Linear Operators

Departamento de Matemáticas

Universidad de los Andes

Queremos agradecer al Departamento de Matemáticas y a la Facultad de Ciencias de Investigacones de la Universidad de Los Andes y la London Mathematical Society por su soporte financiero a esta escuela.

Morning Lectures

Lyonell Boulton (Heriot-Watt University, UK)

Revivals and fractalisation

We will begin the course by considering the one-dimensional free Schrödinger equation with periodic boundary conditions

$$\begin{cases} \partial_t u(t,x) - i\partial_{xx} u(t,x) = 0 \\ u(t,0) = u(t,1) \\ \partial_x u(t,0) = \partial_x u(t,1) \\ u(0,x) = u_0(x) \end{cases}$$

for $x \in \mathbb{R}$ and $x \in [0,1]$. The solution can be found easily by separation of variables. This is one of the simplest partial differential equations, yet only about 30 years ago, Micheal Berry [Ber96] and collaborators [BK96] discovered a remarkably complex behaviour which they named after the Victorian scientist William Henry Fox Talbot [Tal36].

The "Talbot effect" manifests in the system above through a striking pattern of rough "periodicity" in time for simple initial data $u_0(x)$. For example, if $u_0(x)$ is a step function, then

- for t "rational", $u(t,x)$ is a finite linear combination of certain copies of $u_0(x)$, but
- for t "irrational", $u(t,x)$ becomes a rough continuous function of x.

We call the first phenomenon *revival* and the second *fractalisation.*

During the course, we will examine how the classical tools of Fourier analysis and the analysis of PDEs can be used to show striking results abour revivals and fractalisation for a variety of time-evolution boundary problems. The sessions will be devoted to studying five dispersive equations (of first order in time) that support the revival/fractalisation dichotomy, either partially or coompletely. For this, we will cover subjects such as generalised Fourier expansions, spectral analysis of differential operators, the notation of fractal dimensions and perturbation methods for PDEs.

References and further reading

[Ber96] M. V. Berry. Quantum fractals in boxes. *J. Phys. A*, 29(20):6617–6629, 1996.

[BFP21] Lyonell Boulton, George Farmakis, and Beatrice Pelloni. Beyond periodic revivals for linear dispersive PDEs. *Proc. A.*, 477(2251):Paper No. 20210241, 20, 2021.

[BFP24] Lyonell Boulton, George Farmakis, and Beatrice Pelloni. The phenomenon of revivals on complex potential Schrödinger's equation. *Z. Anal. Anwend.*, 43(3-4):401–416, 2024.

[BK96] M. V. Berry and S. Klein. Integer, fractional and fractal Talbot effects. *J. Modern Opt.*, 43(10):2139–2164, 1996.

[BOPS21] Lyonell Boulton, Peter J. Olver, Beatrice Pelloni, and David A. Smith. New revival phenomena for linear integro-differential equations. *Stud. Appl. Math.*, 147(4):1209–1239, 2021.

[CEgT15] Vasilis Chousionis, M. Burak Erdo˘ gan, and Nikolaos Tzirakis. Fractal solutions of linear and nonlinear dispersive partial differential equations. *Proc. Lond. Math. Soc. (3)*, 110(3):543–564, 2015.

[EgT16] M. Burak Erdo˘ gan and Nikolaos Tzirakis. *Dispersive partial differential equations*, volume 86 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2016. Wellposedness and applications.

[Rod00] Igor Rodnianski. Fractal solutions of the Schrödinger equation. In *Nonlinear PDE's, dynamics and continuum physics (South Hadley, MA, 1998)*, volume 255 of *Contemp. Math.*, pages 181–187. Amer. Math. Soc., Providence, RI, 2000.

[Tal36] H.F. Talbot. Facts relating to optical science. No. IV. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 9(56):401–407, 1836.

David Krejčiřík (Czech Technical University in Prague, Czechia)

Geometrical aspects of spectral theory

Spectral theory is an extremely rich field which has found its application in many areas of physics and mathematics. One of the reason which makes it so attractive on the formal level is that it provides a unifying framework for problems in various branches of mathematics, for example partial differential equations, calculus of variations, geometry, stochastic analysis, etc.

The goal of the lecture is to acquaint the students with spectral methods in the theory of linear differential operators coming both from modern as well as classical physics, with a special emphasis put on geometrically induced spectral properties. We give an overview of both classical results and recent developments in the field, and we wish to always do it by providing a physical interpretation of the mathematical theorems.

References

[K] David Krejčiřík. Geometrical aspects of spectral theory. https://nsa.fjfi.cvut.cz/david/other/gspec.pdf

Short Communications

Mauro Artigiani
Universidad Nacional de Colombia, Sede Bogotá, Colombia

A spectral characterization of Maharam flows

Translation surfaces generalize flat tori to higher genus. In this work, we investigate the geodesic flow on $\mathbb{Z}^d$-covers of compact translation surfaces, assuming the existence of a renormalizing (pseudo-Anosov) map on the cover that lifts from a corresponding map on the base surface. Under these conditions, the natural invariant measures are Maharam measures, which can be interpreted as Lebesgue measure on a deformed version of the original surface. Following recent developments in the application of transfer operator techniques to parabolic dynamics, we demonstrate how these measures can be obtained through the analysis of a (twisted) transfer operator on appropriately chosen (anisotropic) Banach spaces.

This work in progress is in collaboration with Roberto Castorrini, Davide Ravotti, and Yuriy Tumarkin.

Mateo Bahos
Universidad de Los Andes, Bogotá, Colombia

Perturbaciones de rango uno no acotadas de un operador autoadjunto

En esta charla vamos a estudiar el espectro de operadores en $L^2(0,1)$ que tienen la forma

$$if'(x) + V(x)f(x) + f(2\pi)k(x)$$

donde V es un potencial acotado y $k(x) \in L^2(0, 2\pi)$. Tales operadores, por medio de un isomorfismo se pueden ver de la forma

$$if'(x) + f(2\pi)K(x)$$

con condiciones de frontera $f(0) = f(2\pi)$. Estos operadores resultan ser perturbaciones de rango no acotadas del operador autoadjunto $if'(x)$. El caso anterior se puede generalizar de la siguiente forma: Sean A un operador autoadjunto con espectro puramente discreto y f un funcional lineal densamente definido y no acotado, las perturbaciones de rango uno no acotadas de A vienen de la forma

$$A + f(\cdot)\psi.$$

Bajo ciertas condiciones sobre f podremos estudiar los autovalores de tales perturbaciones, centrandonos en aspectos como multiplicidades geométricas, algebraicas y comportamiento asintótico de los autovalores en el plano complejo.

Yeison Alejandro Gómez Hernández
Universidad Nacional de Colombia, Sede Manizales, Colombia

Problema Periódico Asociado al Sistema De Zakharov-Rubenchik/Benney-Roskes

En esta charla hablaremos del problema periódico asociado al sistema de Zakharov-Rubenchik/Benney-Roskes unidimensional

$$\begin{cases} \partial_t\psi - \sigma_3\partial_x\psi - i\delta\partial_x^2\psi + i\left\{\sigma_2|\psi|^2 + W\left(\rho + D\partial_x\phi\right)\right\}\psi = 0, \\ \partial_t\rho + \partial_x^2\phi + D\partial_x\left(|\psi|^2\right) = 0 \\ \partial_t\phi + \dfrac{1}{M^2}\rho + |\psi|^2 = 0 \end{cases}$$

el cual ha aparecido en el contexto de física de plasma y ondas de agua. Esencialmente, si el tiempo lo permite, mostraremos que podemos tener soluciones del problema lineal en $H^q(\mathbb{T}) \times H^s(\mathbb{T}) \times H^{s+1}(\mathbb{T})$, $q, s \in \mathbb{R}$, y, siguiendo las ideas de [Lan13], [Obr15] y [SP05], soluciones en $H^s(\mathbb{T}) \times H^s(\mathbb{T}) \times H^{s+1}(\mathbb{T})$, $s > 1/2$, de un modelo no lineal modificado.

Referencias

[Lan13] David Lannes. *The Water Waves Problem: Mathematical Analysis and Asymptotics*, volume 188 of *Mathematical Surveys and Monographs*. AMS, 2013.

[Obr15] Caroline Obrecht. *Sur l'approximation modulationnelle du problème des ondes de surface: Consistance et existence de solutions pour les systèmes de Benney-Roskes/Davey-Stewartson à dispersion exacte*. Thèse de doctorat, Université Paris Sud - Paris XI, 2015. NNT: 2015PA112121.

[SP05] Jean-Claude Saut and Gustavo Ponce. Well-posedness for the Benney-Roskes/Zakharov-Rubenchik system. *Discrete Cont. Dynamical Systems*, 13(3):811–825, 2005.

Edison Leguizamón
Universidad de Los Andes, Bogotá, Colombia

One-dimensional Schrödinger operators with singular perturbations on a non-discrete set

In the theory of Schrödinger operators, it is well known that δ an δ' interactions supported on discrete sets can be modeled via suitable self-adjoint extensions of second-order differential operators, and the spectra of such operators are well understood. However, a natural question arises: what happens when the interactions are supported on a non-discrete set? In this talk, we will present key results concerning the case of a lower bounded potential on the real line, and how the spectrum of the corresponding operator changes when the interactions are supported on a non-discrete set. Finally, we will provide a characterization of the embedded eigenvalues of the perturbed operator.

Andrei-Ionuţ Mohuţ
Universitatea Politehnica Timişoara, Romania.

A computational study of spectral behavior in physics-informed neural networks

This study highlights the connection between spectral theory and machine learning, offering insights into both the potential and limitations of using deep learning for classical mathematical problems. Physics-Informed Neural Networks (PINNs) provide a computational framework in which neural networks are trained to solve differential equations by embedding the equation itself, along with boundary, initial, or other constraints, directly into the loss function.

We present a series of computational experiments exploring how PINNs behave when applied to spectral problems and how spectral aspects can help design neural network architectures. We will examine how PINNs can approximate eigenvalues and eigenfunctions of linear differential operators and observe that their training process tends to favor low-frequency components that are learned more rapidly - a phenomenon known as spectral bias. In order to improve numerical accuracy in solving PDEs, we incorporate Random Fourier Feature (RFF) into neural networks architectures.

These experiments show a promising intersection between deep learning and spectral aspects.

Javier Moreno
Universidad de Los Andes, Bogotá, Colombia

Multiplicities and Discrete spectrum asymptotics for Schrödinger operators with complex potentials on a non-selfadjoint quantum star graph

In this talk, I will present some new spectral properties to a non-selfadjoint Schrödinger operator of the form $-\frac{d^2}{dx^2} + V$ on a quantum star graph Γ with a non-selfadjoint Robin condition at the central vertex with a complex parameter α and Dirichlet boundary conditions on the set of outer vertices. We consider a potential $V \in L^p(\Gamma; \mathbb{C})$ with $p > 2$ and set $(\mu_n(\alpha))_n$ and $(\lambda_n(\alpha))_n$ be the eigenvalues of this Schrödinger operator and of the corresponding unperturbed Laplacian, respectively, repeated according to their algebraic multiplicities such that $(\mathrm{Re}(\lambda_m(\alpha)))_n$ and $(\mathrm{Re}(\mu_n(\alpha)))_n$ are monotonically increasing sequences.

More specifically, I will prove that almost all the eigenvalues $\mu_n(\alpha)$ are simple if we assume that the edge lengths of Γ are incommensurable over $\{-1, 0, 1\}$. Furthermore, we will show that the asymptotic behavior for the real part of these eigenvalues is quadratic. Finally, we will present the following bounds for the eigenvalues of the Schrödinger operator using the eigenvalues of the Laplacian with $\alpha = 0$ (this is, Neumann-Kirchhoff conditions at the central vertex):

$$\lambda_{m-1}(0) + R_m < \mathrm{Re}(\mu_m(\alpha)) < \lambda_{m+1}(0) - R_m \text{ as } m \to \infty$$

with $0 < R_m = o(m)$.

This is a joint work with E. Leguizamón.

Andrés Felipe Patiño
Universidad de Los Andes, Bogotá, Colombia

Complete nonselfadjointness of dissipative Schrödinger operators on a bounded interval

We will discuss the extension theory of dissipative operators of the form $S + iV$ where S is a closed symmetric operator and V is a bounded nonnegative operator. Additionally, we will explore a technique to determine whether these extensions are completely nonselfadjoint. Specifically, we will examine the Schrödinger operator $S + iV$ on an interval, where iV is a dissipative multiplication potential. For this case, we will provide a full description of the dissipative extensions in terms of boundary conditions, and we will study their corresponding reducing selfadjoint subspaces, if they exist.

Leoncio Rodríguez Quiñones
Pontificia Universidad Javeriana, Bogotá, Colombia

Some Shape Optimization Problems

In this talk we will present some shape optimization problems, associated to low eigenvalue problems for the laplacian with different boundary conditions. We will also make a comment on one Escobar's result regarding the first non-zero eigenvalue for the Stekloff problem.

Mitsuru Wilson
TU Ilmenau, Germany

A spectral inclusion property of linear operators and operator pencils

I will present recent results on spectral inclusion for operator pencils and block operator matrices, based on several collaborative projects. The findings provide new insights into the interplay between spectral sets and geometric enclosures in the complex plane. Topics include:

- A spectral enclosure for the essential spectrum of the operator pencil $\lambda A + B$, where A is a closed densely defined linear operator and B is a bounded linear operator on a Banach space. It is shown that the essential spectrum is contained in a sector of the complex plane whose boundaries depend only on the angular sectors containing the essential spectra of A and B.

- A result establishing that the approximate point spectrum of a broad class of block operator matrices is contained in the closure of the quadratic numerical range. These enclosures yield explicit spectral gaps and are sharper than those obtained from classical numerical range techniques.

- A framework for the numerical range and quadratic numerical range for linear relations in Hilbert spaces, along with initial spectral inclusion results in this setting.

Together, these results deepen our understanding of the spectral structure of operator pencils and related operator-theoretic constructions, particularly regarding geometric constraints on the essential and approximate spectra.

Tuesday, June 3rd (Room: Room B-202)

8:00 – 9:00	*Registration*
9:00 – 9:15	*Opening*
9:15 – 10:45	Lyonell Boulton: *Revivals and fractalisation I*
10:45 – 11:15	Break
11:15 – 12:45	David Krejčiřík: *Geometrical aspects of spectral theory I*
12:45 – 14:00	Lunch Break
14:00 – 15:00	Problem Session
15:00 – 15:30	Break
15:30 – 16:10	Leoncio Rodríguez: *Some Shape Optimization Problems*
16:10 – 16:50	Andrei Mohuţ: *A computational study of spectral behavior in physics-informed neural networks*
16:50 – 17:20	Yeison Gómez: *Problema periódico asociado al sistema de Zakharov-Rubenchik/Benney-Roskes*

Wednesday, June 4th (Room: Room B-202)

9:00 – 10:30	David Krejčiřík: *Geometrical aspects of spectral theory II*
10:30 – 11:00	Break
11:00 – 12:30	Lyonell Boulton: *Revivals and fractalisation II*

Thursday, June 5th (Room: Room B-202)

9:00 – 10:30	Lyonell Boulton: *Revivals and fractalisation III*
10:30 – 11:00	Break
11:00 – 12:30	David Krejčiřík: *Geometrical aspects of spectral theory III*
12:30 – 14:00	Lunch Break
14:00 – 15:00	Problem Session
15:00 – 15:30	Break
15:30 – 16:10	Mitsuru Wilson: *A spectral inclusion property of linear operators and operator pencils*
16:10 – 16:50	Javier Moreno: *Multiplicities and discrete spectrum asymptotics for Schrödinger operators with complex potentials on a non-selfadjoint quantum graph*
16:50 – 17:30	Edison Leguizamón: *One-dimensional Schrödinger operators with singular perturbations on a non-discrete set*
17:30 –	Reception

Friday, June 6th (Room: Room B-202)

9:00 – 10:30	David Krejčiřík: *Geometrical aspects of spectral theory IV*
10:30 – 11:00	Break
11:00 – 12:30	Lyonell Boulton: *Revivals and fractalisation IV*
12:30 – 14:00	Lunch Break
14:00 – 15:00	Problem Session
15:00 – 15:30	Break
15:30 – 16:10	Mauro Artigiani: *A spectral characterization of Maharam flows*
16:10 – 16:40	Andrés Patiño: *Complete nonselfadjointness of dissipative Schrodinger operators on graphs*
16:40 – 17:10	Mateo Bahos: *Perturbaciones de rango uno no acotadas de un operador autoadjunto*
17:30 –	Entrega de certificados de asistencia

Contact Information

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