A spectral inclusion property of linear operators and operator pencils

joint work with Trunk, Qui, Qi, Khlif, Kchaou, Aydi, Gernandt

The SPIRIT of science

TECHNISCHE UNIVERSITÄT ILMENAU

spectral inclusion

Let H be a Hilbert space and

$$A : H \to H$$

be an operator. The spectral inclusion refers to a set $W \subset \mathbb{C}$ such that $\sigma(A) \subset W$. Typical examples include:

- $\star$ $\sigma(A) \subset B_{||A||}(0)$ if A is bounded
- $\star$ $\sigma(A) \subset \mathbb{R}$ if $A = A^*$
- $\star$ $\sigma(\Delta_D) \subset [0, \infty]$, $H = L^2(a, b)$.

Spectrum

Definition (Spectrum)

$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda$ has a bounded inverse$\}$ is the resolvent of an operator $A : H \to H$ and the spectrum of A is defined as

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

- $\star$ $\lambda \in \sigma_p(A) \iff \exists x \neq 0$ such that $Ax = \lambda x$
- $\star$ $\lambda \in \sigma_c(A) \iff A - \lambda$ is injective and has a dense range but not surjective
- $\star$ $\lambda \in \sigma_r(A) \iff A - \lambda$ is injective but its range is not dense range
- $\star$ $\lambda \in \sigma_{app}(A) \iff \exists (x_n),\ \|x_n\| = 1$ such that $\|(A - \lambda)x_n\| \to 0$
- $\star$ $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_{disc}(A)$

Spectral inclusion

Numerical range Let H be a Hilbert space and

$$A : H \to H$$

be an operator.

Definition (Toeplitz 1918)

The numerical range is defined as

$$W(A) = \{(Af, f) : f \in \operatorname{dom} A, \|f\| = 1\}$$

is called the numerical range

Theorem

The numerical range $W(A)$ is connected in $\mathbb{C}$.

Spectral inclusion

Theorem

If A is self adjoint, then $W(A)$ and $\sigma(A)$ are real and the norm of the resolvent satisfies the following bound.

$$\|(A-\lambda)^{-1}\| \leq 1/dist(\lambda, \sigma(A)), \quad \lambda \in \rho(A).$$

Theorem

For unbounded linear operators A, the inclusion $\sigma(A) \subset \overline{W}(A)$ of the spectrum prevails if every component of $\mathbb{C} \setminus W(A)$ contains at least one point of $\rho(A)$. Moreover,

$$\|(A-\lambda)^{-1}\| \leq 1/dist(\lambda, W(A)), \quad \lambda \notin \overline{W}(A).$$

Block matrix operator

- ⋆ Let $H_1 \oplus H_2$ be a Hilbert space and

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : D_{A,C} \oplus D_{B,D} \longrightarrow H_1 \oplus H_2$$

  be a linear operator where $D_{X,Y} = \mathrm{dom}\, X \cap \mathrm{dom}\, Y$.
- ⋆ One can leverage the matrix structure to capture more localized spectral inclusion than the numerical range.

Quadratic numerical range

Definition (Quadratic numerical range)

Let $\mathcal{A}$ be a block operator matrix and $(f, g) \in \operatorname{dom}(\mathcal{A})$.

$$\mathcal{A}_{f,g} := \begin{pmatrix} (Af, f) & (Bg, f) \\ (Cf, g) & (Dg, g) \end{pmatrix} \in M_2(\mathbb{C}).$$

The quadratic numerical range of $\mathcal{A}$ is defined as

$$W^2(\mathcal{A}) := \bigcup_{\|f\|=\|g\|=1} \sigma_p(\mathcal{A}_{f,g}).$$

Quadratic numerical range

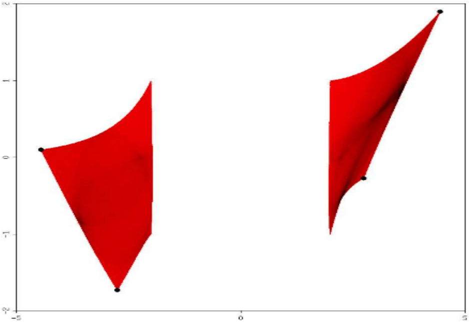


Abbildung: Simulated with $\begin{pmatrix} 2 & i & 1 & 3+i \\ i & 2 & 3+i & 1 \\ 1 & 3+i & -2 & i \\ 3+i & 1 & i & -2 \end{pmatrix}$

Some known results

Theorem (Theorems 2.5.3, 2.5.4, and 2.5.9 [tretter2008spectral])

For the class of block operator matrices $\mathcal{A}_\pm = \begin{pmatrix} 0 & B \\ -B^* & \pm D \end{pmatrix}$,

$$\sigma_p(\mathcal{A}_\pm) \subset W^2(\mathcal{A}_\pm) \subset W(\mathcal{A}_\pm).$$

If, in addition, $\dim H_1, \dim H_2 > 1$*, then*

$$W(\pm D) \cup \{0\} \subset W^2(\mathcal{A}_\pm).$$

Bounded case

Suppose $\mathcal{A}_\pm = \begin{pmatrix} 0 & B \\ -B^* & \pm D \end{pmatrix}$ is bounded. Then,

Theorem (H. Langer, A. Markus, V. Matsaev, C. Tretter (2001))

$$\sigma(\mathcal{A}_\pm) \subset \overline{W^2(\mathcal{A}_\pm)} \subset \overline{W(\mathcal{A}_\pm)}.$$

Proposition (W. Qiu, Y. Qi, C. Trunk, M.W. (2023))

$$\sigma(\mathcal{A}_\pm) \subset \{\lambda \in \mathbb{C} |\, 0 \leq \pm \operatorname{Re} \lambda \leq \gamma,\ |\operatorname{Im} \lambda| \leq 2b + c\}.$$

$$b = \sup_{f \in H_1,\, g \in H_2,\, \|f\|^2 + \|g\|^2 = 1} |\operatorname{Im}(f, Bg)|, \quad c = \sup_{g \in H_2,\, \|g\| = 1} |\operatorname{Im}(Dg, g)|$$

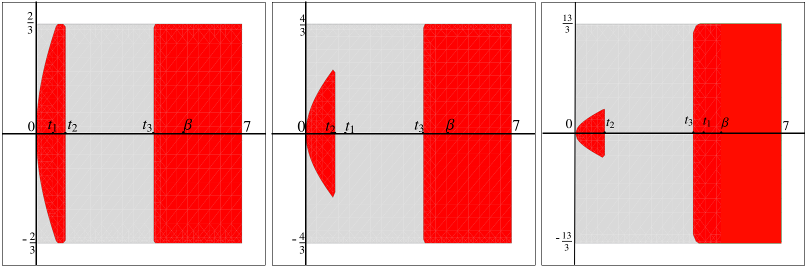


Abbildung: $c = \frac{1}{3}$ and $t_1 = \frac{1}{2}$ (left). $c = 1$ and $t_1 = \frac{20}{13}$ (center). $c = 4$ and $t_1 = \frac{169}{41}$ (right).

Unbounded case

Suppose

$$\mathcal{A}_{\pm} = \begin{pmatrix} 0 & B \\ -B^* & \pm D \end{pmatrix} : \mathrm{dom}(-B^*) \oplus \mathrm{dom}(B) \to H_1 \oplus H_2$$

is an off-diagonally dominant unbounded block operator matrix where B is a densely defined closed operator, D is B-bounded and accretive in H_2; *i.e.*
$\mathrm{Re}(Dg, g) \geq 0, g \in \mathrm{dom}(B)$.

Theorem (W. Qiu, Y. Qi, C. Trunk, M.W. (2023))

$$\sigma_{ap}(\mathcal{A}_{\pm}) \subset \overline{W^2(\mathcal{A}_{\pm})}.$$

Unbounded case

Theorem (W. Qiu, Y. Qi, C. Trunk, M.W. (2023))

Let $\mathcal{A}_\pm$ as before. If a component Ω of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}_\pm)}$ contains a point $\mu \in \rho(\mathcal{A}_\pm)$, then $\Omega \subset \rho(\mathcal{A}_\pm)$; in particular if every component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}_\pm)}$ contains a point $\mu \in \rho(\mathcal{A}_\pm)$, then

$$\sigma(\mathcal{A}_\pm) \subset \overline{W^2(\mathcal{A}_\pm)}.$$

Moreover,

$$\sigma(\mathcal{A}_\pm) \subset \left\{ \lambda \in \mathbb{C} \,\middle|\, 0 \leq \pm \operatorname{Re}\lambda \leq \gamma,\ |\operatorname{Im}\lambda| \leq \begin{cases} \frac{k|\operatorname{Re}\lambda|}{1-\frac{2}{\beta}|\operatorname{Re}\lambda|} & \pm\operatorname{Re}\lambda \in [0, \frac{\beta}{2}) \\ \infty & \pm\operatorname{Re}\lambda \in [\frac{\beta}{2}, \frac{\gamma}{2}] \\ \frac{k|\operatorname{Re}\lambda|}{\frac{2}{\gamma}|\operatorname{Re}\lambda|-1} & \pm\operatorname{Re}\lambda \in (\frac{\gamma}{2}, \gamma] \end{cases} \right.$$

Theorem (W. Qiu, Y. Qi, C. Trunk, M.W. (2023))

Suppose that D is a bounded and self-adjoint operator in $\mathcal{A}_\pm$. If there are $\pm\lambda_1, \pm\lambda_2 \in \rho(\mathcal{A}_\pm)$ with $\pm\lambda_1 < 0$ and $\pm\lambda_2 > \|D\|$, then

$$\sigma(\mathcal{A}_\pm) \subset \{\lambda \in \mathbb{C} | \, 0 \le \pm \operatorname{Re} \lambda \le \|D\|/2\} \cup [\|D\|/2, \|D\|]. \tag{1}$$

Essential spectrum of an operator pencil

Definition

Let X be a Banach space and $A : X \to X$ be a closed operator and $B : X \to X$ be a bounded operator and

$$\begin{aligned} P : \mathbb{C} &\to C(X) \\ \lambda &\mapsto \lambda A + B, \end{aligned}$$

be an operator pencil associated to A and B.

The essential spectrum of the operator pencil P is defined by

$$\sigma_\epsilon(P) = \{\lambda \in \mathbb{C} \mid 0 \in \sigma_{ess}(\lambda A + B)\}$$

where $\sigma_{ess}(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \notin \Phi(X)\}$.

Essential spectrum of an operator pencil

Assumption

Let S_1 and S_2 be two sectors,

$$S_1 = \{\lambda \mid \varphi_1 \leq \arg\lambda \leq \varphi_2\} \quad \text{and} \quad S_2 = \{\lambda \mid \theta_1 \leq \arg\lambda \leq \theta_2\}$$

with $\varphi_1, \varphi_2, \theta_1, \theta_2 \in [0, 2\pi)$ such that

(a) $\varphi_1 \leq \varphi_2 < \theta_1 \leq \theta_2$, that is, $S_1 \cap S_2 = \{0\}$,

(b) $\sigma_e(A) \subset S_1$ and $\sigma_e(B) \subset S_2$

(c) + some technical assumptions

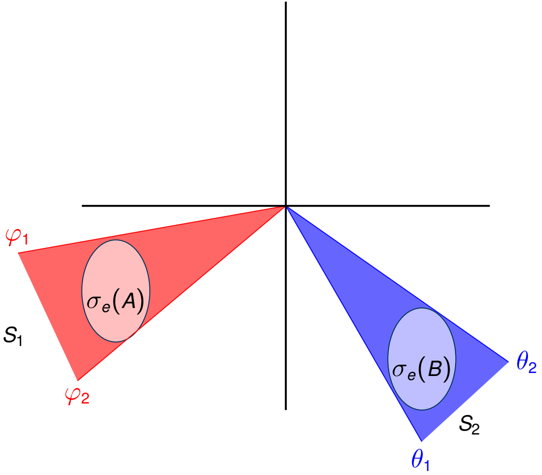


Abbildung: An example of a pair of wedges for $\varphi_1 = 190^\circ$, $\varphi_2 = 220^\circ$, $\theta_1 = 300^\circ$, $\theta_2 = 325^\circ$.

Results

Theorem (C. Trunk, H. Khlif, M. W. (2024))

Let A and B be as above. Then,

$$\sigma_e(A + B) \subseteq \sigma_e(A) + \sigma_e(B).$$

Lemma (C. Trunk, H. Khlif, M. W. (2024))

Let $\alpha \in \mathbb{C}$. Then, $\alpha\sigma_\epsilon(A) \subset \mathbb{C} \setminus (-S_2)$ implies that

$$\alpha \notin \sigma_\epsilon(P).$$

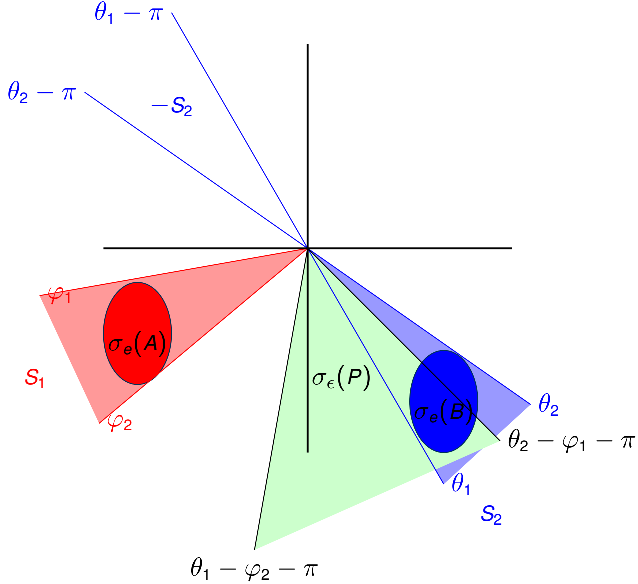
The main result

Theorem (C. Trunk, H. Khlif, M. W. (2024))

Suppose that the assumption holds for A and B. Then, the essential spectrum $\sigma_\epsilon(P)$ of the operator pencil

$$P(\lambda) = \lambda A + B$$

is contained in the sector Σ defined by the angles between $\theta_1 - \varphi_2 - \pi$ and $\theta_2 - \varphi_1 - \pi$ oriented counterclockwise, where these angles lie between $-\pi$ and π.


$\theta_1 - \pi$
$\theta_2 - \pi$
$-S_2$
φ_1
$\sigma_e(A)$
S_1
φ_2
$\sigma_\epsilon(P)$
$\sigma_e(B)$
θ_2
$\theta_2 - \varphi_1 - \pi$
θ_1
S_2
$\theta_1 - \varphi_2 - \pi$

Linear relations

Definition (Linear relations)

Any subspace $T \subset H \times H$ is called a linear relation.

Example

Let $A : \operatorname{dom} A \to H$ be an operator, then

$$\Gamma(A) = \{(x, y) \in H \times H : Ax = y\} \subset H \times H$$

is a subspace.

Linear relations

- $\operatorname{dom}(T) := \{x \in H : \exists y \in H \text{ such that } (x, y) \in T\}$
- $\operatorname{ran}(T) := \{y \in H : \exists x \in H \text{ such that } (x, y) \in T\}$
- $\ker(T) := \{x \in H : (x, 0) \in T\}$
- $\operatorname{mul}(T) := \{y \in H : (0, y) \in T\}$

The multivalue part

Suppose $y_1 \neq y_2$ and T is a multtivalued map $T : x \mapsto y_1$ and y_2. Then,

$$y_1 - y_2 = Tx - Tx = 0 = T0.$$

In other words, "$(0, y_1) \in \Gamma(T)$".

Linear relations

Theorem

If $\operatorname{mul}(T) = \{0\}$*, then* T *is a graph associated to an operator.*

In the above, this operator is also denoted by T.

Moreover, one can define the operator part T_{op} of T as follows:

$$T_{op} = \{(x, y) \in T : x \neq 0\}$$
$$\implies T_{op}x = y \text{ is a unique value}$$

Intuitively, $T/\operatorname{mul}(T)$ is a graph of an operator.

Theorem

Let T *be a linear relation in* H*. Then, there exist operators* E *and* A *such that*

$$Ax = Ey \qquad \text{for all } (x, y) \in T.$$

Spectrum of linear relations

Definition (Spectrum of linear relations)

The point spectrum $\sigma_p(T)$ of a linear relation T in H is defined as

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : (x, \lambda x) \in T\}$$
$$\sigma_{ap}(T) := \{\lambda\mathbb{C} : \exists\, (x_n, \tilde{x}_n) \in \mathcal{L}, \text{ such that } \|x_n\| = 1, \lim_{n\to\infty} \|\tilde{x}_n - \lambda x_n\| = 0\}$$

and the numerical range is defined as

$$W(T) := \{\langle y, x\rangle \; : \; (x, y) \in \mathcal{L}, \quad \|x\| = 1\}$$

Spectral inclusion for linear relatioins

Definition

Let T be a linear relation in $H_1 \oplus \cdots \oplus H_n$. The block linear relation is defined as

$$T = \left\{ (x, y) : y_j \in \bigoplus_{k=1}^{n} A_{jk} x_k \right\}.$$

Consequently,

$$W^n(T) = \{\lambda \in \sigma_p(T_x) \; : \; x \in X, \; \|x_j\| = 1\}.$$

where $T_x = \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_1, x_n) \\ & \vdots & \\ (A_{n1}x_n, x_1) & \cdots & (A_{nn}x_n, x_n) \end{pmatrix} \subset \mathbb{C}^2$ is the induced linear relation

Spectral inclusion for linear relatioins

Theorem (O. Kchaou, R. Aydi, H. Gernandt, M.W.)

- *For $n \geq 1$,*

$$W^{n+1}(T) \subseteq W^n(T)$$

- $$\sigma_{ap}(T) \subset \overline{W^n(T)}.$$

- $$W_e(T) := \Big\{\lambda \in \mathbb{C} \mid \exists\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset T \text{ with } \|x_n\| = 1, x_n \xrightarrow{w} 0, \langle y_n, x_n\rangle \to \lambda\} \supset W^n(T).$$

An example and a followup question

Suppose $M, C, K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $M\ddot{u} + C\dot{u} + Ku = 0$

$$\implies \frac{d}{dt} \underbrace{\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}}_{A} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ -c & -k \end{pmatrix}}_{-B} \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

1. Computing a spectral inclusion of the pencil $\lambda A + B$.
2. Leverage the matrix structure to achieve a tighter inclusion.
3. Estimates on the tightness of the inclusions.

Thank you!