Schrödinger operators with complex potentials on a non-selfadjoint quantum star graph

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June 05, 2025.

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1. A **graph** $\Gamma = (V, E)$ where V is the set of **vertices** and E is the set of **edges**. Each edge e connects a pair of vertices (v_i, v_j).
2. Each edge e is associated with a finite interval $[0, a_e]$, giving the graph a metric structure.

3. The Hilbert space

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- Recently however, there is growing interest in the spectral theory of non-selfadjoint quantum graphs. Such operators may be used to model open physical systems where certain quantities are not conserved.
- Some notable authors in this area include Hussein, Krejčiřík, Kurasov, Rivière, Royer, Siegl, among others.

Laplacians on a non-selfadjoint quantum star graph

We work with a non-selfadjoint Schrödinger operator on a quantum star graph Γ with a non-selfadjoint Robin condition at the central vertex and Dirichlet boundary conditions on the set of outer vertices.

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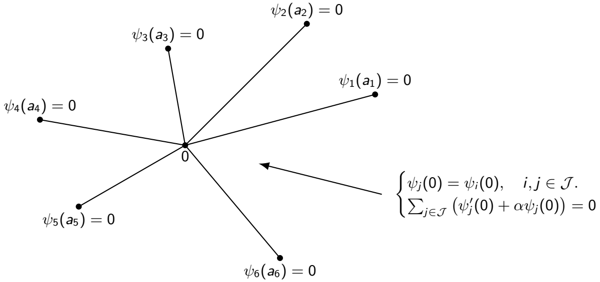

Figure: Quantum star graph with Robin condition at the central vertex.

Let $\mathcal{H} = \bigoplus_{j=1}^{n} L^2(0, a_j)$, with $a_j > 0$ for $j = 1, \ldots, n$, $\mathcal{J} = \{1, 2, \ldots, n\}$. For $\alpha \in \mathbb{C} \cup \{\infty\}$

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$$D(L_\alpha) := \left\{ \psi = (\psi_j)_{j=1}^{n} \in \bigoplus_{j=1}^{n} H^2(0, a_j) : \begin{array}{ll} \psi_j(0) = \psi_i(0), & i, j \in \mathcal{J}. \\ \psi_j(a_j) = 0, & j \in \mathcal{J} \\ \sum_{j \in \mathcal{J}} \left(\psi_j'(0) + \alpha \psi_j(0) \right) = 0 & \end{array} \right\}$$

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Remark

- If $\alpha = 0$, the conditions at the central vertex are called Neumann-Kirchhoff conditions

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Remark (Hussein, Krejčiřík, Siegl, 2015)

L_α has a **Riesz basis with parentheses** of generalized eigenfunctions, for all $\alpha \in \mathbb{C}$. That is, L_α is similar to an orthogonal direct sum of finite dimensional operators.

For $\alpha \in \mathbb{C} \cup \{\infty\}$ we denote the eigenvalues of L_α by $\lambda_m(\alpha)$, $m \in \mathbb{N}$, repeated according to their algebraic multiplicities and ordered such that

$$\mathrm{Re}(\lambda_m(\alpha)) \leq \mathrm{Re}(\lambda_{m+1}(\alpha)), \qquad m \in \mathbb{N}.$$

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Proposition (Rivière, Royer, 2020)

For $\alpha \in \mathbb{C}$, there exists $\gamma_\alpha \geq 0$ such that

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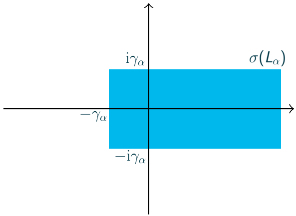

Figure: Spectral enclosure for the spectrum of L_α.

Definition

The lengths $a_1, a_2, \ldots, a_n$ are called **incommensurable** over $\{-1, 0, 1\}$ if only the trivial linear combination of $a_1, a_2, \ldots, a_n$ with these coefficients vanishes.

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Theorem (Rivière, Royer, 2020)

Let $\alpha \in \mathbb{C}$. If $a_1, a_2, \ldots, a_n$ are incommensurable over $\{-1, 0, 1\}$ then

$$\lambda_m(\alpha) = \lambda_m(0) + O\left(\frac{1}{\lambda_m(0)}\right).$$

Corollary

If $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then L_α is similar to a non-selfadjoint normal operator.

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Proof:

- L_α has a Riesz basis with parentheses of generalized eigenfunctions.
- In particular, we have that L_α is a **spectral operator**.
- It is well known that a discrete spectral operator with semi-simple eigenvalues (this is, the geometric and algebraic multiplicity of their eigenvalue are equal) is similar to a normal operator $\square$

Schrödingers operator with complex potentials

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$$S \equiv S_\alpha = L_\alpha + V$$

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Definition

We say that A is T-**bounded** if $D(T) \subseteq D(A)$ and there exist $a, b \geq 0$ such that

$$\|Ax\|^2 \leq a^2\|x\|^2 + b^2\|Tx\|^2, \qquad x \in D(T). \tag{1}$$

The infimum δ_A of all $b \geq 0$ such that there is an $a \geq 0$ which (1) holds is called the T-**bound** of A.

Definition

Let $p \in [0,1]$. We say that A is p-**subordinate** to T if $D(T) \subseteq D(A)$ and there exists $c \geq 0$ such that

$$\|Ax\| \leq c\|x\|^{1-p}\|Tx\|^{p}, \quad x \in D(T). \tag{2}$$

The infimum over all constants $c \geq 0$ such that (2) holds is called the p-**subordination bound** of A to T.

Remark

If A is p-subordinate to T with $p < 1$, then A is T-bounded with T-bound equal to zero and

$$\|Ax\| \leq \epsilon^{\frac{1}{p}} cp\|Tx\| + \epsilon^{\frac{1}{p-1}} c(1-p)\|x\|, \qquad \forall \epsilon \geq 0.$$

Spectral and Pseudospectral gaps.

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Definition

Let X a Banach space and $S(X \to X)$ a closed operator. If $\delta > 0$, the **δ-pseudospectrum** of S is defined by

$$\sigma_\delta(S) := \left\{ z \in \mathbb{C} : \|(S-z)^{-1}\| \geq \frac{1}{\delta} \right\},$$

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We have that

$$\sigma(S) \subseteq \sigma_\delta(S), \quad \text{for all } \delta > 0.$$

Abstract results

Theorem (Moreno, Winklmeier, 2023)

Let T be a normal operator and let A be T-bounded with T-bound < 1 and let $a, b \geq 0$, $b < 1$, as in (1), i.e. $\|Ax\|^2 \leq a\|x\|^2 + b\|Tx\|^2$.

- *If* $\operatorname{Im}\sigma(T) \subseteq [-\gamma_T, \gamma_T]$, *then*

$$\boxed{\left\{ z \in \mathbb{C} : (|\operatorname{Im} z| - \gamma_T)^2 > \frac{a^2 + b^2\left(|\operatorname{Re} z|^2 + \gamma_T^2\right)}{1 - b^2} \right\} \subseteq \rho(T + A).}$$

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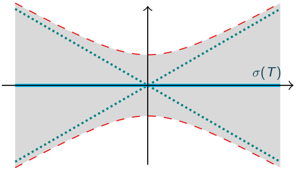
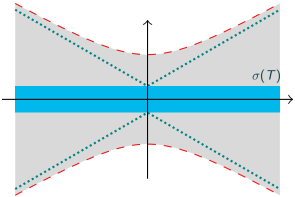


Spectral enclosure for $\gamma_T = 0$.



Spectral enclosure for $\gamma_T > 0$.

- If in addition T has a spectral free strip $(\alpha_T, \beta_T) + i\mathbb{R} \subset \mathbb{C}$ and if

$$\sqrt{a^2 + b^2\gamma_T^2 + b^2\alpha_T^2} + \sqrt{a^2 + b^2\gamma_T^2 + b^2\beta_T^2} < \beta_T - \alpha_T.$$

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Then also $T + A$ has a spectral free strip

$$\boxed{\{z \in \mathbb{C} : \alpha'_{T+A} < \operatorname{Re} z < \beta'_{T+A}\} \subseteq \rho(T + A)}$$

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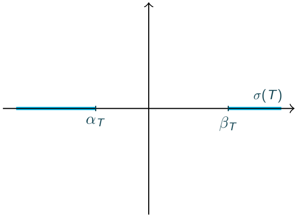

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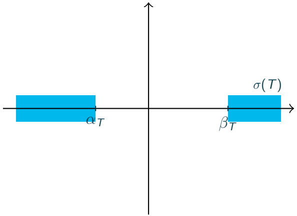

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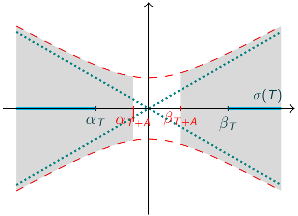

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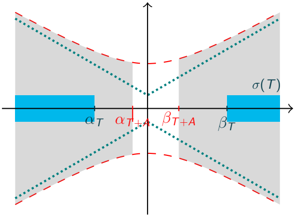

Spectral enclosure for $\gamma_T > 0$.

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Since

$$(T + A - z) = (I + A(T - z)^{-1})(T - z),$$

we obtain that $z \in \rho(T + A)$.

Since T is normal, we have the estimates

$$\|(T-z)^{-1}\| = \frac{1}{\operatorname{dist}(z,\sigma(T))}$$

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Hence

$$\begin{aligned}\|A(T-z)^{-1}\|^2 &\leq a^2\|(T-z)^{-1}\|^2 + b^2\|T(T-z)^{-1}\|^2\\ &\leq \frac{a^2+b^2((\operatorname{Re} z)^2+\gamma_T^2)}{(|\operatorname{Im} z|-\gamma_T)^2} + b^2.\end{aligned}$$

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Hence

$$\begin{aligned}\|A(T-z)^{-1}\|^2 &\leq a^2\|(T-z)^{-1}\|^2 + b^2\|T(T-z)^{-1}\|^2 \\ &\leq \frac{a^2+b^2((\operatorname{Re} z)^2+\gamma_T^2)}{(|\operatorname{Im} z|-\gamma_T)^2} + b^2.\end{aligned}$$

Observe that

$$\frac{a^2+b^2((\operatorname{Re} z)^2+\gamma_T^2)}{(|\operatorname{Im} z|-\gamma_T)^2} + b^2 < 1 \iff \frac{a^2+b^2((\operatorname{Re} z)^2+\gamma_T^2)}{1-b^2} < (|\operatorname{Im} z|-\gamma_T)^2.$$

So, for all $z \in \mathrm{Hyp}_{\gamma_T} := \{z \in \mathbb{C} : \frac{a^2+b^2((\mathrm{Re}\, z)^2+\gamma_T^2)}{1-b^2} < (|\,\mathrm{Im}\, z| - \gamma_T)^2\}$, we have that $\|A(T-z)^{-1}\| < 1$, So, for all $z \in \mathrm{Hyp}_{\gamma_T} := \{z \in \mathbb{C} : \frac{a^2+b^2((\mathrm{Re}\, z)^2+\gamma_T^2)}{1-b^2} < (|\operatorname{Im} z| - \gamma_T)^2\}$, we have that $\|A(T-z)^{-1}\| < 1$, hence $z \in \rho(T+A)$.

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We can also prove that

$$\alpha'_{T+A} < \mathrm{Re}\, z < \beta'_{T+A} \quad \Longrightarrow \quad \|A(T-z)^{-1}\| < 1.$$

So, for all $z \in \mathrm{Hyp}_{\gamma_T} := \{z \in \mathbb{C} : \frac{a^2+b^2((\mathrm{Re}\, z)^2+\gamma_T^2)}{1-b^2} < (|\operatorname{Im} z| - \gamma_T)^2\}$, we have that $\|A(T-z)^{-1}\| < 1$, hence $z \in \rho(T+A)$. That is $\boxed{\mathrm{Hyp}_{\gamma_T} \subseteq \rho(T+A).}$

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$$\boxed{\{z \in \mathbb{C} : \alpha'_{T+A} < \mathrm{Re}\, z < \beta'_{T+A}\} \subseteq \rho(T+A).} \qquad \square$$

Theorem (Moreno, Winklmeier, 2023)

Let T be a normal operator in a Hilbert space H, let A be p-subordinate to T with $0 \leq p \leq 1$, let $c \geq 0$ as in (2) and let $\operatorname{Im}\sigma(T) \subseteq [-\gamma_T, \gamma_T]$ with $\gamma_T \geq 0$.

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then $T + A$ has a spectral free strip, i.e.

$$\boxed{\{z \in \mathbb{C} : \alpha'_{T,p} < \operatorname{Re} z < \beta'_{T,p}\} \subseteq \rho(T + A)}$$

with

$$\begin{aligned} \alpha'_{T,p} &:= \alpha_T + c \cdot \max\{|\alpha_T + \mathrm{i}\gamma_T|^p, |\beta_T + \mathrm{i}\gamma_T|^p\}, \\ \beta'_{T,p} &:= \beta_T - c \cdot \max\{|\alpha_T + \mathrm{i}\gamma_T|^p, |\beta_T + \mathrm{i}\gamma_T|^p\}. \end{aligned}$$

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Note that T_α and L_α have the same spectra.

Since T_α is a normal operator, we have

$$\sigma_\delta(T_\alpha) = \Big\{z \in \mathbb{C} : \mathrm{dist}(z, \sigma(T_\alpha)) \leq \delta\Big\}$$

and so

$$\sigma_\delta(L_\alpha) = \Big\{z \in \mathbb{C} : \mathrm{dist}(z, \sigma(T_\alpha)) \leq \delta\kappa_\alpha\Big\}$$

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where $\kappa_\alpha := \|J_\alpha\|\|J_\alpha^{-1}\|$. This means that T_α and L_α have infinitely many δ-pseudospectral gaps for all $\delta > 0$.

Lemma

Let $\alpha \in \mathbb{C} \setminus \{\alpha_\Gamma\}$, with $\alpha_\Gamma := \frac{1}{n}\sum_{j\in\mathcal{J}} \frac{1}{a_j}$. If $V = (V_j)_{j=1}^n \in L^q(\Gamma)$ for some $q \geq 2$, then $\mathcal{M}_V$ is $\frac{1}{q}$-subordinate to L_α where $\mathcal{M}_V$ is the operator of multiplication defined by V.

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Set $A_V := J_\alpha \mathcal{M}_V J_\alpha^{-1}$. If $V \in L^q(\Gamma)$, then the above lemma shows that A_V is $\frac{1}{q}$-subordinate with respect to T_α.

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Note that $T_\alpha + A_V$ and $L_\alpha + V$ have the same spectra.

The Weyl law proved by Rivière and Royer shows that

$$\mathrm{Re}(\lambda_m(\alpha)) = \frac{\pi^2}{|\Gamma|^2} m^2 + O(m) \text{ as } m \to \infty$$

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Let $R_m(\alpha) := c \cdot \max\left\{ |\,\mathrm{Re}(\lambda_m(\alpha)) + \mathrm{i}\gamma_\alpha|^{\frac{1}{q}}, |\,\mathrm{Re}(\lambda_{m+1}(\alpha)) + \mathrm{i}\gamma_\alpha|^{\frac{1}{q}} \right\}$. Note that,

$$\frac{R_m(\alpha)}{\mathrm{Re}(\lambda_{m+1}(\alpha)) - \mathrm{Re}(\lambda_m(\alpha))} \leq K \frac{m^{\frac{2}{q}}}{m} \longrightarrow 0 \text{ as } m \to \infty.$$

Theorem (Moreno, Winklmeier, 2023)

Let $\alpha \in \mathbb{C} \setminus \{\alpha_\Gamma\}$. Let $\delta > 0$ and $V \in L^q(\Gamma)$ with $q > 2$. Then at most finitely many of the vertical δ-pseudospectral free strips of L_α close under perturbation by V.

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Remark

For all $z \in \mathrm{Hyp}_{\gamma_\alpha} := \{z \in \mathbb{C} : \frac{a^2+b^2((\mathrm{Re}\, z)^2+\gamma_\alpha^2)}{1-b^2} < (|\,\mathrm{Im}\, z| - \gamma_\alpha)^2\}$, we have that

$$(T_\alpha + A_V - z)^{-1} = (T_\alpha - z)^{-1} \sum_{k=0}^{\infty} (-A_V (T_\alpha - z)^{-1})^k$$

and since T_α has compact resolvent, we obtain that $T_\alpha + A_V$ and $L_\alpha + V$ also have compact resolvent.

Multiplicities and asymptotics for the discrete spectrum

For $\alpha \in \mathbb{C} \cup \{\infty\}$ we denote the eigenvalues of $S_\alpha = L_\alpha + V$ by $\mu_m(\alpha)$, $m \in \mathbb{N}$, repeated according to their algebraic multiplicities and ordered such that

$$\mathrm{Re}(\mu_m(\alpha)) \leq \mathrm{Re}(\mu_{m+1}(\alpha)), \qquad m \in \mathbb{N}.$$

Theorem (Moreno, Winklmeier, 2023)

Let T be a normal operator in a Hilbert space H, let A be p-subordinate to T with $0 \leq p \leq 1$, let $c \geq 0$ as in (2) and let $\operatorname{Im} \sigma(T) \subseteq [-\gamma_T, \gamma_T]$ with $\gamma_T \geq 0$.

Theorem (Moreno, Winklmeier, 2023)

Let T be a normal operator in a Hilbert space H, let A be p-subordinate to T with $0 \leq p \leq 1$, let $c \geq 0$ as in (2) and let $\operatorname{Im} \sigma(T) \subseteq [-\gamma_T, \gamma_T]$ with $\gamma_T \geq 0$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are N isolated eigenvalues of T repeated according to their multiplicities and ordered such that

$$\operatorname{Re}(\lambda_k) \leq \operatorname{Re}(\lambda_{k+1}), \quad k = 1, 2, \ldots, N-1.$$

Assume that $\{z \in \mathbb{C} : \operatorname{Re} \lambda_1 \leq \operatorname{Re} z \leq \operatorname{Re} \lambda_N\} \cap \sigma(T) = \{\lambda_k\}_{k=1}^N$ and suppose that T satisfies the following spectral inclusions

$$\sigma(T) \cap \Big((\alpha, \operatorname{Re} \lambda_1) + \mathrm{i}\mathbb{R} \Big) = \emptyset \quad \text{and} \quad \sigma(T) \cap \Big((\operatorname{Re} \lambda_N, \beta) + \mathrm{i}\mathbb{R} \Big) = \emptyset,$$

for some $\alpha, \beta \in \mathbb{R}$.

If

$$2R_1 := 2c \cdot \max\{|\operatorname{Re}\lambda_1 + \mathrm{i}\gamma_T|^p, |\alpha + \mathrm{i}\gamma_T|^p\} < \operatorname{Re}\lambda_1 - \alpha$$

and

$$2R_N := 2c \cdot \max\{|\beta + \mathrm{i}\gamma_T|^p, |\operatorname{Re}\lambda_N + \mathrm{i}\gamma_T|^p\} < \beta - \operatorname{Re}\lambda_N$$

then the set

$$\sigma(T) \cap ([\operatorname{Re}\lambda_1 - R_1,\ \operatorname{Re}\lambda_N + R_N] + \mathrm{i}\mathbb{R})$$

contains exactly N isolated eigenvalues of $T + A$, counted with algebraic multiplicity.

Proposition

Let $\alpha \in \mathbb{C} \setminus \alpha_\Gamma$. If $a_1, a_2, \ldots, a_n$ are incommensurable over $\{-1, 0, 1\}$ then the algebraic multiplicity of the eigenvalue $\mu_m(\alpha)$ is 1 for almost all $m \in \mathbb{N}$.

Proposition

Let $\alpha \in \mathbb{C} \setminus \alpha_\Gamma$. If $a_1, a_2, \ldots, a_n$ are incommensurable over $\{-1, 0, 1\}$ then the algebraic multiplicity of the eigenvalue $\mu_m(\alpha)$ is 1 for almost all $m \in \mathbb{N}$. Moreover,

$$\boxed{\operatorname{Re}\lambda_m(\alpha) - R_{m-1}(\alpha) \leq \operatorname{Re}\mu_m(\alpha) \leq \operatorname{Re}\lambda_m(\alpha) + R_m(\alpha) \text{ as } m \to \infty.}$$

.

Since $R_m(\alpha) = o(m)$,

$$\mathrm{Re}(\lambda_m(\alpha)) = \frac{\pi^2}{|\Gamma|^2} m^2 + O(m) \text{ as } m \to \infty$$

and

$$\mathrm{Re}\,\lambda_m(\alpha) = \lambda_m(0) + O\left(\frac{1}{\lambda_m(0)}\right),$$

we obtain the following results.

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we obtain the following results.

Corollary

Let $\alpha \in \mathbb{C} \setminus \alpha_\Gamma$. If $a_1, a_2, \ldots, a_n$ are incommensurable over $\{-1, 0, 1\}$, then

$$\mathrm{Re}(\mu_m(\alpha)) = \frac{\pi^2}{|\Gamma|^2} m^2 + O(m) \textit{ as } m \to \infty.$$

Corollary

Let $\alpha \in \mathbb{C} \setminus \alpha_\Gamma$. If $a_1, a_2, \ldots, a_n$ are incommensurable over $\{-1, 0, 1\}$, then

$$\lambda_m(0) - \mathcal{R}_m \leq \mathrm{Re}(\mu_m(\alpha)) \leq \lambda_m(0) + \mathcal{R}_m \text{ as } m \to \infty$$

with $0 \leq \mathcal{R}_m = o(m)$.

Remark

In particular, we also have

$$\lambda_{m-1}(0) \leq \mathrm{Re}(\mu_m(\alpha)) \leq \lambda_{m+1}(0) \text{ as } m \to \infty.$$

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