A Computational Study of Spectral Behavior in Physics-Informed Neural Networks

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June 3, 2025

Motivation

- Differential equations are fundamental tools in modeling physical systems — from quantum mechanics to fluid dynamics.
- Analytical solutions are often intractable; numerical methods provide approximations but introduce discretization and stability challenges.
- Advances in deep learning offer new, flexible tools to solve differential equations through data-driven or physics-informed models.
- Spectral theory connects linear operators, eigenvalue problems, and function spaces — crucial in understanding wave behavior, stability, and modal dynamics.
- Recent research suggests that spectral insights can help design better neural architectures and explain training behavior (e.g., spectral bias).
- This project aims to explore the intersection of deep learning, differential equations, and spectral theory through computational experiments.

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Summary

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A Single Dense Neuron for Regression Tasks

Artificial Intelligence (AI): Simulates intelligent behavior (e.g., reasoning, planning).

Machine Learning (ML): Learns patterns from data for prediction and decision-making.

Deep Learning (DL): Uses multi-layer neural networks to model complex functions.

$DL \subset ML \subset AI$

Examples:

- AI: Robotics, game agents
- ML: Spam filters, recommendations
- DL: Image classification, text-to-speech

House Price Prediction (Single Neuron) Input features:

- x_1 Size (m^2)
- x_2 Number of rooms
- x_3 Location score
- x_4 Year built
- x_5 Distance to center (km)

Single house prediction:

$$\hat{y} = w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 + w_5 x_5 + b$$

Matrix form for n houses:

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{w} + b$$

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{15} \\ x_{21} & x_{22} & \cdots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n5} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_5 \end{bmatrix}$$

Output $\hat{\mathbf{y}} \in \mathbb{R}^n$ gives predicted prices.

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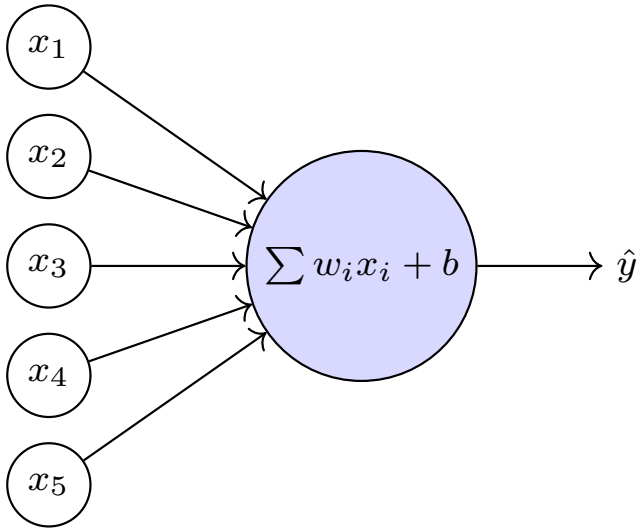
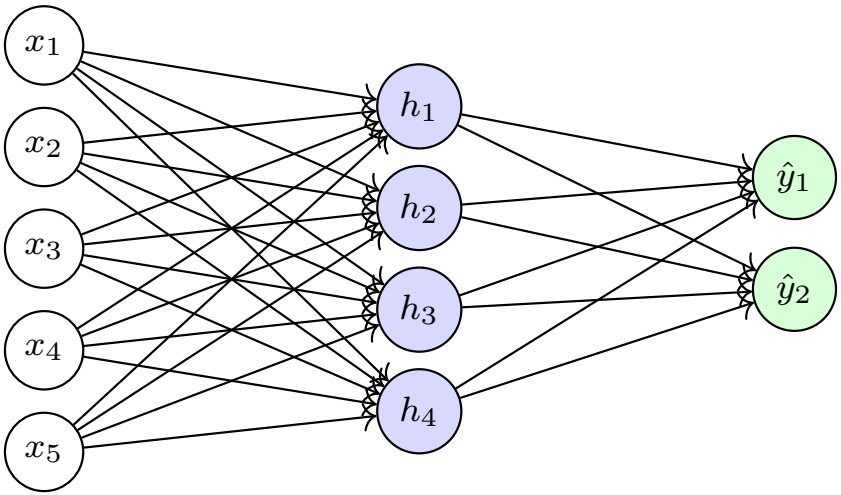
From a Single Neuron to a Small Neural Network

Single Neuron (Fully Connected)



Neural Network (1 Hidden Layer, 2 Outputs)



Single Neuron Equation:

$$\hat{y} = \sum_{i=1}^{5} w_i x_i + b \quad \text{or} \quad \hat{y} = \mathbf{x}^\top \mathbf{w} + b$$

Neural Network (Layered Form):

$$\mathbf{h} = \mathbf{W}_1 \mathbf{x} + \mathbf{b}_1$$

$$\hat{\mathbf{y}} = \mathbf{W}_2 \mathbf{h} + \mathbf{b}_2$$

Shapes:

- $\mathbf{x} \in \mathbb{R}^5$
- $\mathbf{W}_1 \in \mathbb{R}^{4\times 5}$, $\mathbf{b}_1 \in \mathbb{R}^4$
- $\mathbf{W}_2 \in \mathbb{R}^{2\times 4}$, $\mathbf{b}_2 \in \mathbb{R}^2$
- $\hat{\mathbf{y}} \in \mathbb{R}^2$

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Why Layers Alone Are Not Enough

Without activation functions:

$$\mathbf{h} = \mathbf{W}_1\mathbf{x} + \mathbf{b}_1, \quad \hat{\mathbf{y}} = \mathbf{W}_2\mathbf{h} + \mathbf{b}_2$$

Substituting:

$$\hat{\mathbf{y}} = \mathbf{W}_2(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2 = \underbrace{\mathbf{W}_2\mathbf{W}_1}_{\mathbf{W}_{\text{eq}}}\mathbf{x} + \underbrace{\mathbf{W}_2\mathbf{b}_1 + \mathbf{b}_2}_{\mathbf{b}_{\text{eq}}}$$

$$\Rightarrow \quad \hat{\mathbf{y}} = \mathbf{W}_{\text{eq}}\mathbf{x} + \mathbf{b}_{\text{eq}}$$

Conclusion: multiple layers without nonlinearities collapse to a single linear map.

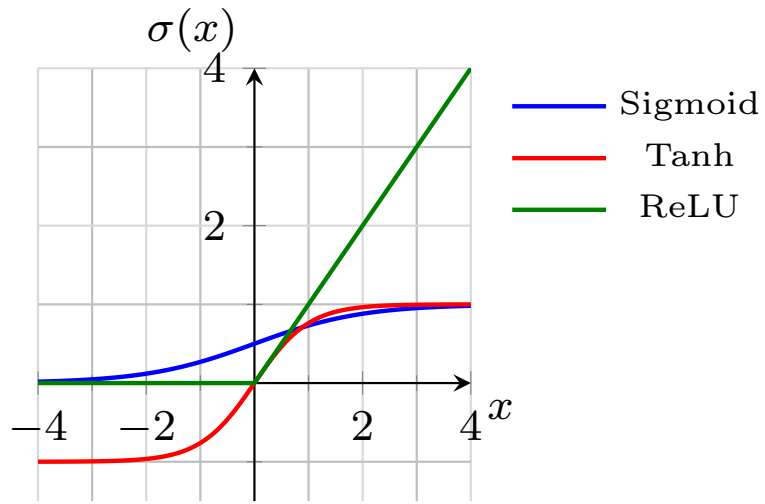
To model non-linear relationships:

$$\mathbf{h} = \sigma(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1), \quad \hat{\mathbf{y}} = \mathbf{W}_2\mathbf{h} + \mathbf{b}_2$$

Now the network can learn non-linear functions.

Common Nonlinearities:

- Sigmoid: $\sigma(x) = \frac{1}{1+e^{-x}}$
- Tanh: $\tanh(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$
- ReLU: $\text{ReLU}(x) = \max(0, x)$



Nonlinearities add expressive power beyond linear systems.

Neural Networks as Universal Approximators

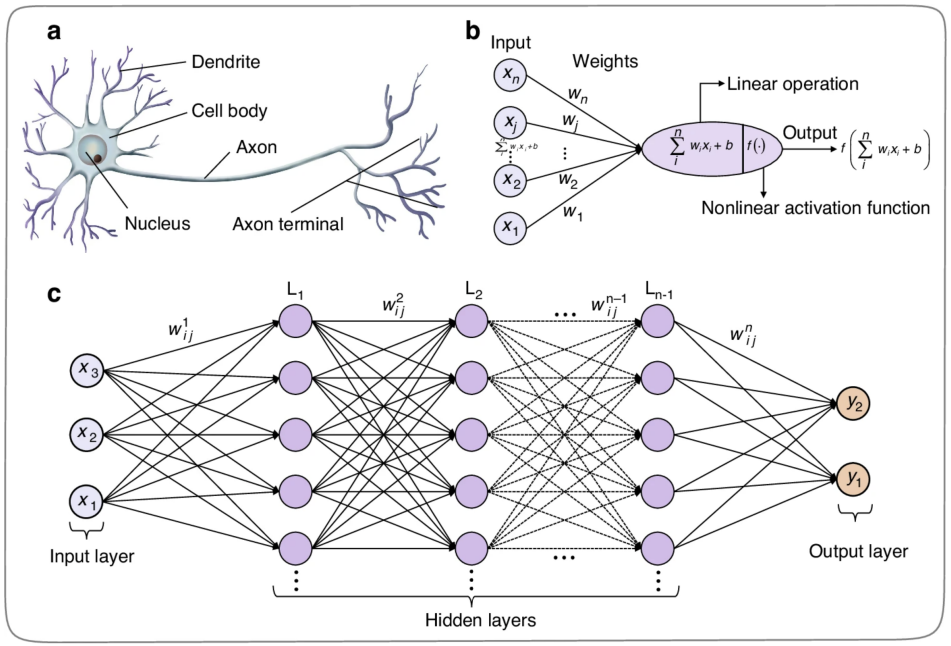


Figure 1: Biological and mathematical abstraction of a neural network

Source: Fu, T., Zhang, J., Sun, R. et al. Optical neural networks: progress and challenges. *Light Sci Appl* **13**, 263 (2024). https://doi.org/10.1038/s41377-024-01590-3

Universal Approximation Theorem

Consider a fixed activation function $\sigma : \mathbb{R} \to \mathbb{R}$ that is not a polynomial. For any integers d, D, any continuous function $f : \mathbb{R}^d \to \mathbb{R}^D$, any compact set $K \subset \mathbb{R}^d$, and any $\epsilon > 0$, there exists a neural network function $f_\epsilon : \mathbb{R}^d \to \mathbb{R}^D$ such that:

$$f_\epsilon = W_2 \circ \sigma \circ W_1$$

where W_1, W_2 are affine maps and:

$$\sup_{x \in K} \|f(x) - f_\epsilon(x)\| < \epsilon$$

Implication: Neural networks can approximate any continuous function on compact domains arbitrarily well.

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Learning as Optimization in Neural Networks

Training data:

$$\{(x^{(i)}, y^{(i)})\}_{i=1}^{n}, \quad x^{(i)} \in \mathbb{R}^d, \; y^{(i)} \in \mathbb{R}^D$$

Prediction:

$$\hat{y}^{(i)} = f_\theta(x^{(i)}), \quad \theta = \{W_k, b_k\}$$

Loss (MSE):

$$\mathcal{L}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \|\hat{y}^{(i)} - y^{(i)}\|^2$$

Training (Gradient Descent):

$$\theta_{t+1} = \theta_t - \eta \nabla_\theta \mathcal{L}(\theta_t)$$

Why not solve analytically?

- Non-convex loss landscape
- Millions of parameters
- No closed-form for $\nabla_\theta \mathcal{L} = 0$

Gradient Computation:
Backpropagation applies the chain rule layer-by-layer:

$$\frac{\partial \mathcal{L}}{\partial W_k} = \frac{\partial \mathcal{L}}{\partial a_k} \cdot \frac{\partial a_k}{\partial W_k}$$

where a_k is the activation at layer k

Implementation:

- Gradients are computed via **automatic differentiation**
- **No manual derivatives** needed

Hardware Acceleration:

- Matrix operations run on GPUs/TPUs
- Enables fast training over large datasets

Goal: Learn parameters θ that minimize loss and generalize to unseen data

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PINNs: Solving PDEs through Optimization

Physics-Informed Neural Networks (PINNs) solve PDEs by embedding physical laws into the loss function. The network approximates $u(x,t)$ by minimizing residuals of the governing equations, boundary, and initial conditions.

Governing PDE: Describes the physical system (e.g., heat, wave, Burgers' equation).

$$\mathcal{D}[u(x,t);\lambda] = f(x,t), \quad x \in \Omega, \quad t \in [0,+\infty)$$

Boundary Conditions: Applied on the spatial domain boundary $\partial\Omega$.

$$\mathcal{B}_k[u(x,t)] = g_k(x,t), \quad x \in \Gamma_k \subseteq \partial\Omega$$

Initial Conditions: Needed for time-dependent problems to set initial state.

$$u(x,0) = u_0(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = u_0'(x)$$

Total Loss Function: Combines all physics constraints into one objective.

$$L(\omega) = L_p(\omega) + L_b(\omega) + L_i(\omega)$$

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PINNs: Solving PDEs through Optimization

The loss terms guide the network to approximate a solution that respects:

- the PDE dynamics at interior points,
- the boundary values at domain edges,
- and the initial state at $t = 0$.

PDE Residual Loss: Enforces the differential equation at collocation points.

$$L_p(\omega) = \frac{1}{N_p} \sum_{i=1}^{N_p} \|\mathcal{D}[NN(x_i, t_i; \omega)] - f(x_i, t_i)\|^2$$

Boundary Loss: Penalizes mismatch between predicted and exact boundary values.

$$L_b(\omega) = \sum_k \frac{1}{N_b} \sum_{j=1}^{N_b} \|\mathcal{B}_k[NN(x_j, t_j; \omega)] - g_k(x_j, t_j)\|^2$$

Initial Loss: Ensures the network respects initial conditions of u and $\partial u/\partial t$.

$$L_i(\omega) = \|NN(x, 0) - u_0(x)\|^2 + \left\| \frac{\partial NN(x, 0; \omega)}{\partial t} - u_0'(x) \right\|^2$$

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PINNs: Solving PDEs through Optimization

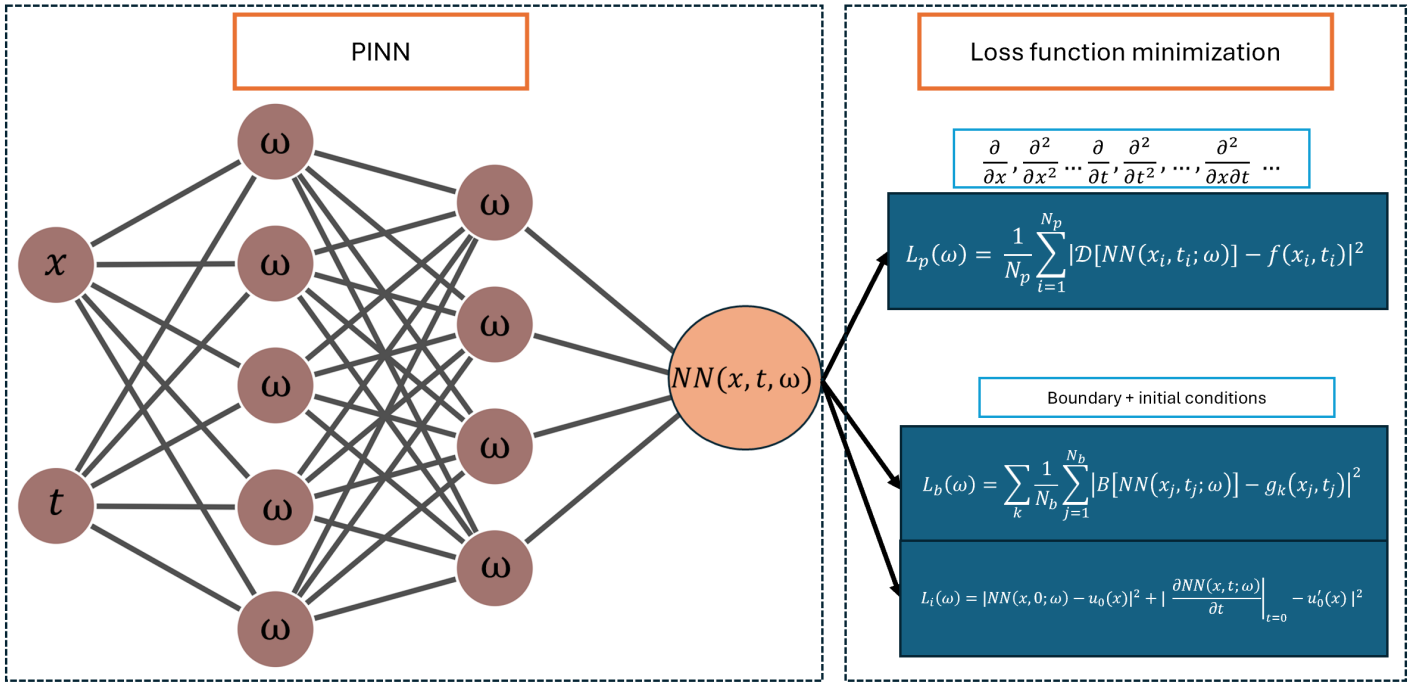

Figure 2: The network minimizes the residuals of the governing equation, boundary conditions, and initial conditions through loss functions $L_p(\omega)$, $L_b(\omega)$, and $L_i(\omega)$, respectively.

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Why PINNs Are Different from Traditional Neural Networks

Traditional Neural Networks:

- Require labeled data: $(x^{(i)}, y^{(i)})$
- Learn to interpolate from training data
- Generalize only within the domain of seen data

Physics-Informed Neural Networks (PINNs):

- Do not require a labeled dataset
- Instead, use physical laws (PDEs) to generate training signals
- Can extrapolate to unseen domains by obeying the governing equations
- Directly minimize residuals of the PDE + boundary/initial conditions

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Problem Setup: Solving an ODE with PINNs

We consider the first-order ODE:

$$\frac{du}{dt} = \sin(t) - u, \quad u(0) = 1$$

Analytical solution:

$$u(t) = \frac{1}{2}(\sin(t) - \cos(t)) + \frac{3}{2}e^{-t}$$

- Domain: $t \in [0, 5]$
- Used for benchmarking PINNs against Euler, RK2, RK4

PINN Setup:

- Neural network: 2 layers, 128 neurons each, Tanh activation
- Physics loss = PDE residual + initial condition
- Trained for 10k and 20k epochs on collocation points in $[0, 5]$

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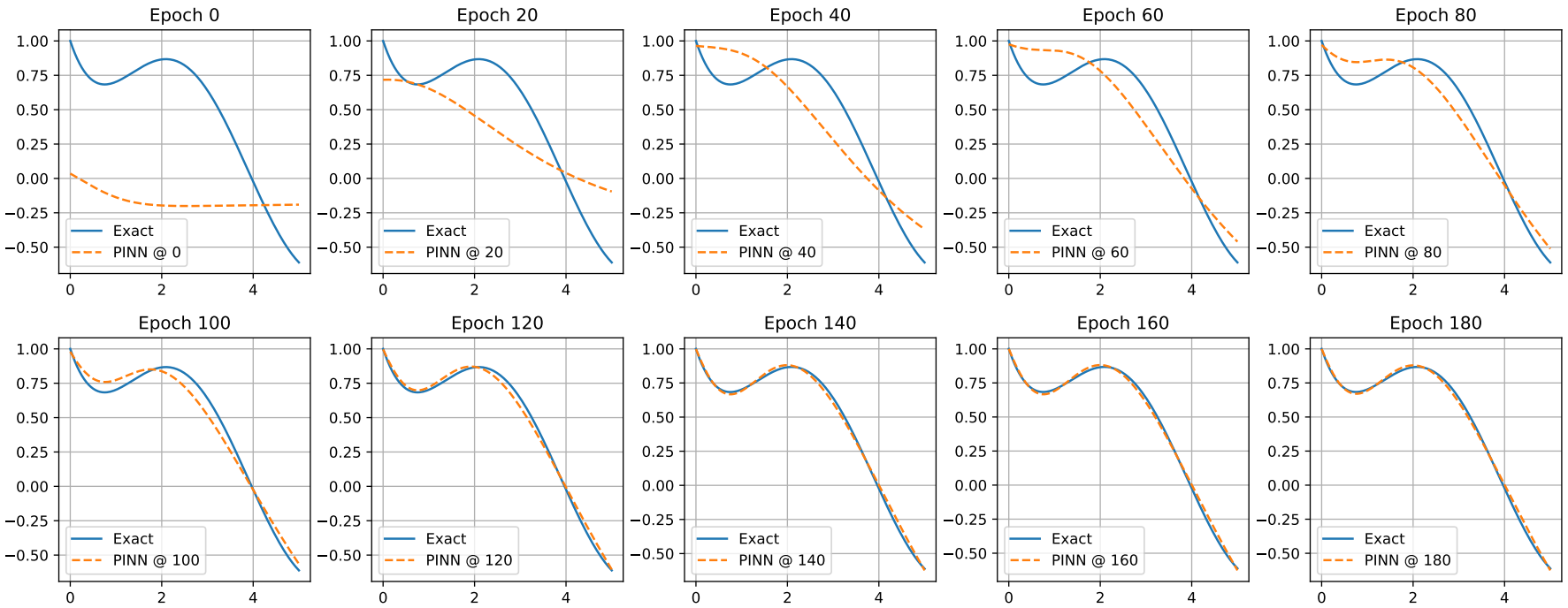
Learning Dynamics of the PINN



- PINN solution shown every 20 epochs, from epoch 0 to 180
- Initial residual is large; network gradually learns the dynamics
- Visible convergence around epoch 100–150

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Accuracy Comparison

Method	MSE	Max Error
Euler	1.9888×10^{-5}	8.4372×10^{-3}
RK2	1.7121×10^{-9}	7.8139×10^{-5}
RK4	1.0440×10^{-18}	1.8441×10^{-9}
PINN @10k	1.8149×10^{-7}	1.6839×10^{-3}
PINN @20k	1.1398×10^{-8}	4.0000×10^{-4}

Table 1: Comparison of mean squared and maximum error for all methods.

Observations:

- RK4 yields highest accuracy (used as benchmark)
- PINN improves significantly with longer training
- PINNs learn a **continuous function** over time
- Can be evaluated at any t — even outside training domain
- Traditional methods compute only on fixed grids

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Lane-Emden Equation: A Showcase for PINNs

- The Lane-Emden equation is a dimensionless form of the Poisson equation used in astrophysics to model polytropic stellar structures.
- It describes the structure of a self-gravitating, spherically symmetric star in hydrostatic equilibrium, where pressure and density obey the relation: $P \propto \rho^{1+1/n}$.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0$$

- Here, ξ is the dimensionless radius, $\theta(\xi)$ is the normalized density, and n is the polytropic index.
- Boundary conditions at the stellar center:

$$\theta(0) = 1, \quad \theta'(0) = 0$$

- Only a few analytical solutions exist:

$$\theta_{n=0} = 1 - \frac{\xi^2}{6}, \quad \theta_{n=1} = \frac{\sin\xi}{\xi}, \quad \theta_{n=5} = \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$$

Historical references: Lane (1870), Emden (1907), Chandrasekhar (1957)

PINN Formulation for the Lane-Emden Equation (1/2)

- The classical Lane-Emden equation models stellar structure under a polytropic assumption:

$$\frac{d^2\theta}{dt^2} + \frac{2}{t}\frac{d\theta}{dt} + \theta^n = 0$$

- We reformulate it as a 2D PINN problem:

$$\theta(t, n), \quad \text{with } t = \text{radius}, \quad n = \text{polytropic index}$$

  This enables the neural network to generalize across families of stellar models, not just one.

- Derivatives $\partial\theta/\partial t$ and $\partial^2\theta/\partial t^2$ are computed via automatic differentiation.
- The residual function enforced at each collocation point (t_i, n_j) is:

$$R(t_i, n_j) = \frac{\partial^2\theta}{\partial t^2} + \frac{2}{t_i}\frac{\partial\theta}{\partial t} + \theta^{n_j}$$

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PINN Loss Formulation for the Lane-Emden Equation

- The total loss combines:
  - the residual of the PDE
  - boundary conditions at $t = 0$

$$\mathcal{L} = \mathcal{L}_{\text{residual}} + \lambda_1 \mathcal{L}_{\theta(0,n)} + \lambda_2 \mathcal{L}_{\frac{\partial\theta}{\partial t}(0,n)}$$

$$\mathcal{L}_{\text{residual}} = \frac{1}{N_b} \sum_{i=1}^{m} \sum_{j=1}^{p} R(t_i, n_j)^2$$

$$\mathcal{L}_{\theta(0,n)} = \sum_{j=1}^{p} (\theta(0, n_j) - 1)^2$$

$$\mathcal{L}_{\frac{\partial\theta}{\partial t}(0,n)} = \sum_{j=1}^{p} \left(\frac{\partial\theta}{\partial t}(0, n_j) \right)^2$$

Here, λ_1 and λ_2 are weighting coefficients controlling the emphasis on boundary conditions relative to the residual. These may be tuned manually or learned adaptively during training. The conditions reflect physical requirements:

$$\theta(0, n) = 1, \quad \frac{d\theta}{dt}(0, n) = 0$$

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PINN Solutions to the Lane-Emden Equation

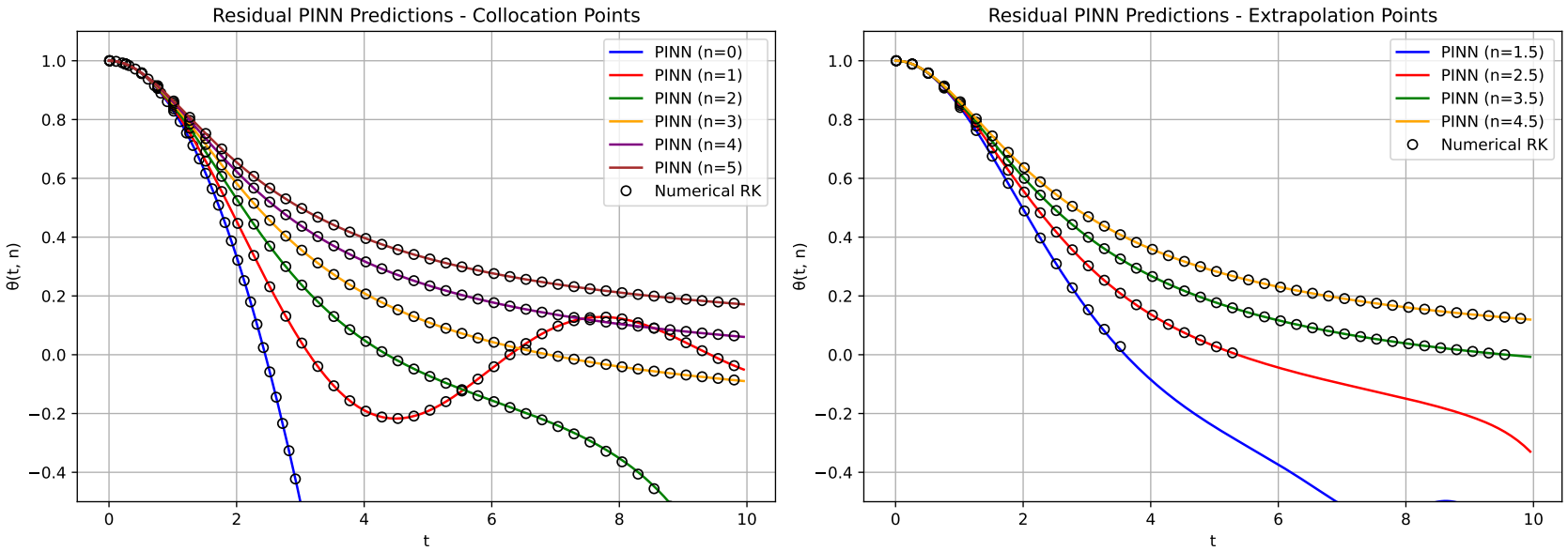

Figure 3: Residual-based PINN predictions (solid lines) versus classical Runge-Kutta (RK) solutions (dashed lines) for the Lane-Emden equation. **Left:** PINN interpolates accurately over collocation points for integer indices $n = 0$ to 5. **Right:** The PINN generalizes to non-integer indices $n = 1.5$ to 4.5 beyond the trained region. PINNs exhibit strong interpolation and extrapolation capabilities for singular ODEs.

Source: A.-I. Mohut, C.-A. Popa, *Residual-Based PINNs for Accurate Solutions of the Lane-Emden Equation*, in preparation (2025).

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Spectral PINN: 1D Laplace Eigenproblem

- We consider the eigenvalue problem:

$$-\frac{d^2u}{dx^2} = \lambda u, \quad x \in (0,1), \quad u(0) = u(1) = 0$$

- The true eigenvalues and eigenfunctions are:

$$\lambda_n = n^2\pi^2, \quad u_n(x) = \sin(n\pi x), \quad n = 1, 2, 3, \ldots$$

- The PINN is trained to minimize:

$$\mathcal{L} = \|u''(x) + \lambda u(x)\|^2 + \|u(0)\|^2 + \|u(1)\|^2$$

- We optimize both $u_\theta(x)$ and λ jointly.
- Goal: observe how well the PINN captures different eigenmodes and how frequency affects convergence.

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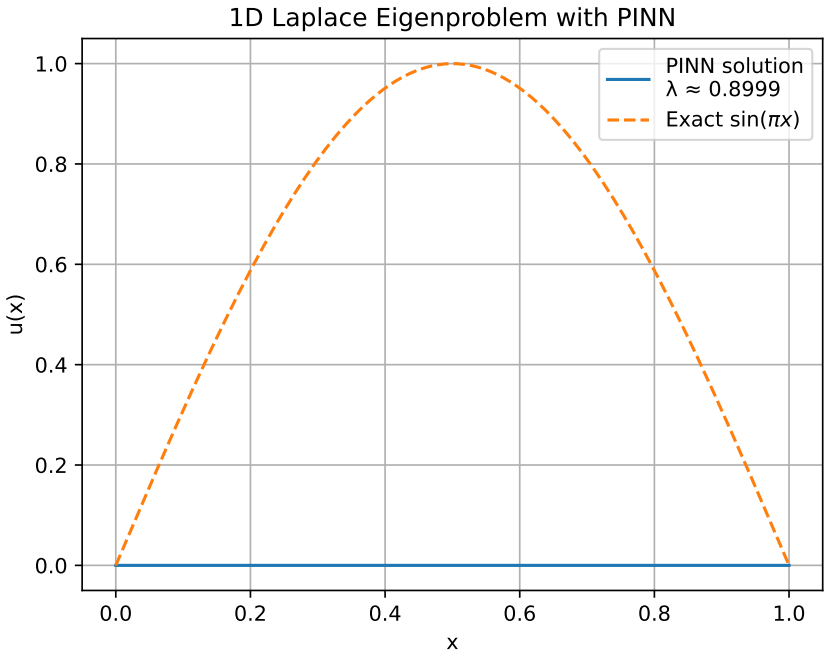
Spectral PINN Failure Case: Trivial Solution

- The loss function penalizes deviation from the PDE and the Dirichlet boundary conditions:

$$\mathcal{L} = \|u''(x) + \lambda u(x)\|^2 + \|u(0)\|^2 + \|u(1)\|^2$$

- We optimize both the function $u_\theta(x)$ and the eigenvalue λ jointly using automatic differentiation and Adam optimizer.
- However, in this example, the PINN collapsed to a trivial solution $u(x) \approx 0$, which minimizes the loss but is not an eigenfunction.



Spectral PINN: Normalization & Implicit BCs

- Instead of enforcing boundary conditions via the loss, we embed them into the neural network output via an **ansatz**:

$$u(x) = x(1-x) \cdot \hat{u}_\theta(x) \Rightarrow u(0) = u(1) = 0 \text{ automatically}$$

- To avoid trivial solutions $u(x) \approx 0$, we normalize the output:

$$\mathcal{L}_{\text{norm}} = \left(\|u\|^2 - 1\right)^2$$

- The total loss becomes:

$$\mathcal{L} = \|u''(x) + \lambda u(x)\|^2 + \alpha \cdot \mathcal{L}_{\text{norm}}$$

  where α is a weighting parameter (e.g., 100).

- This yields a valid eigenfunction shape and eigenvalue without explicitly enforcing boundary conditions.

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Spectral PINN: Normalization & Implicit BCs

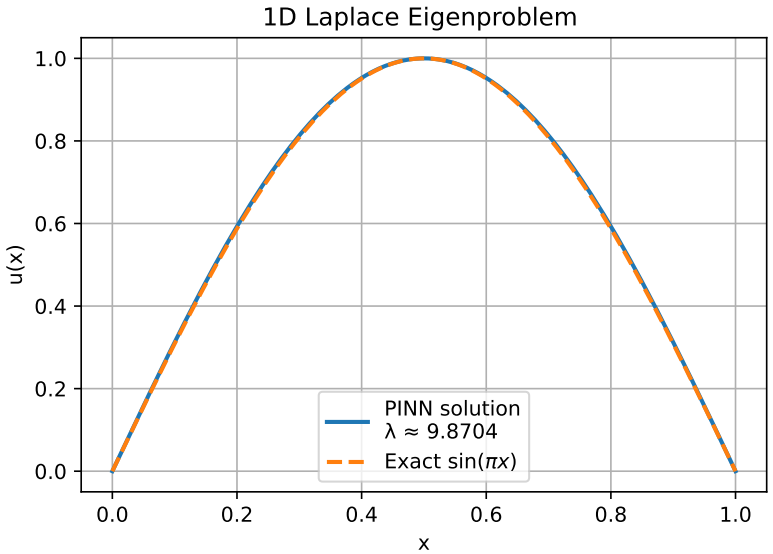
- Eigenfunctions are only defined up to sign: both $\sin(\pi x)$ and $-\sin(\pi x)$ are valid.
- We align the sign during training using a **phase constraint**:

$$\mathcal{L}_{\text{phase}} = \text{ReLU}(-u(0.5))^2$$

- For fair comparison with ground truth, we normalize:

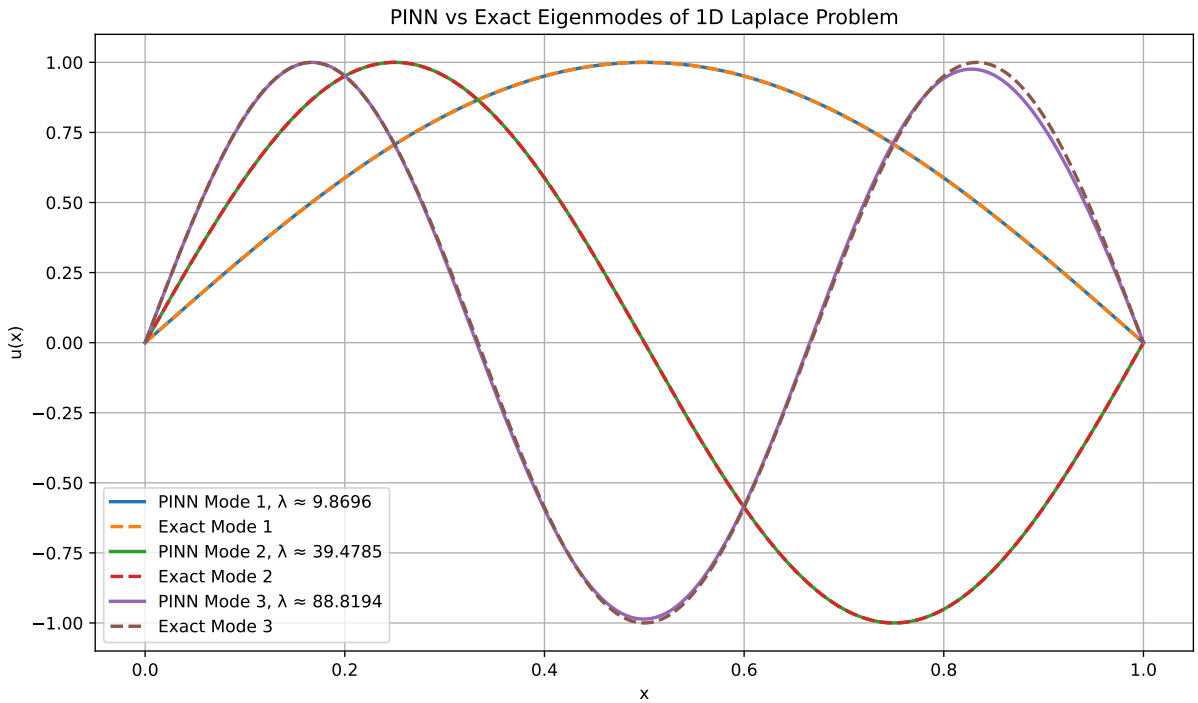
$$u(x) \leftarrow \frac{u(x)}{\max |u(x)|}$$

- The PINN now learns the correct frequency and shape, with clean output thanks to ansatz and normalization.



Spectral PINN: Learning Higher Eigenmodes

- To compute higher modes ($n > 1$), we train separate PINNs with:
  1. **Implicit BCs:** Ansatz $u(x) = x(1-x) \cdot \hat{u}_\theta(x)$
  2. **Normalization:** Enforce $\|u\|_{L^2} \approx 1$
  3. **Orthogonality:** Enforce $\langle u_n, u_k \rangle \approx 0$ for all $k < n$
- Sign ambiguity is expected (both $\sin(n\pi x)$ and $-\sin(n\pi x)$ are valid eigenfunctions). We postprocess by flipping u_n to align with the ground truth.



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Eigenvalue Problem: Setup and Numerical Methods

Problem:

$$-u''(x) = \lambda u(x), \quad x \in [0, 2], \quad u(0) = 0, \quad u'(2) = 0$$

- Analytical solution: $u(x) = \sin\left(\frac{\pi x}{4}\right), \quad \lambda = \left(\frac{\pi}{4}\right)^2 \approx 0.6169$

Methods Compared:

- **PINN** (Physics-Informed Neural Network)
- **FDM** (Finite Differences)
- **Shooting Method**
- **Rayleigh Quotient**
- **Galerkin Method**

Each method aims to compute the lowest eigenvalue λ_1 and associated eigenfunction $u(x)$, compared to the known analytical ground state.

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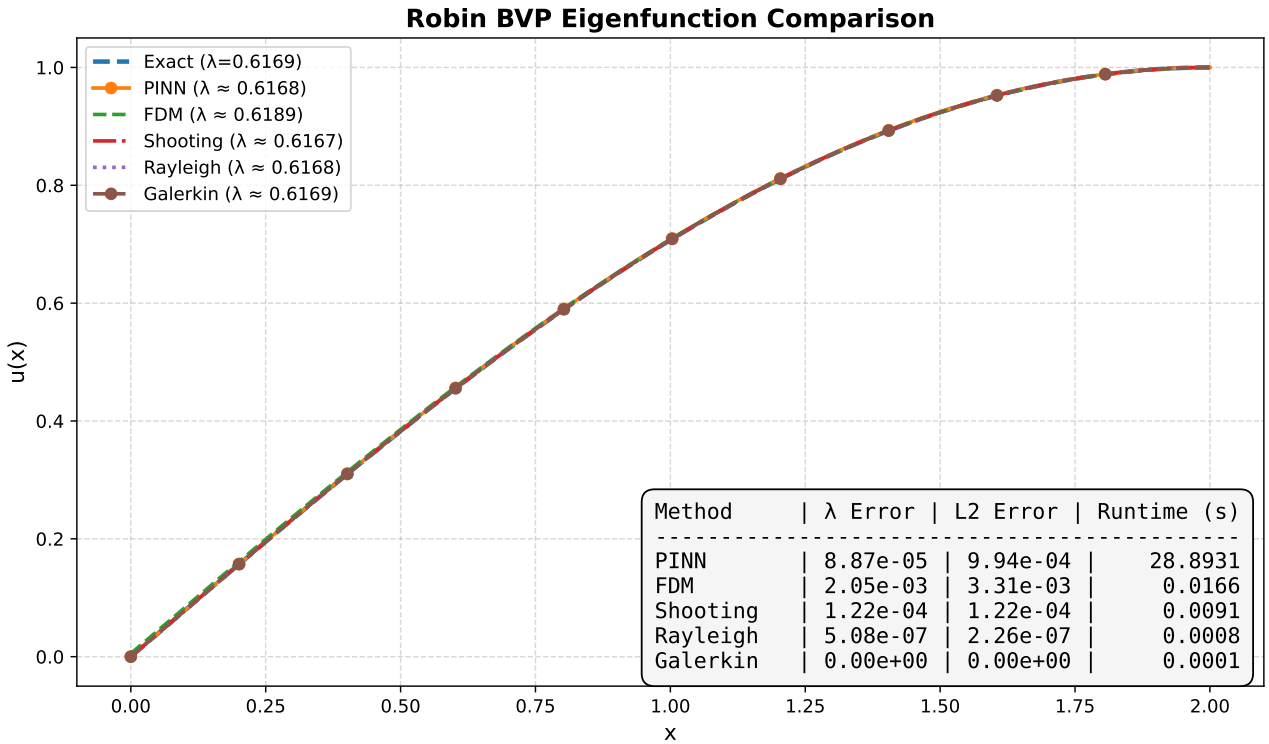
Numerical Results and Accuracy



- All methods approximate the ground state eigenvalue accurately.
- Galerkin and Rayleigh methods nearly exact for this specific problem.

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Strengths and Weaknesses of Methods

Method	**Strengths**	**Weaknesses**
PINN	Flexible, can handle complex geometries and physics; learns λ automatically	High runtime; requires tuning and many epochs
FDM	Easy to implement; fast runtime	Less accurate near boundaries; needs fine grids
Shooting	Very accurate for 1D problems; fast	Hard to generalize to higher dimensions or multiple eigenmodes
Rayleigh	Extremely fast; accurate with good trial function	Accuracy depends heavily on choice of trial function
Galerkin	Can achieve exact result with appropriate basis	Requires symbolic or analytic knowledge of solution space

Table 2: Qualitative comparison of numerical methods

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Spectral Bias in PINNs: Motivation & Setup

- **Goal:** Demonstrate the **spectral bias** of PINNs — their tendency to learn low-frequency components of a solution faster than high-frequency ones.
- We construct a known multi-frequency solution using only odd modes:

$$u(x) = \sum_{n=1,3,5,7,9} \frac{1}{n^2} \sin(n\pi x)$$

- We plug this into:

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0$$

to obtain a custom ODE with known forcing term $f(x)$, ensuring full control over the solution.

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PINN Setup and Frequency Tracking

- We solve the equation using a PINN with no ansatz:
  - Boundary conditions $u(0) = u(1) = 0$ are imposed directly via the loss.
- The total loss includes:

$$\mathcal{L} = \mathcal{L}_{\text{PDE}} + \mathcal{L}_{\text{BC}}$$

- During training, the PINN solution is projected onto the Fourier basis:

$$\text{coeff}_n = \langle u_\theta, \sin(n\pi x) \rangle$$

  and compared to the true amplitudes $a_n = \frac{1}{n^2}$.
- A mode is considered converged when:

$$\left| \frac{\text{coeff}_n - a_n}{a_n} \right| < 1\% \quad \text{for at least 100 epochs}$$

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Results: Convergence of Fourier Modes

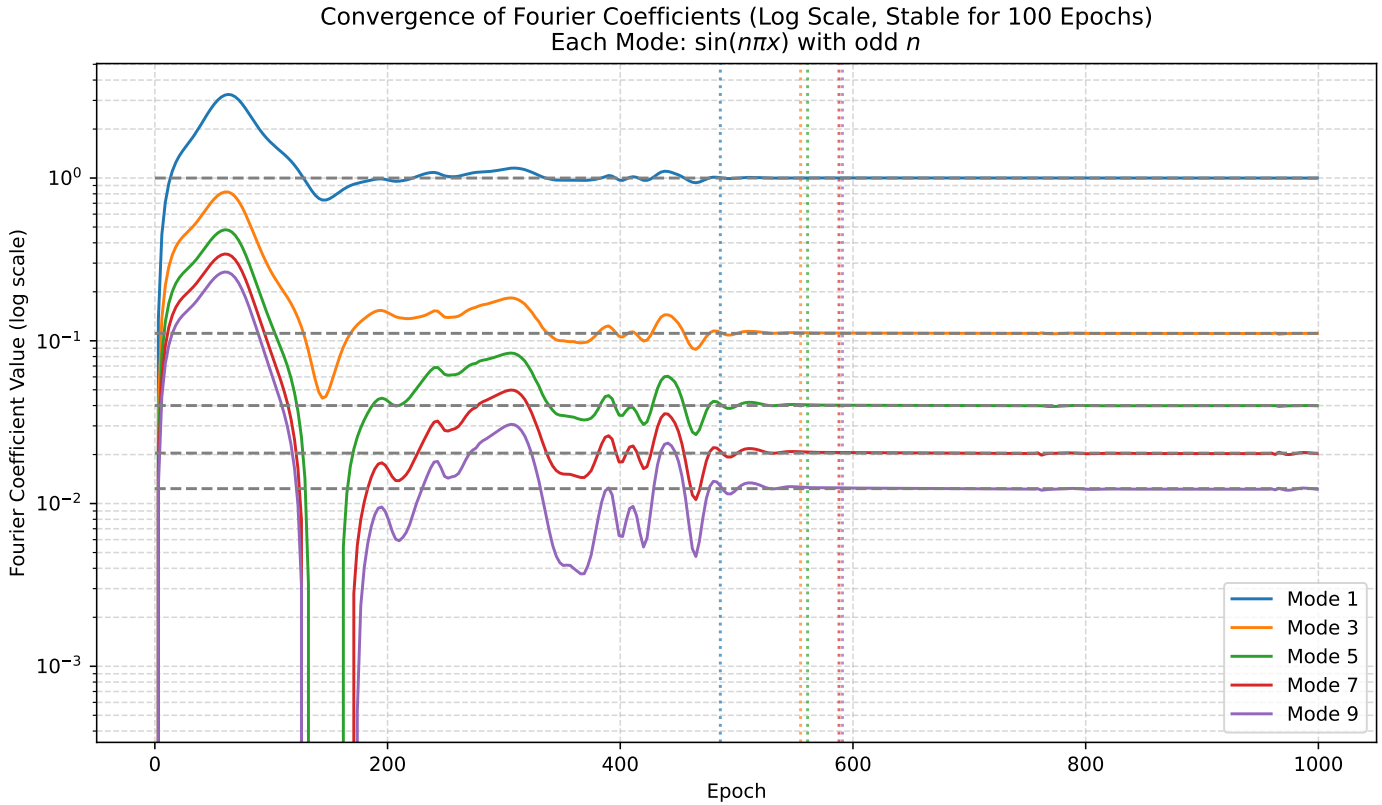

Figure 4: Learned Fourier amplitudes (solid lines) compared to target values (gray dashed). Dotted vertical lines show when each mode's error drops and stays below 1% for 100 epochs.

Interpreting Spectral Bias in PINNs

- **Spectral bias observed:** lower-frequency modes (e.g., $\sin(\pi x)$) converge first.
- Higher modes (e.g., $\sin(7\pi x), \sin(9\pi x)$) take significantly longer to stabilize.
- This behavior reveals a frequency-ordering in how PINNs absorb information.
- **This is spectral bias:**
  - A known behavior of neural networks trained via gradient descent
  - More pronounced with smooth activations (e.g., `tanh`)
- **Why it matters:**
  - Explains why PINNs may struggle with high-frequency PDEs
  - Suggests using enriched input encodings (e.g., Fourier features, positional encoding)

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Random Fourier Feature (RFF) Embedding for PINNs

- **Motivation:** Due to spectral bias, PINNs learn low-frequency features faster — making high-frequency learning inefficient.
- **Solution:** Use a **Random Fourier Feature (RFF)** embedding to enrich the input space with high-frequency content.
- The original input vector $x = (t, n)$ is transformed as:

$$\Phi(x) = \begin{bmatrix} \cos(Bx) \\ \sin(Bx) \end{bmatrix}$$

  where:

  - $B \in \mathbb{R}^{D \times 2}$, with $B_{ij} \sim \mathcal{N}(0, s^2)$
  - s is a hyperparameter controlling the frequency scale

- The PINN is then trained on $\Phi(t, n)$ instead of (t, n), allowing the network to access multi-scale representations.
- **Effect:** Accelerates learning of oscillatory (high-frequency) components that would otherwise converge slowly due to spectral bias.

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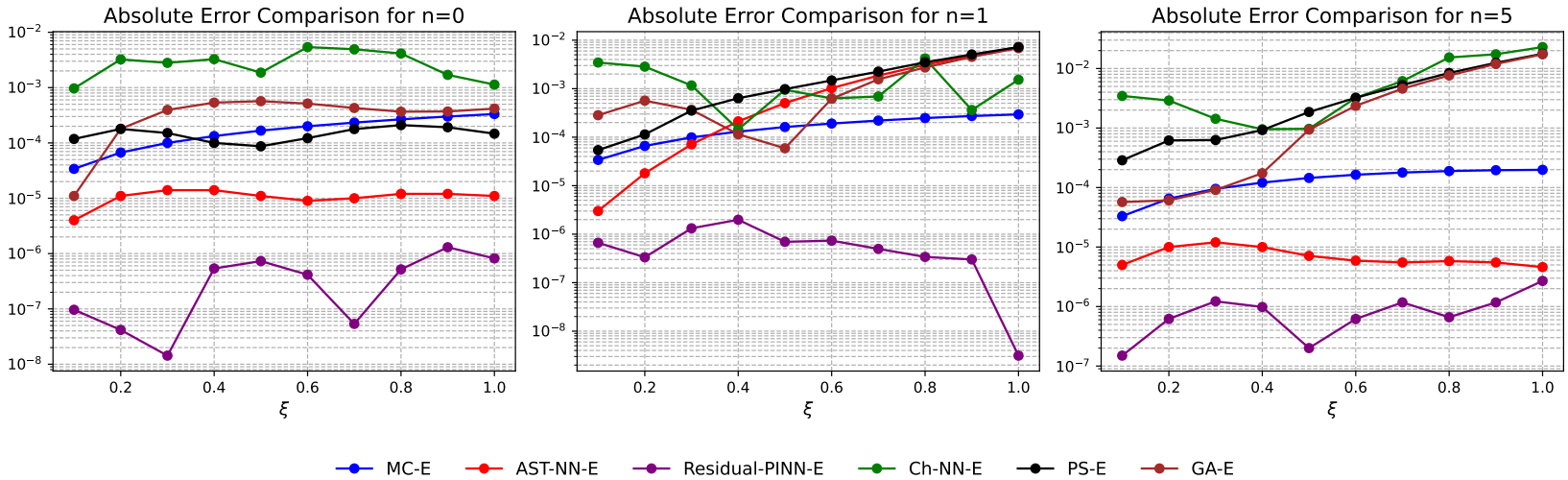
Improving Lane-Emden PINNs with RFF Embeddings

- **Goal:** Reduce absolute error in PINN solutions of the Lane-Emden equation by mitigating spectral bias.
- **Method:** Use a Random Fourier Feature (RFF) embedding to enrich the input (t, n) space with high-frequency components.
- **Observation:** RFF-PINNs yield substantial accuracy improvements across all tested polytropic indices n, including singular and stiff regimes.



Key Takeaways

- **Neural networks** can approximate any continuous function — but nonlinear activations are critical to their power.
- **PINNs** embed differential equations directly into the loss function — allowing solution of ODEs/PDEs without labeled data.
- PINNs are **flexible** but **optimization-heavy** — high runtime and convergence sensitivity are key challenges.
- **Spectral PINNs** solve eigenvalue problems by combining physics constraints, normalization, and ansatz-driven architectures.
- **Spectral bias** causes PINNs to learn low-frequency modes first — high-frequency accuracy requires enriched encodings.
- **Random Fourier Features (RFFs)** improve high-frequency learning and address limitations of smooth activations like `tanh`.

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State of the Art: PINNs and Spectral Approaches

- **PINNs** have been extended to:
  - Multiphysics (e.g., Navier-Stokes, Maxwell, Schrödinger)
  - Irregular and multi-domain geometries
  - Inverse problems and data assimilation
- **Key Innovations:**
  - *Adaptive weighting* of loss terms
  - *Domain decomposition* for scaling to large domains
  - *Fourier feature embeddings* for frequency-rich solutions
  - *Complex PINN architectures*
- **Limitations:**
  - Struggle with stiff or chaotic dynamics
  - Require extensive hyperparameter tuning
  - Slow convergence compared to classical solvers

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Thank you!

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