A SPECTRAL CHARACTERIZATION OF MAHARAM MEASURES

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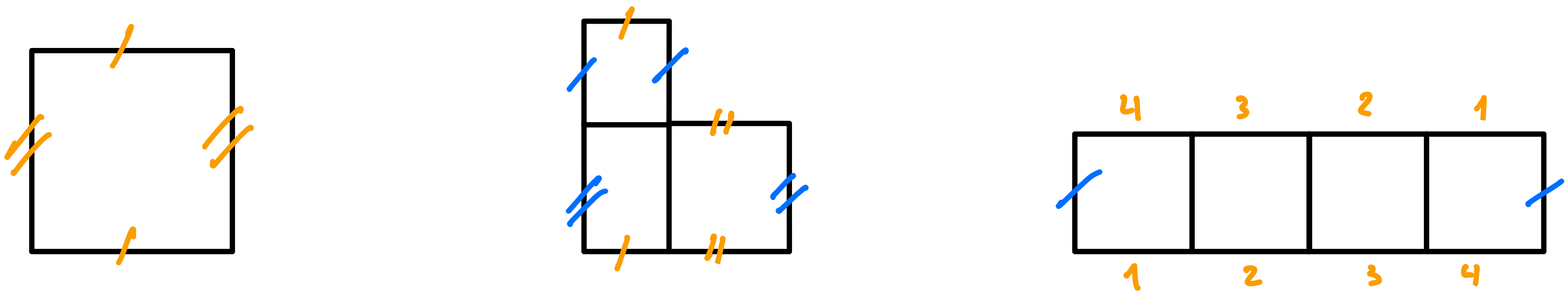
Aspects of Spectral Theory for Linear Operators – 06/06/2025

COAUTHORS

WIP with Roberto Castorrini (Scuola Normale Superiore, Pisa), Davide Ravotti (Université de Lille), Yuriy Tumarkin (University of Zurich).

TRANSLATION SURFACES

DEFINITION



A compact translation surface is a pair (S, ω) where S is a compact Riemann surface and ω is a (non identically zero) holomorphic 1-form on S.

Local integration of ω defines a **Euclidean** metric on S with conical singularities at Σ, the zeros of ω. In particular, we have two orthonormal vector fields X and Y on $S_0 = S \setminus \Sigma$.

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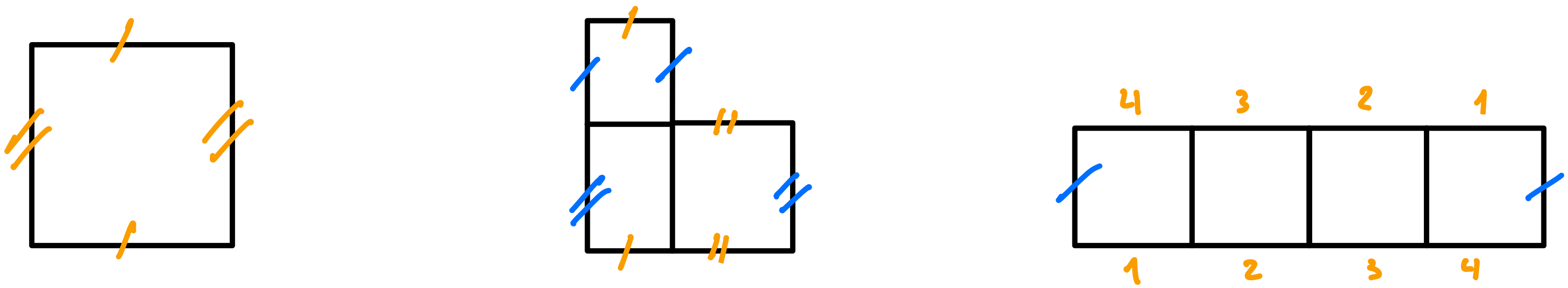

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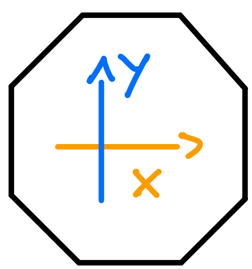
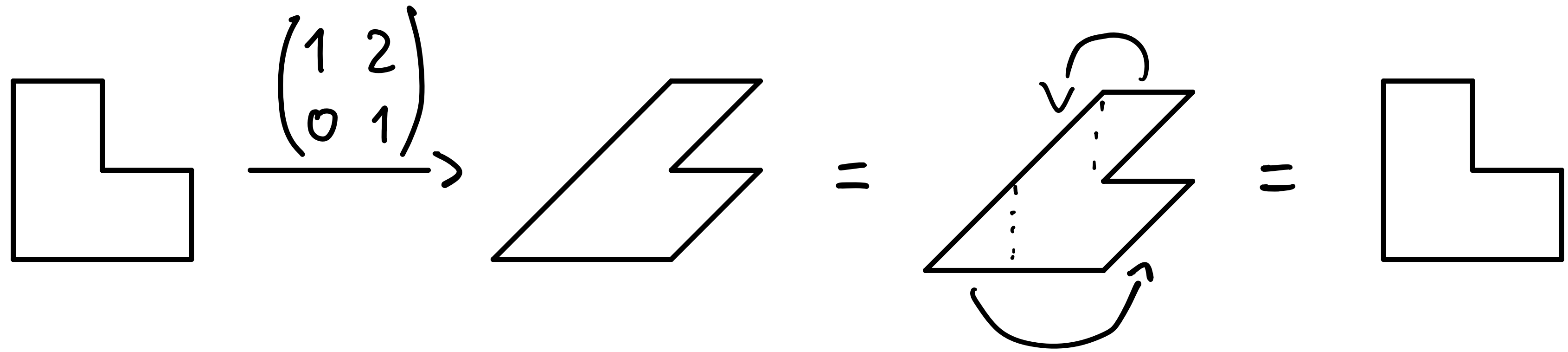
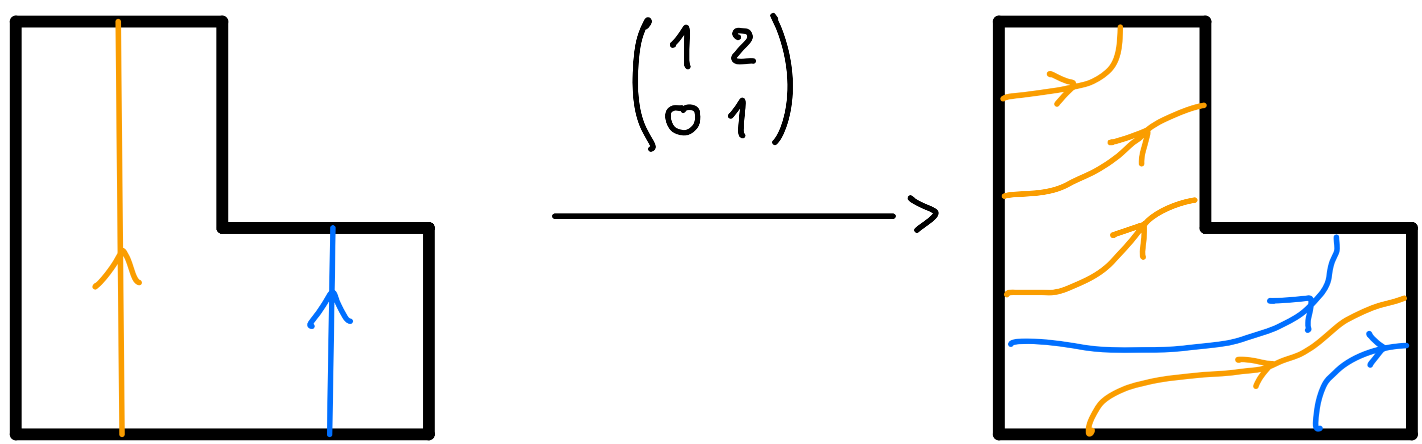


AFFINE AUTOMORPHISMS

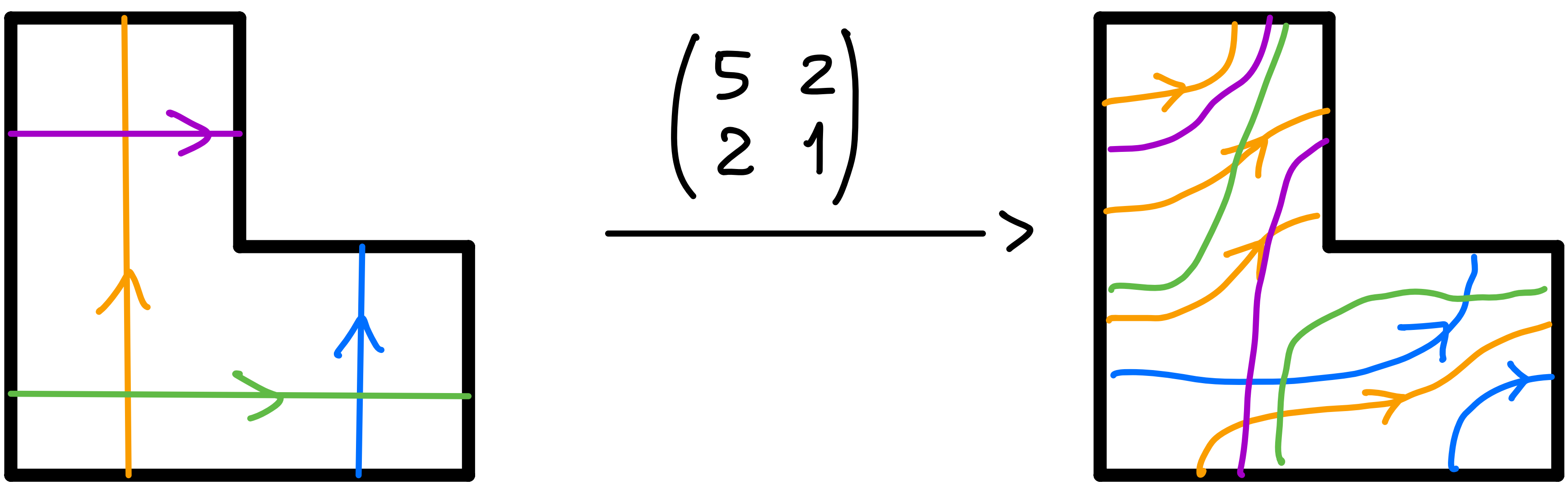


An homeomorphism $T\colon S \to S$ is called an affine automorphism if it fixes the singularities and it is affine in local coordinates.



PSEUDO-ANOSOV AUTOMORPHISMS

We will be interested in pseudo-Anosov affine automorphisms: DT is an **hyperbolic** matrix.



RENORMALIZATION

If T is a pseudo-Anosov automorphism, then there exist two orthogonal foliations, called the stable and unstable foliations, which:

- are **invariant** under T.
- DT **contracts** the stable foliation and **expands** the unstable foliation (by the same factor $\lambda > 1$).

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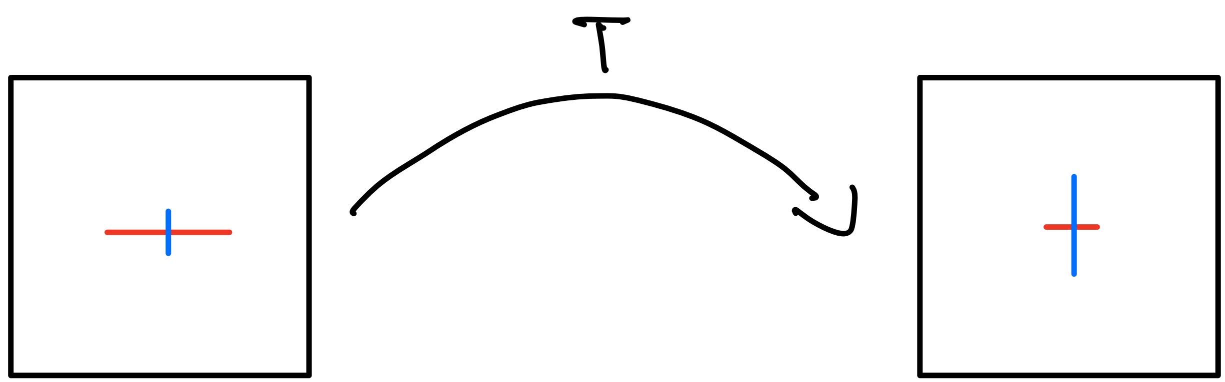

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Let ϕ_t be the flow along the stable foliation. We have the renormalization equation:

$$T \circ \phi_t = \phi_{\lambda^{-1}t} \circ T.$$

We will always assume that the stable foliation is the **horizontal** one.

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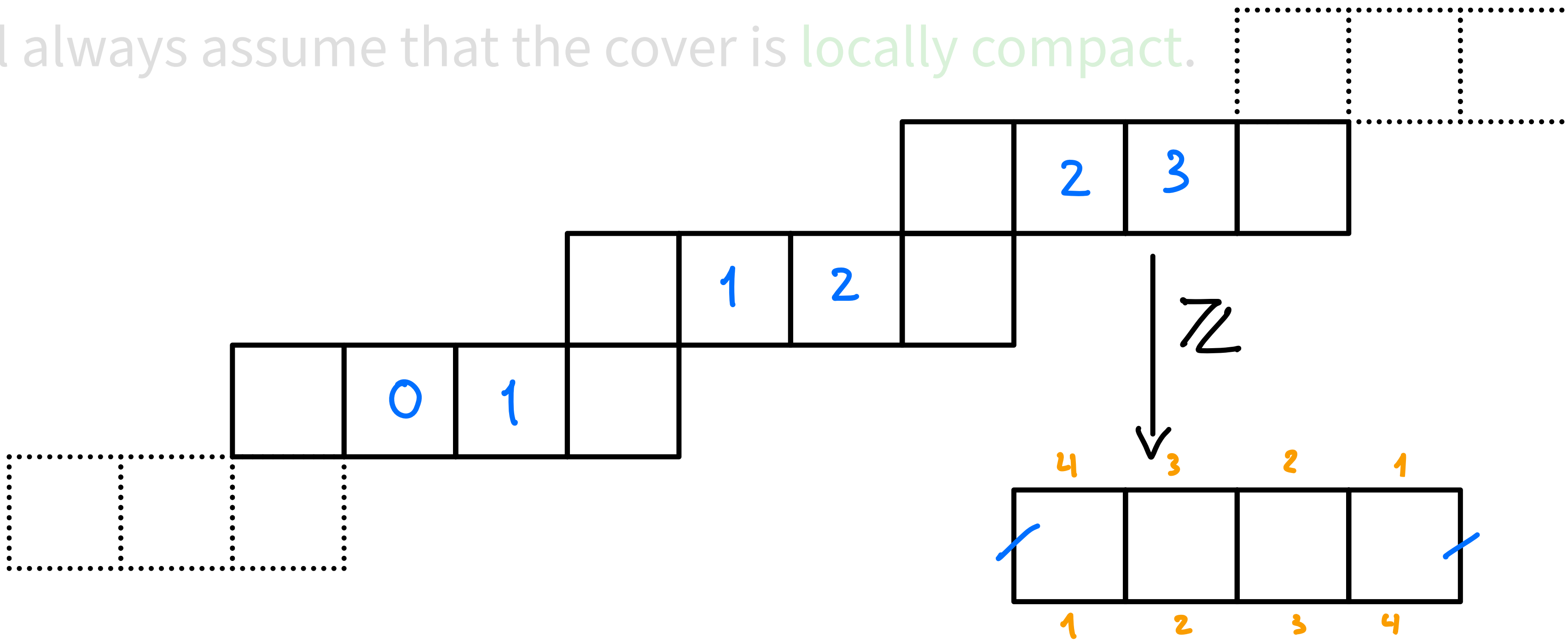
COVERS OF TRANSLATION SURFACES AND MAHARAM MEASURES

$\mathbb{Z}^d$-COVERS

Consider a cover $p: \widetilde{S} \to S$ of translation surfaces, which is given by translation in charts, with deck group isomorphic to $\mathbb{Z}^d$.

We will always assume that the cover is locally compact.



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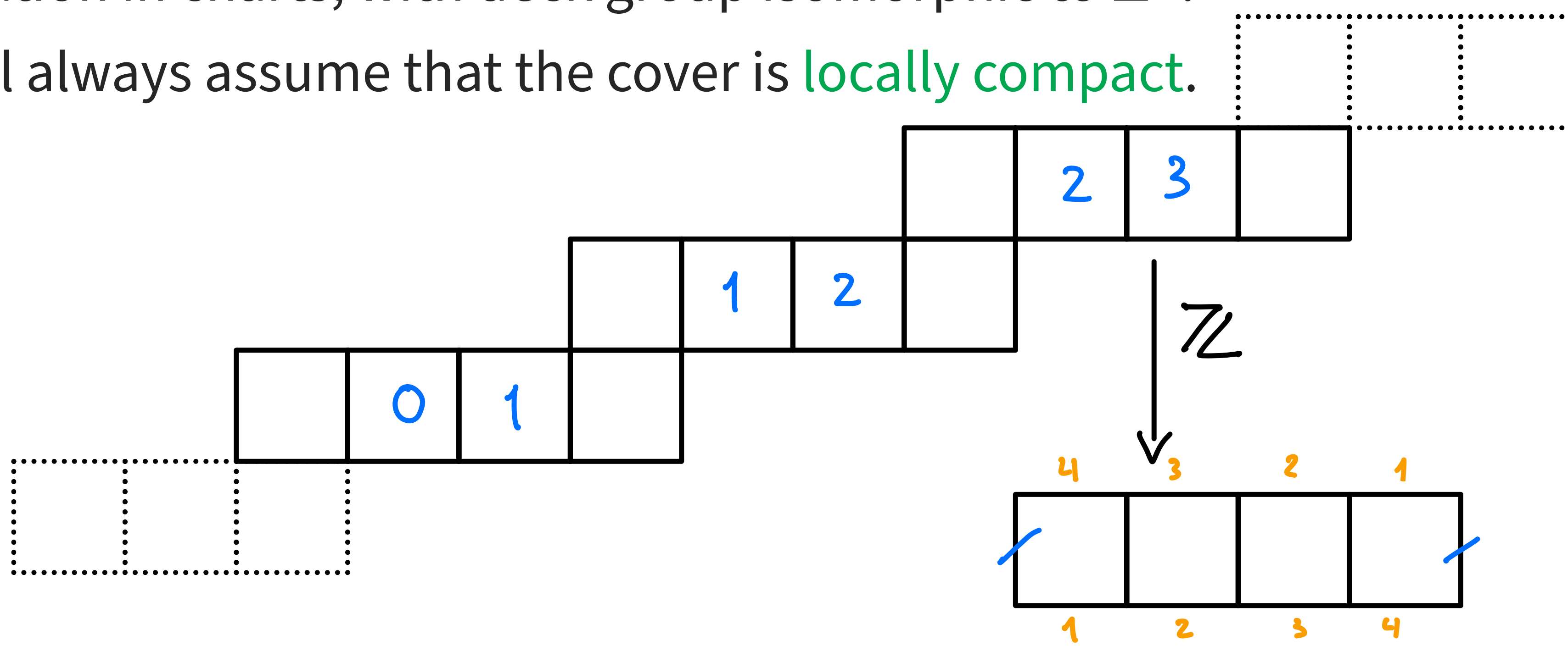
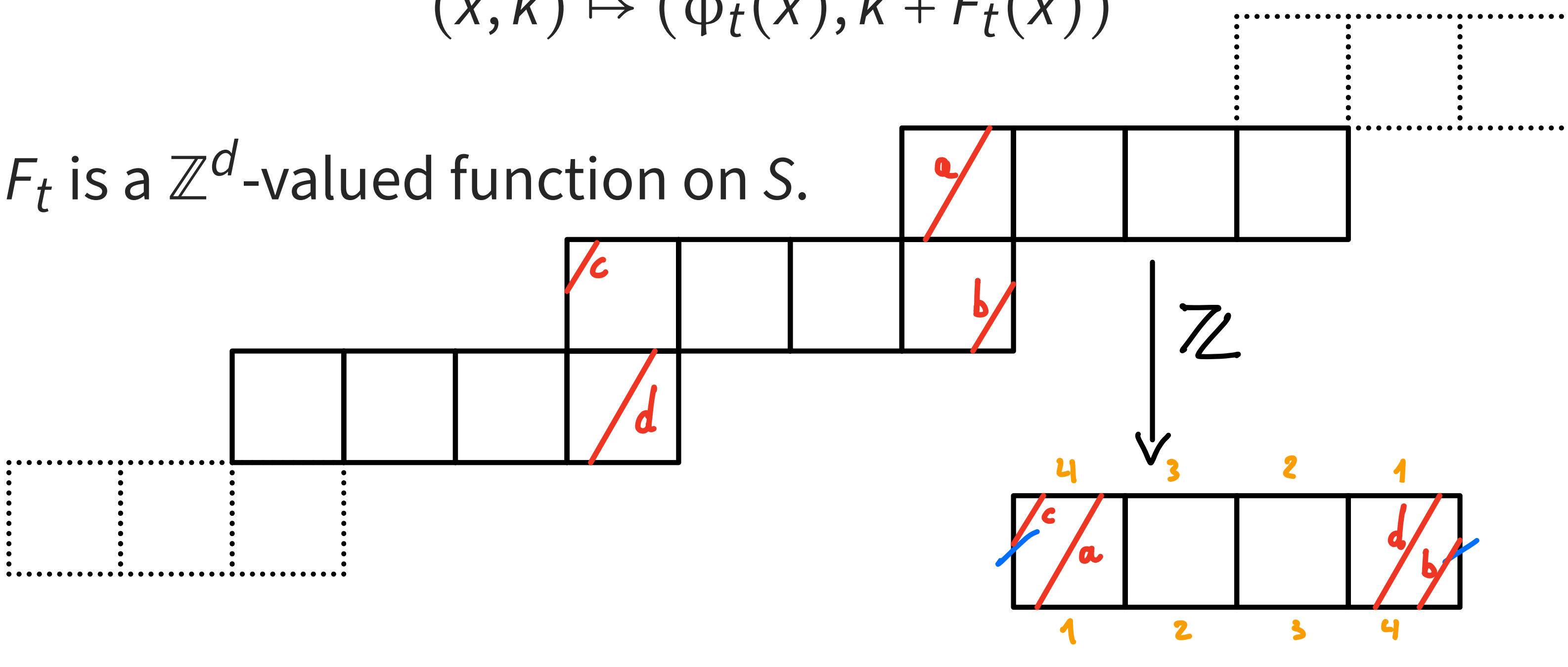


MAHARAM MEASURES

The geodesic flow $\tilde{\phi}_t$ on $\widetilde{S}$ can be dynamically described as a $\mathbb{Z}^d$-skew-product:

$$\tilde{\phi}_t \colon S \times \mathbb{Z}^d \to S \times \mathbb{Z}^d$$
$$(x, k) \mapsto (\phi_t(x), k + F_t(x))$$

where F_t is a $\mathbb{Z}^d$-valued function on S.



MAHARAM MEASURES

For every $\eta \in \mathbb{R}_+$, we can define a Maharam measure μ_η on $\widetilde{S}$ as:

$$d\mu_\eta(x, k) = e^{-k\eta}\, dm(x),$$

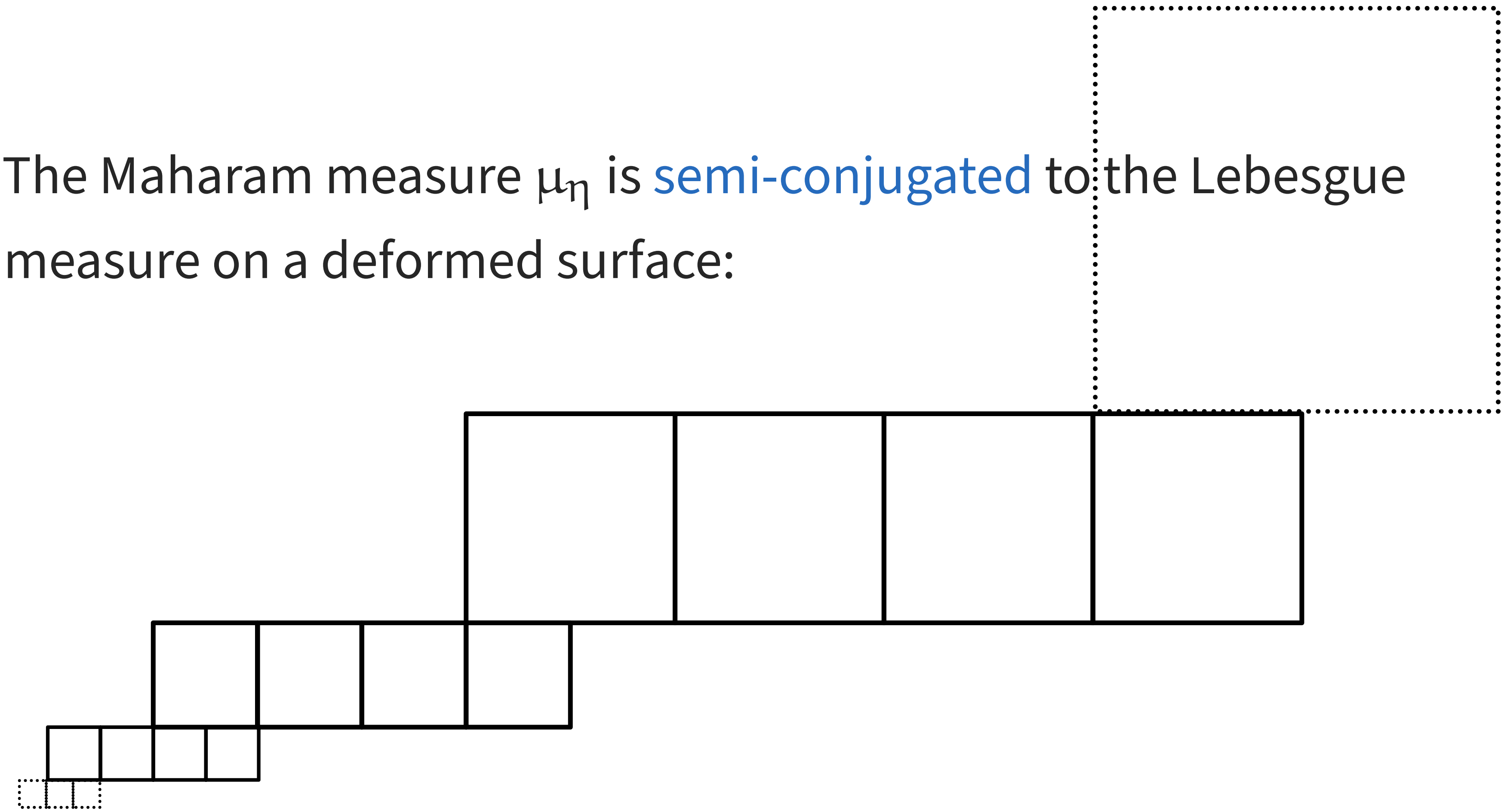
where dm is a measure on S which is quasi-invariant under the flow:

$$\frac{d\phi_{s*}m}{dm} = e^{\eta}.$$

These are the relevant locally finite Borel measures on $\widetilde{S}$.

MAHARAM MEASURES, GEOMETRICALLY

The Maharam measure μ_η is semi-conjugated to the Lebesgue measure on a deformed surface:



THE RESULT

THE SETTING

Let T be a pseudo-Anosov automorphism of a compact translation surface S, and ϕ_t the flow along the stable foliation of T.

Let $p: \widetilde{S} \to S$ a locally compact $\mathbb{Z}^d$-cover of S, and $\tilde{\phi}_t$ the flow defined by $p \circ \widetilde{\phi}_t = \phi_t \circ p$.

Assume that there exists a pseudo-Anosov automorphism $\widetilde{T}$ of $\widetilde{S}$ such that $p \circ \widetilde{T} = T \circ p$ and $\widetilde{T}$ commutes with deck transformations.

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THE RESULT

A distribution $\mathcal{D}$ is called Maharam if there exists a homomorphism $\chi\colon \mathrm{Deck} \to \mathbb{R}$ such that, for all $D \in \mathrm{Deck}$, and $f \in C_c^\infty(\widetilde{S})$,

$$\mathcal{D}(f \circ D) = e^{\chi(D)}\mathcal{D}(f).$$

Theorem (ACRT, '25+)
There exists an infinite countable set $\Xi = \{\mu_i \,:\, i \in \mathbb{N}\}$, described explicitly, of complex numbers μ_i, with $|\mu_i| \leq 1$ so that, for any $\mu_i \in \Xi$, there exists an uncountable family of invariant Maharam distributions $\mathcal{D}_{\mu_i,\chi}$, parametrized by $\chi \in \mathrm{Hom}(\mathrm{Deck}, \mathbb{R}) \simeq \mathbb{R}$.

We have $\mu_1 = 1$, which is associated to a Maharam measure.

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(SOME) PREVIOUS RESULTS

The geodesic flow on a translation surface can be seen has a suspension over a transformation called Intervale Exchange Transformations (IETs). $\mathbb{Z}^d$-skew products over IETs have been studied:

- Aaronson, Nakada, Sarig and Solomyak: if the IET is a rotation, the ergodic Radon measures are Maharam
- Hooper, Hubert and Weiss gave a geometric interpretation.
- Pollicott and Sharp: ANSS for pseudo-Anosov, under heavier hypothesis.
- Tumarkin extended Pollicott and Sharp to our setting.
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A CRASH COURSE ON ANISOTROPIC BANACH SPACES

POINTERS TO THE LITERATURE

Anisotropic Banach spaces have been introduced in *hyperbolic* dynamics by Blank, Keller and Liverani in 2001.
After that, there has been a flurry of activity, and many different constructions have been implemented by: Baladi, Faure, Gouëzel, Tsuji, Demers,...

Via renormalization, the have been used for *parabolic* dynamics by: Giulietti-Liverani, Faure-Gouëzel-Lanneau, Forni, Castorrini-Ravotti.

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THE IDEA

Define a Banach space $\mathcal{B}_{p,q}$ of distributions which are:

- smooth in the **horizontal** direction (we can take p derivatives)
- "dual smooth" in the **vertical** one: we can integrate them against functions which can be derived q times in the vertical direction.

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THE IDEA

We show that the transfer operator $\mathcal{L}_T(f) = w \cdot f \circ T^{-1}$ acts on $\mathcal{B}_{p,q}$.

We think of $\mathcal{B}_{p,q}$ as the **strong** space and of $\mathcal{B}_{p-1,q+1}$ as the **weak** one. We show that:

$$\mathcal{B}_{p,q} \hookrightarrow \mathcal{B}_{p-1,q+1}$$

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THE IDEA

The key technical lemma is a Lasota-Yorke inequality: there exists some $C \geq 0$ and $\sigma < 1$ such that for all $f \in \mathcal{B}_{p,q}$ and $n \geq 0$:

$$\|\mathcal{L}_T^n(f)\|_{p,q} \leq C\sigma^n \|f\|_{p,q} + C\|f\|_{p-1,q+1}.$$

A theorem of Hennion implies that the transfer operator $\mathcal{L}_T$ is quasi-compact on $\mathcal{B}_{p,q}$:

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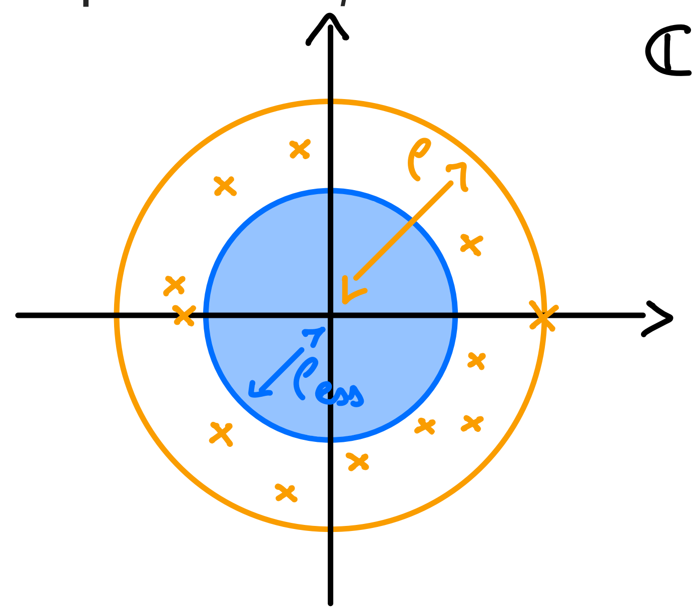

OUR IMPLEMENTATION OF THE GENERAL STRATEGY, IDEAS

A FOURIER DECOMPOSITION

Following [CR, '24] we can decompose smooth functions with compact support on $\widetilde{S}$ with a Fourier decomposition:

$$f(x) = \int_{\mathbb{T}} \pi_{r,\theta}(f)(x)\, d\theta,$$

for $r \in \mathbb{R}$, $\mathbb{T} = S^1$ and $\pi_{r,\theta}(f) \in C(S, r, \theta)$ changes by a character (depending on θ and D) when composed with a deck transformation D.

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Functions on $\widetilde{S}$ which are invariant under deck transformations can be identified with functions on S.

We show that the transfer operator on $C(S, r, \theta)$ can be identified with a twisted transfer operator on functions on S.

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THE BANACH SPACES

We use Sobolev spaces of distributions.

Given $v \in H^{p+2}(S)$ and $x \in S$ we define $\Gamma v(x) \in C^{-q}$ by:

$$\langle \Gamma v(x), u \rangle = \int_0^1 v \circ \phi_t(x) \cdot u(t)\, dt,$$

for $u \in C^q$.

The map $x \mapsto \Gamma v(x)$ is continuous and p times differentiable, so we define

$$\mathcal{B}_{p,q} = \mathrm{cl}_p(\langle \Gamma v : v \in H^{p+2}(S) \rangle),$$

where we take the closure in $C^p(S, C^{-q})$.

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THE TRANSFER OPERATOR

We fix a real function F, which is smooth and bounded on S (the Frobenius function), and for any $z \in \mathbb{C}$ we define

$$\mathcal{L}_{z,F} f(x) = (e^{zF} \cdot f) \circ T^{-1}(x).$$

We prove the Lasote-Yorke inequality for this operator.

We analyze the peripheral spectrum and show that, if $z \in \mathbb{R}$, then the maximal eigenvalue is simple and there is no other eigenvalue on the circle $|\rho(\mathcal{L}_{z,F})|$.

The associated eigenvector is actually a function in $L^1(S)$, and its dual is a measure (the Maharam measure).

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