Some Shape Optimization Problems

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Outline

In general, we should consider the operator L define on some suitable space of functions (for example $H^1(\Omega)$ but in general it can be a Banach or Hilbert Space). This operator L has the form

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u \tag{1}$$

where we assume the following conditions:

- Derivatives are in the sense of distributions.
- $a_{ij}(x)$, $i, j = 1, \ldots, N$ are bounded functions on a bounded open set $\Omega \subset \mathbb{R}^N$ with Lipchitz boundary.
- $a_0(x)$ is bounded function defined on Ω.
- We assume that $\exists \alpha > 0$, such that $\forall \zeta \in \mathbb{R}^N$, $\forall x \in \Omega$ $\sum_{i,j=1}^{N} a_{i,j=1}(x)\xi_i\xi_j \geq \alpha \|\xi\|^2$
- $\forall x \in \Omega$, $\forall i, j$ $a_{ij}(x) = a_{ji}(x)$

Introduction

Our focus is when $L = -\Delta$, and we consider the following boundary value problems:

- **Dirichlet Boundary Condition** Let $u \in H_0^1(\Omega)$ and we look at the boundary value problem:

$$-\Delta u = \lambda(\Omega) u \text{ in } \Omega \tag{2}$$
$$u = 0 \text{ on } \partial\Omega \tag{3}$$

- **Neumann Boundary Condition** Let $u \in H^1(\Omega)$ and we look at the boundary value problem:

$$-\Delta u = \mu(\Omega) u \text{ in } \Omega \tag{4}$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{5}$$

- **Stekloff Boundary Condition** Let $u \in H^1(\Omega)$ and we look at the boundary value problem:

$$-\Delta u = 0 \text{ in } \Omega \tag{6}$$
$$\frac{\partial u}{\partial n} = \sigma(\Omega) u \text{ on } \partial\Omega \tag{7}$$

Variational characterization of these eigenvalues

- **Dirichlet eigenvalues**

$$\lambda_k(\Omega) = \min_{S_k \in \mathcal{S}} \max_{v \in S_k, v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 dx}{\int_\Omega v^2(x) dx} \tag{8}$$

  where $\mathcal{S}$ denotes de collection of all subspaces of $H_0^1(\Omega)$ with dimension k

- **Neumann Boundary Condition**

$$\mu_k(\Omega) = \min_{S_k \in \mathcal{S}} \max_{v \in S_k, v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 dx}{\int_\Omega v^2(x) dx} \tag{9}$$

  where $\mathcal{S}$ denotes de collection of all subspaces of $H^1(\Omega)$ with dimension k

- **Stekloff Boundary Condition**

$$\sigma_k(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla v(x)|^2 dx}{\int_{\partial\Omega} v^2(x) dx} : v \in H^1(\Omega), \int_{\partial\Omega} v u_j ds = 0, j = 1, \ldots, k-1 \right\} \tag{10}$$

All these eigenvalues are nonnegative and can be arranged in an increasing sequence converging toward infinity. That is.

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_n(\Omega) \le \cdots \to \infty$$
$$0 \le \mu_1(\Omega) \le \mu_2(\Omega) \le \cdots \le \mu_n(\Omega) \le \cdots \to \infty$$
$$0 \le \sigma_1(\Omega) \le \sigma_2(\Omega) \le \cdots \le \sigma_n(\Omega) \le \cdots \to \infty$$

We will be interested in the cases $\lambda_1(\Omega)$, $\mu_2(\Omega)$ and $\sigma_2(\Omega)$ where the unknown is Ω.

$$\lambda_1(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 ds} : v \in H_0^1, v \neq 0(\Omega) \right\} \tag{11}$$

$$\sigma_2(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\partial\Omega} v^2 ds} : v \in H^1(\Omega), v \neq 0, \int_{\partial\Omega} vds = 0 \right\} \tag{12}$$

Notice the similarity with the first positive **Neumann** eigenvalue given by

$$\mu_2(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx} : v \in H^1(\Omega), v \neq 0, \int_\Omega vdx = 0 \right\} \tag{13}$$

(One can study other eigenvalues or even function of eigenvalues)

Eigenvalues and the geometry of the domains

Problems connecting the shape of a domain with the sequence of eigenvalues of an elliptic operator are very interesting!

For example, the most classical question in spectral geometry is **"Can you hear the shape of a drum?"**

In other words, is it possible to determine the shape of a rigid domain having knowledge of the eigenvalues?

Shapes and extremal values of eigenvalues

Our interest is to find shapes that yield extremal values of eigenvalues. For example, for the eigenvalue problems mentioned above we find questions like:

Question

Among all drums of given area, which is the one that produces de deepest bass note?

Equivalently what is the domain (with fixed area) that minimizes $\lambda_1(\Omega)$

Question

What is the biggest value of the n^{th} Neumann eigenvalue on a simply-connected planar domain of given area?

Equivalently, What is the biggest value of $\mu_n(\Omega)|\Omega|$?

Question

What is the biggest value of the n^{th} Steklov eigenvalue on a simply-connected planar domain of given perimeter?

Equivalently, What is the biggest value of $\sigma_n(\Omega)|\partial\Omega|$?

But why for these eigenvalues the questions ask for the maximum and not for the minimum?

In the case of **Neumann** eigenvalues, consider the domain $(0, L) \times (0, \ell)$, its n^{th} eigenvalue is given by $\mu_n = (n-1)^2\pi^2/L^2$. Thus letting $L \to \infty$ one sees that

$$\inf\{\mu_n(\Omega), |\Omega| = A\} = 0, \quad [12]$$

In the case of the **Steklov** eigenvalues, it is possible to construct a planar domain $\Omega_\delta = \mathbb{D}_1 \cup P_\delta \cup \mathbb{D}_2$ satisfying

$$\lim_{\delta \to 0} \sigma_n(\Omega_\delta) = 0 \text{ for all } n = 1, 2, \dots,$$

Where P_δ is a rectangular passage of length δ and width δ^3 [13].

Related work

- 1894 – 1896, **Lord Rayleigh** conjectured that for simply connected bounded planar domains, the 1^{st} Dirichlet eigenvalue λ_1, satisfies the inequality

$$\lambda_1(\Omega)|\Omega|^{1/2} \geq \sqrt{\pi}\lambda_1(\mathbb{D})$$

  and equality is attained when Ω is a disk. Here $\lambda_1(\mathbb{D}) = 2.4048$ is the least positive zero of the **Bessel** function $J_0(r)$.

- 1923, **Faber** and **Krahn** proved Lord Rayleigh conjecture in what is known as the **Faber-Krahn** inequality.

Theorem 1 (Faber-Krahn)

Let c be a positive number and B the ball of volume c. Then,
$\lambda_1(B) = \min\{\lambda_\Omega, \Omega \textit{ open subset of } \mathbb{R}^N, |\Omega| = c\}$

Remark: The ball is not the unique minimizer. In $\mathbb{R}^2$ you can remove a finite number of points from a disk and still get a minimizer. In general, you can remove from Ω a set of zero *capacity* and still get a minimizer[12]

Two interesting problems

Instead of looking at open sets with only a volume constraint. Consider looking for a solution to the following problem.

$$\min\{\lambda_1(\Omega), \Omega \subset D, |\Omega| = A \text{ (given) }\}[12] \tag{14}$$

That is, domains Ω lying in a given box D, with a given volume. **Remark**: Solution to this problem is positive in the class of *quasi-open* sets. This result is due to Butazzo and Dal Maso see Them 2.4.6 in [12]

Question

Let $\Omega^* \subset \mathbb{R}^N$ be a minimizer for problem (14). Prove what is the regularity of Ω^* when $N \geq 3$? For example, is it analytic? [12]

Question

Let $\Omega^* \subset \mathbb{R}^N$ be a minimizer for the problem (14). Can we show that if D is convex (or star shaped) then Ω^* is convex (or star shaped)

Related work

- 1954, **Szegö** proved that on simply connected planar domains

$$\mu_2(\Omega)|\Omega| \leq \mu_2(\mathbb{D})\pi$$

  and the equality holds if and only if Ω is a disk. Here $\mu_2(\mathbb{D})$ is the square of the first zero of the derivative of the first **Bessel** function of the first kind.
- 1956, **Weinberger** proved that the last inequality holds without the assumption of Ω being simply connected.

Theorem 2 (Szegö-Weinberger)

The ball maximizes the second Neumann eingevalue among (Lipschitz) open sets of given volume. Moreover, it is the only maximizer in this class.

Polya conjecture and Kröger result

Pólya conjectured that for any **regular** bounded planar domain Ω, the **Neumann** eigenvalues satisfy

$$\mu_k(\Omega)|\Omega| \leq 4(k-1)\pi, \quad k \geq 2 \tag{15}$$

In 1992, **Kröger** proved that on bounded domains with **piecewise smooth boundary** the best that one could obtain is

$$\mu_k(\Omega)|\Omega| \leq 8(k-1)\pi, \quad k \geq 3 \tag{16}$$

Actually, **Girouard**,**Nadirashvili** and **Polterovich** in [5] improved this result for $k = 2$ when Ω is simply connected

$$\mu_3(\Omega)|\Omega| < 2\mu_2(\mathbb{D})\pi \approx 6.78\pi \tag{17}$$

Question

Can we show that there exist an open set (of given volume) which maximizes the $k-th$ Neumann eigenvalue, with $k \geq 3$. is it possible to identify this maximizer? [12]

Steklov eigenvalues of the unit disk

Using polar coordinates and separation of variables, we can find that the **Steklov** eigenvalues of the unit disk are given by

$$0, 1, 1, 2, 2, \dots, n, n, \dots \tag{18}$$

with corresponding eigenfunctions

$$1, r\sin\theta, r\cos\theta, \dots, r^n\sin(n\theta), r^n\cos(n\theta), \dots \tag{19}$$

Thus note that for this case $\sigma_{2n+1}(\mathbb{D}) = \sigma_{2n}(\mathbb{D}) = n,\ n = 1, 2, \dots$

Weinstock inequality

In 1954 **Weinstock**,proved that when Ω is a simply-connected bounded planar domain with Lipschitz boundary. Then

$$\sigma_2(\Omega)|\partial\Omega| \leq 2\pi\sigma_2(\mathbb{D}) = 2\pi$$

with equality if and only if Ω is a disk.

Theorem 3 (Weinstock, Brock)

The ball maximizes the second Stekloff eigenvalue among open sets of given volume.

In 2012, **Girouard** and **Polterovich**, proved an estimate in [4] for bounded planar domains Ω that are **not necessarily** simply-connected.

$$\sigma_n(\Omega)|\partial\Omega| \leq 2\pi\ell n.$$

Where ℓ is the number of its boundary components.

$\ell = 2$

It turns out that unlike **Szegö** inequality, **Weinstock's** inequality does not hold **for non-simply-connected planar domains**.

Let us consider for example, bounded planar connected domains with $\ell = 2$. Typical examples of these type of domains are **annuli**.

For the Steklov problem on $\Omega_\epsilon = \mathbb{D} \setminus B(0, \epsilon)$, $\epsilon \in (0, 1)$ the there are two simple eigenvalues one of which is zero and the other one is given by

$$\sigma = \frac{1 + \epsilon}{\epsilon \ln (1/\epsilon)}$$

Notice that for ϵ small enough this σ can be large. However, we are not focusing in this particular eigenvalue.

The the remaining Steklov eigenvalues are the solutions of

$$\sigma^2 - \frac{\sigma n}{\epsilon}(\epsilon + 1)\frac{1+\epsilon^{2n}}{1-\epsilon^{2n}} + \frac{n^2}{\epsilon} = 0$$

For $n = 1$ we obtain

$$\sigma_2(\Omega_\epsilon) = \frac{1+\epsilon^2}{2\epsilon(1-\epsilon)}\left[1 - \sqrt{1 - 4\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right)^2}\right] \tag{20}$$

and for $\epsilon > 0$ small enough $\sigma_2(\Omega_\epsilon)|\partial\Omega_\epsilon| > 2\pi\sigma_2(\mathbb{D})$, [3].

Therefore **Weinstock** inequality fails!

Some interesting questions

Question

*How large can σ_2 be on a **non-simply** connected bounded planar domain with two boundary components and given perimeter?[3]*

Since **Weinstock's** result does not hold for **non-simply-connected** planar domains

Question ([3])

Is the supremum of $\sigma_2(\Omega)|\partial\Omega|$ among all planar domains of fixed perimeter finite?

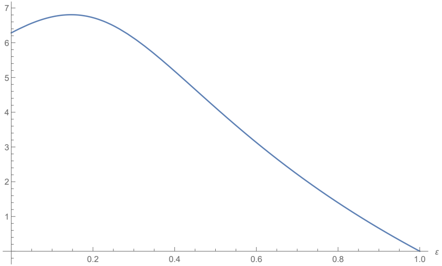
Colbois,**El Soufi** and **Girouard**, answered this question in [7] by showing that there exists a universal constant C for which

$$\sigma_n(\Omega)|\partial\Omega| \leq C(\gamma+1)n$$

Can we improve this bound on planar annuli?

Conjecture (Girouard-Polterovich, [3])

When restricting to bounded connected planar domains with two boundary components, the expectation is that the best planar annulus is the one that realizes the max on the curve of the function $\sigma_2(\Omega_\epsilon)|\partial\Omega_\epsilon|$, *where* $\sigma_2(\Omega_\epsilon)$ *is given by equation (20)*



Normalized eigenvalue $\sigma_2(\Omega_\epsilon)|\partial\Omega_\epsilon|$. The max is attained at the solution ϵ_0 of $\frac{d(\sigma_2(\Omega_\epsilon)|\partial\Omega_\epsilon|)}{d\epsilon} = 0$. Numerically we get $\epsilon_0 \approx 0.146721$

Remark: I tried to prove this in [17]. I think this was fully answered recently in [14] by Lagacé and Karpukhin?

A Stability Result?

Question

Let Ω be a planar simply–connected domain such that the difference $2\pi - \sigma_2(\Omega)|\partial\Omega|$ is small. Show that Ω must be close to a disk (in the sense of Fraenkel asymmetry) $\mathcal{F}(\Omega) = \inf_{x\in\mathbb{R}^N} |(B+x)\Delta\Omega|$ (here $|B| = |\Omega|$)

Tools to work with this type of problems

- Analysis.(Potential Theory and the idea of capacity)
- Differential geometry tools. (Sprectral theory of the Laplacian on Riemannian Manifolds)
- Calculus of Variations (Shape and topological derivatives.)

On general manifolds

Theorem 4 (Escobar)

Let (M^2, g) be a compact Riemannian manifold with boundary. Assume M has non-negative Gaussian curvature, K,and that the geodesic curvature of ∂M, k_g satisfies $k_g \geq k_0 > 0$. Then the first non-zero eigenvalue of the Stekloff problem σ_2, satisfies $\sigma_2 \geq k_0$. Equality holds only of the Euclidean ball of radious k_0^{-1}[15]

Theorem 5 (Escobar)

Let (M^n, g) be a compact Riemannian manifold with boundary and dimension $n \geq 3$. Assume that $Ric(g) \geq 0$ and the second fundamental form π satisfies $\pi \geq k_o I$ on ∂M, $k_0 > 0$. Then $\sigma_2 > \frac{k_0}{2}$. where I is an isoperimetric constant.[15]

Proposition 2 in Escobar's paper [15] reads as follow:

Proposition 1 (Escobar)

$\sigma_2(g) \geq \max_{x \in \partial M} e^{-f(x)} \sigma_2(g_0)$ *where* g *and* g_0 *are conformally related.* $g = e^{2f} g_0$

Remark: Escobar claims that this inequality holds for any smooth f however, we were able to show that this result is only true when it is an equality and only happens when f is a constant [16]. That is,

$$\sigma_2(g) = e^{-f} \sigma_2(g_0) \tag{21}$$

only when f is a constant on $\partial\Omega$

Thanks!

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