Complete nonselfadjointness of dissipative Schrödinger-type operators on a bounded interval

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Aspects of Spectral Theory for Linear Operators

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Outline

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Dissipative operators

Definition

An operator A is called **dissipative** if and only if

$$\mathrm{Im}\langle f, Af\rangle \geq 0$$

for all $f \in D(A)$.

Example

Any symmetric (in particular, selfadjoint) operator S is dissipative because

$$\mathrm{Im}\langle f, Sf\rangle = \frac{\langle f, Sf\rangle - \overline{\langle f, Sf\rangle}}{2i} = 0$$

for all $f \in D(S)$.

Dissipative operators

Definition

An operator V is called **nonnegative** $(V \geq 0)$ if and only if

$$\langle f, Vf \rangle \geq 0$$

for all $f \in D(V)$.

Remark

If $V \geq 0$ is selfadjoint then there exists a unique square root $V^{1/2}$ and

$$J : \mathrm{Rg}(V^{1/2}) \to H, \quad J(V^{1/2} f) = f$$

for $f \in \overline{\mathrm{Rg}(V^{1/2})}$ is an injective nonnegative well-defined operator. We will use the notation

$$V^{-1/2} := J.$$

Dissipative operators

Example

A matrix $A \in \mathbb{C}^{n\times n}$ is dissipative if and only if $\mathrm{Im}(A) := \frac{1}{2i}(A - A^*)$ is nonnegative.

Example

If S is symmetric and $V \geq 0$ then $A = S + iV$ is dissipative.

Maximally dissipative operators

Definition

A dissipative operator A is called **maximally dissipative** if and only if there is no nontrivial dissipative extension of A.

Example

Any selfadjoint operator is a maximally dissipative operator.

Dissipative extensions of dissipative operators

Question 1

Let S be a symmetric Laplacian defined on $D(S) \subset H^2(0,1)$. How can we describe all dissipative extensions of the dissipative operator

$$A := S + iV$$

where V is a bounded nonnegative operator?

Complete nonselfadjointness

Definition

Let A be a dissipative operator. A closed subspace $\mathcal{M} \subset H$ is called a **reducing subspace** of A if

$$D(A) = (D(A) \cap \mathcal{M}) \oplus (D(A) \cap \mathcal{M}^{\perp}),$$

$$A(D(A) \cap \mathcal{M}) \subset \mathcal{M} \quad \text{and} \quad A(D(A) \cap \mathcal{M}^{\perp}) \subset \mathcal{M}^{\perp}.$$

Remark

$D(A) \cap \mathcal{M}$ is dense in $\mathcal{M}$, hence $(A|_{\mathcal{M}})^*$ is well-defined in $\mathcal{M}$ and

$$(A|_{\mathcal{M}})^* = A^*|_{\mathcal{M}}.$$

Complete nonselfadjointness

Definition

Let A be a dissipative operator. A closed subspace $\mathcal{M} \subset H$ is called a **reducing subspace** of A if

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Definition

We say that $\mathcal{M}$ is a **reducing selfadjoint subspace** if $A|_{\mathcal{M}}$ is selfadjoint in $\mathcal{M}$.

Definition

A dissipative operator A is called **completely nonselfadjoint (c.n.s.a.)** if there is no nontrivial reducing selfadjoint subspace of A.

Complete nonselfadjointness

Definition

The **selfadjoint space** of a dissipative operator A is

$$H_{\mathrm{sa}}(A) = \overline{\mathrm{span}}\,\{M : M \text{ is a reducing selfadjoint subspace of } A\}$$

and the **symmetric space** of A is

$$H_{\mathrm{sym}}(A) = \ker(A - A^*).$$

Properties

- $H_{\mathrm{sa}}(A)$ reducing selfadjoint subspace. Actually, $H_{\mathrm{sa}}(A)$ is the largest set with this property.
- $H_{\mathrm{sa}}(A) \cap D(A) \subset H_{\mathrm{sym}}(A)$.

Complete nonselfadjointness

Theorem (Behrndt, Hassi, de Snoo)

Let S be a symmetric operator and assume that there is a selfadjoint operator $\widehat{S} \supset S$ such that $\sigma(\widehat{S}) = \sigma_p(\widehat{S})$. Then

$$S \text{ is c.n.s.a} \iff \sigma_p(S) = \emptyset.$$

Theorem (Kurasov, Muller, Naboko)

Let A be a maximally dissipative operator such that $\sigma(A) \cap \mathbb{R}$ is purely discrete. Then

$$A \text{ is c.n.s.a} \iff \sigma(A) \cap \mathbb{R} = \emptyset.$$

Complete nonselfadjointness

Question 2

Let S be a symmetric Laplacian defined on $D(S) \subset H^2(0,1)$ and let V be a bounded nonnegative operator.

Let $\widehat{A}$ be a given maximally dissipative extension of the dissipative operator

$$A := S + iV.$$

Then

- is $\widehat{A}$ completely nonselfadjoint?
- If not, what is its selfadjoint space $H_{\mathrm{sa}}(\widehat{A})$?

Outline

Example: Schrödinger-type operators on $L^2(0,1)$.

Let us consider the following operators.

Laplacian on $H_0^2(0,1)$

$$S_{\min}: \quad D(S_{\min}) = H_0^2(0,1), \quad S_{\min} f = -f''$$

where

$$H_0^2(0,1) = \left\{ f \in H^2(0,1) : f(0) = f(1) = f'(0) = f'(1) = 0 \right\}.$$

Schrödinger-type operator on $H_0^2(0,1)$

$$A_{\min} := S_{\min} + iV$$

where V is a multiplication operator for a nonnegative essentially bounded function $V(\cdot)$ in $L^2(0,1)$.

Theorem (Characterization of maximally dissipative extensions)

The maximally dissipative extensions of $A_{\min} = S_{\min} + iV$ are fully characterized in five cases. We focus on one of them.

(1) $\widehat{A}_{B,K} = \widehat{S}_{B,K} + iV$ where

$$D(\widehat{S}_{B,K}) = \left\{ f \in H^2(0,1) : \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix} = B \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \right\}$$

$$\widehat{S}_{B,K} f = -f'' + f(0)k_1 + f(1)k_2,$$

$K = (k_1, k_2)$, $k_1, k_2 \in \mathrm{Rg}(V^{1/2})$, $B \in \mathbb{C}^{2\times 2}$ and

$$\mathrm{Im}(B) - \frac{1}{4} M_K \geq 0$$

with

$$M_K = \begin{pmatrix} \|V^{-1/2}k_1\|^2 & \langle V^{-1/2}k_1, V^{-1/2}k_2 \rangle \\ \langle V^{-1/2}k_2, V^{-1/2}k_1 \rangle & \|V^{-1/2}k_2\|^2 \end{pmatrix}.$$

Case **(1)** of extensions

Recall: $\widehat{A}_{B,K} = \widehat{S}_{B,K} + iV$ where

$$D(\widehat{S}_{B,K}) = \left\{ f \in H^2(0,1) : \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix} = B \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \right\},$$
$$\widehat{S}_{B,K} f = -f'' + f(0)k_1 + f(1)k_2$$

and

$$\mathrm{Im}(B) - \frac{1}{4} M_K \geq 0.$$

Some useful operators for **(1)**

$$\widehat{S}_B : \quad D(\widehat{S}_B) = D(\widehat{A}_{B,K}), \quad \widehat{S}_B f = -f'',$$
$$\widetilde{S}_B = \widehat{S}_B \Big|_{H_{\mathrm{sym}}(\widehat{S}_B)}.$$

Some useful operators for **(1)**

$$\widehat{S}_B : \quad D(\widehat{S}_B) = D(\widehat{A}_{B,K}), \quad \widehat{S}_B f = -f'',$$
$$\widetilde{S}_B = \widehat{S}_B\Big|_{H_{\mathrm{sym}}(\widehat{S}_B)} .$$

Properties

- $\widehat{S}_B$ is maximally dissipative.
- $\widetilde{S}_B$ is symmetric.
- $S_{\mathrm{min}} \subset \widetilde{S}_B \subset \widehat{S}_B$.
- $S_{\mathrm{min}} + iV \subset \widetilde{S}_B + iV \subset \widehat{S}_{B,K} + iV$

Outline

Theorem

Let $\widehat{A}$ be a dissipative extension of $A = S + iV$ where S is symmetric and $V \geq 0$. If S is c.n.s.a. and if

$$H_{\mathrm{sym}}(\widehat{A}) = H_{\mathrm{sym}}(A)$$

then $\widehat{A}$ is c.n.s.a.

Remark

$H_{\mathrm{sym}}(A) = H_{\mathrm{sym}}(S + iV) = D(S) \cap \ker(V)$.

Theorem

$$H_{\mathrm{sym}}(\widehat{S}_{B,K} + iV) = \left\{ f \in D(\widehat{S}_B) : \begin{array}{l} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \in \ker(\mathrm{Im}(B) - \frac{1}{4} M_K), \\ Vf = \frac{i}{2}(f(0)k_1 + f(1)k_2) \end{array} \right\}.$$

Theorem

$$H_{\mathrm{sym}}(\widehat{S}_{B,K} + iV) \cap \ker V = D(\widetilde{S}_B) \cap \ker V.$$

Example

- If $\mathrm{rank}(\mathrm{Im}(B) - \frac{1}{4} M_K) = 2$ then $\widehat{S}_{B,K} + iV$ is c.n.s.a.
- If $k_1 = k_2 = 0$ then $S_{B,K} + iV$ is c.n.s.a.
- If $\widehat{S}_B$ is selfadjoint then $\mathrm{Im}(B) = 0$, $k_1 = k_2 = 0$ and $\widehat{S}_{B,K} + iV$ is c.n.s.a.

Selfadjoint spaces of the extensions

Theorem

$$H_{\mathrm{sym}}(\widehat{S}_{B,K} + iV) \cap \ker(V) = D(\widetilde{S}_B) \cap \ker(V).$$

Theorem

There exists a space $\mathcal{U}$ with $\dim(\mathcal{U}) \leq 2$ and $\mathcal{U} \cap \ker(V) = \{0\}$ such that

$$H_{\mathrm{sym}}(\widehat{S}_{B,K} + iV) = (D(\widetilde{S}_B) \cap \ker(V)) \dot{+} \mathcal{U}.$$

Theorem

If $\widetilde{S}_B$ is c.n.s.a. then

$$\dim(H_{\mathrm{sa}}(\widehat{S}_{B,K} + iV)) \leq \dim(\mathcal{U}) \leq 2.$$

Theorem

Let $r_1, r_2 \in \mathbb{R}$ with $r_1 \neq r_2$ and nonzero functions $f, g \in H^2(0,1)$ such that

$$(S_{\min} + iV)^* f = -f'' - iVf = r_1 f, \; f(1) = 0,$$
$$(S_{\min} + iV)^* g = -g'' - iVg = r_2 g, \; g(0) = 0.$$

Then there exists a maximally dissipative extension of the form $\widehat{S}_{B,K} + iV$ such that $H_{\mathrm{sa}}(\widehat{S}_{B,K} + iV) = \mathrm{span}\{f, g\}$

Proof.

From the eigenvalue equation, it follows that $f(0) \neq 0$ and $g(1) \neq 0$. Therefore, the proof follows by choosing

$$B := \begin{pmatrix} \frac{f'(0)}{f(0)} & \frac{g'(0)}{g(1)} \\ \frac{f'(1)}{f(0)} & \frac{g'(1)}{g(1)} \end{pmatrix} \quad \text{and} \quad K := \left(\frac{2}{f(0)} Vf, \frac{2}{g(1)} Vg \right).$$

$\square$

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¡Gracias!

(2) $\widehat{A}_{b_{12},b_{22},k} = \widehat{S}_{b_{12},b_{22},k} + iV$ where

$$D(\widehat{S}_{b_{12},b_{22},k}) = \left\{ f \in H^2(0,1) : \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = \begin{pmatrix} -b_{22} & 0 \\ b_{12} & \frac{1}{\overline{b_{22}}} \end{pmatrix} \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix} \right\}$$

$$\widehat{S}_{b_{12},b_{22},k} f = -f'' + f(1)k$$

$b_{12}, b_{22} \in \mathbb{C},\ b_{22} \neq 0,\ k \in \mathrm{Rg}(V^{1/2})$ and

$$\mathrm{Im}(b_{12}\overline{b_{12}}) - \frac{1}{4}\|V^{-1/2}k\|^2 \geq 0.$$

(3) $\widehat{A}_{b_{11},k} = \widehat{S}_{b_{11},k} + iV$ where

$$D(\widehat{S}_{b_{11},k}) = \left\{ f \in H^2(0,1) : \begin{pmatrix} f(1) \\ f'(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_{11} & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ -f'(1) \end{pmatrix} \right\}$$
$$\widehat{S}_{b_{11},k} f = -f'' + f(0)k$$

$b_{11} \in \mathbb{C}$, $k \in \mathrm{Rg}(V^{1/2})$ and

$$\mathrm{Im}(b_{11}) - \frac{1}{4}\|V^{-1/2}k\|^2 \geq 0.$$

(4) $\widehat{A}_{b_{12},k} = \widehat{S}_{b_{12},k} + iV$ where

$$D(\widehat{S}_{b_{12},k}) = \left\{ f \in H^2(0,1) : \begin{pmatrix} f(0) \\ -f'(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b_{12} \end{pmatrix} \begin{pmatrix} f'(0) \\ f(1) \end{pmatrix} \right\}$$

$$\widehat{S}_{b_{12},k} f = -f'' + f(1)k$$

$b_{12} \in \mathbb{C}$, $k \in \mathrm{Rg}(V^{1/2})$ and

$$\mathrm{Im}(b_{12}) - \frac{1}{4}\|V^{-1/2}k\|^2 \geq 0.$$

(5) $\widehat{A} = \widehat{S} + iV$ where

$$D(\widehat{S}) = \left\{ f \in H^2(0,1) : \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix} \right\}$$

$$\widehat{S} f = -f''.$$

Schrödinger operator on $H^2(0,1)$ with $V = \rho\langle\phi,\cdot\rangle\phi$

Assume $V = \rho\langle\phi,\cdot\rangle\phi$ with $\rho > 0$ and $\|\phi\| = 1$. Let us also assume that

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

with

$$\mathrm{Im}(b_{11}) > 0,\ \mathrm{Im}(b_{22}) > 0,\ b_{12} = \overline{b_{21}}.$$

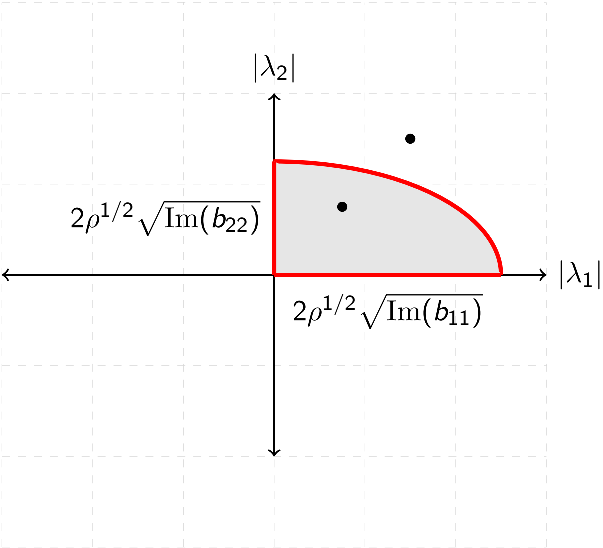
Observe that, $\mathrm{Rg}(V^{1/2}) = \mathrm{span}\{\phi\}$, so $k_1 = \lambda_1\phi$, $k_2 = \lambda_2\phi$ and because of

$$V^{-1/2}\phi = \rho^{-1/2}\phi$$

we can show that $\widehat{S}_{B,K} + i\rho\langle\phi,\cdot\rangle\phi$ is maximally dissipative if and only if

$$\frac{|\lambda_1|^2}{4\rho\mathrm{Im}(b_{11})} + \frac{|\lambda_2|^2}{4\rho\mathrm{Im}(b_{22})} \leq 1$$

Rank-One Perturbation: Ellipse



Maximally dissipative operators

Definition

A dissipative operator A is called **maximally dissipative** if and only if there is no nontrivial dissipative extension of A.

Theorem

Let A be a dissipative operator. Then the following statements are equivalent

- A is maximally dissipative
- There is a $\lambda \in \mathbb{C}^-$ such that $\lambda \in \rho(A)$.
- $\mathbb{C}^- \subset \rho(A)$
- $-A^*$ is dissipative.
- iA is the generator of a strongly continuous semigroup of contractions.

Nonnegative Matrix

A matrix $M \in \mathbb{C}^{n\times n}$ is nonnegative (or $M \geq 0$) if and only if

$$\langle \vec{x}, M\vec{x} \rangle \geq 0$$

for all $\vec{x} \in \mathbb{C}^n$

Remarks

- $M \geq 0$ if and only if M is semi-definite positive.
- $M \geq 0$ if and only if its eigenvalues are nonnegative.
- If $n = 2$ and $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ then $M \geq 0$ if and only if

$$m_{11} \geq 0, \; m_{22} \geq 0 \text{ and } \det(M) = m_{11}m_{22} - m_{12}m_{21} \geq 0.$$