Jumps, cusps and fractals in time-evolution models

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Bogotá, 3-6 June 2025

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

LONDON MATHEMATICAL SOCIETY EST. 1865

In this cursillo

In this cursillo

I will first show you the classical results

In this cursillo
I will first show you the classical results

... as in 1990's classical

... then

... then

I will show how these results change

... then

I will show how these results change

when we modify the equation
in different ways

I want to highlight

I want to highlight

the role of spectral theory in all this

... but,

... but,

we will use techniques from

Harmonic analysis
Number theory
Fractal geometry
and PDEs

Context.

Schrödinger's equation

Schrödinger on $\mathbb{T} \equiv (-\pi, \pi]$

regularity of the solution for fixed $t \in \mathbb{R}$

$$\begin{aligned} i\partial_t u(x,t) &= -\partial_x^2 u(x,t) && x \in \mathbb{T} \quad t \in \mathbb{R} \\ u(x,0) &= f(x) && x \in \mathbb{T} \end{aligned} \tag{A}$$

[1] P. J. Olver, *The American Mathematical Monthly* **117**, 599–610 (2010)

[2] I. Rodnianski, *Contemporary Mathematics* **255**, 181–188 (2000) or M. B. Erdoğan, N. Tzirakis, *Dispersive PDEs*, (Cambridge University Press, 2016)

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Theorem A. [Co/re-discovered since 1990s]

a)[1] For all $f \in \mathrm{L}^2(\mathbb{T})$ and co-prime $p, q \in \mathbb{N}$,

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} f\Big(x - 2\pi\frac{k}{q}\Big).$$

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$$\begin{aligned} i\partial_t u(x,t) &= -\partial_x^2 u(x,t) \qquad && x \in \mathbb{T} \quad t \in \mathbb{R} \\ u(x,0) &= f(x) && x \in \mathbb{T} \end{aligned} \tag{A}$$

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b)[2] For $f \in \mathrm{BV}(\mathbb{T})$, $u(\cdot, t) \in \mathrm{C}^{\frac{1}{2}-}(\mathbb{T})$ for a.e. $t \in \mathbb{R}$. And if additionally $f \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T}) \setminus \mathrm{H}^{\frac{1}{2}+}(\mathbb{T})$, then the graph of $\mathrm{Re}\, u(\cdot, t)$ has fractal dimension $\frac{3}{2}$ for a.e. $t \in \mathbb{R}$.

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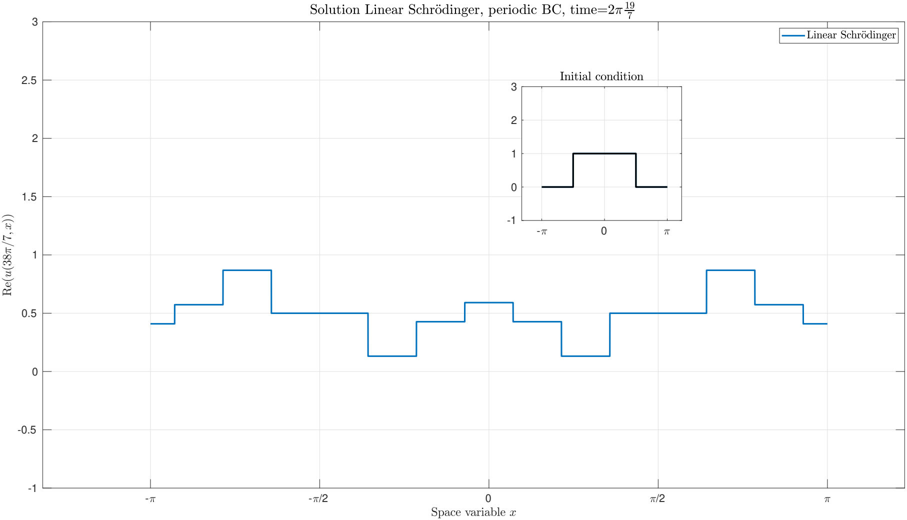
[1] P. J. Olver, *The American Mathematical Monthly* **117**, 599–610 (2010)

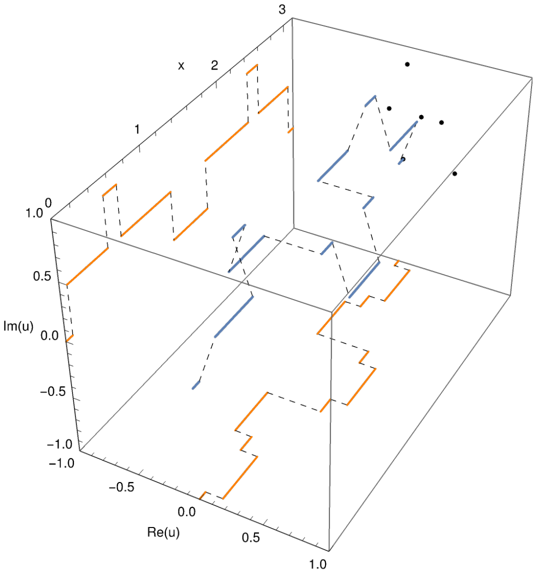
[2] I. Rodnianski, *Contemporary Mathematics* **255**, 181–188 (2000) or M. B. Erdoğan, N. Tzirakis, *Dispersive PDEs*, (Cambridge University Press, 2016)

Theorem A - a)

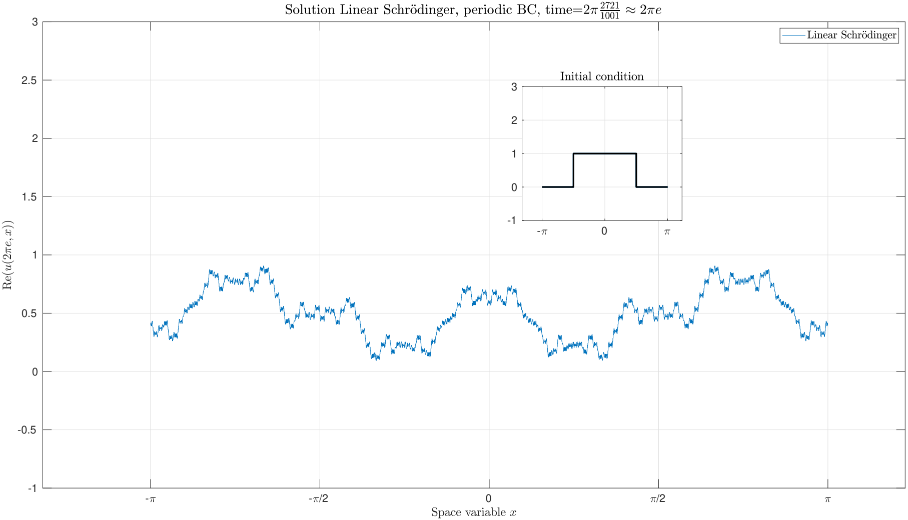


Solution Linear Schrödinger, periodic BC, time=$2\pi\frac{19}{7}$
Linear Schrödinger
Initial condition
Re$(u(38\pi/7, x))$
Space variable x
-π
-π/2
0
π/2
π
3
2.5
2
1.5
1
0.5
0
-0.5
-1

Theorem A - a)

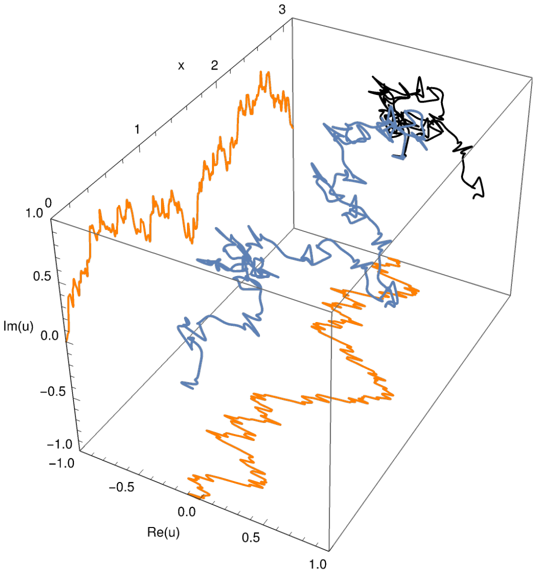


Theorem A - b)



Solution Linear Schrödinger, periodic BC, time=$2\pi\frac{2721}{1001} \approx 2\pi e$
Linear Schrödinger
Initial condition
3
2.5
2
1.5
1
0.5
0
-0.5
-1
Re($u(2\pi e, x)$)
-π
-π/2
0
π/2
π
Space variable x

Theorem A - b)



Changes.

Boundary conditions

Using symmetries

The effect of changing boundary conditions in $(0, \pi)$

$$
\begin{aligned}
&i\partial_t u(x,t) = -\partial_x^2 u(x,t)\\
&u(0,t) = u(\pi,t) = 0\\
&u(x,0) = f(x)
\end{aligned}
\quad \Rightarrow \quad u\left(x, 2\pi\frac{p}{q}\right) = \frac{1}{q}\sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} f^{\mathrm{o}}\left(x - 2\pi\frac{k}{q}\right)
$$

$$
\begin{aligned}
&i\partial_t u(x,t) = -\partial_x^2 u(x,t)\\
&u'(0,t) = u'(\pi,t) = 0\\
&u(x,0) = f(x)
\end{aligned}
\quad \Rightarrow \quad u\left(x, 2\pi\frac{p}{q}\right) = \frac{1}{q}\sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} f^{\mathrm{e}}\left(x - 2\pi\frac{k}{q}\right)
$$

Breaking the symmetry in the boundary conditions[4,5]

Shadows of revivals

Fix $b \in (0,1)$ and consider

$$\begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ &bu(a,t) = (1-b)u'(a,t) && a = 0,\pi \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in (0,\pi) \end{aligned} \tag{B}$$

[3] *Proc Royal Soc A* **477**, 20210241 (2021)

[4] P. J. Olver *et al.*, *Q Appl Math* **78**, 161–192 (2020)

[5] G. Farmakis, *Revivals in Time Evolution Problems*, (PhD Thesis HWU, 2022)

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Theorem B.[3]

Let $g(x) = \mathrm{e}^{\frac{b}{1-b}x}$. For p, $q \in \mathbb{N}$ co-prime, the solution

$$u\left(x, \frac{2\pi p}{q}\right) = c_0 \mathrm{e}^{-i\frac{b^2}{(1-b)^2}2\pi\frac{p}{q}} \langle f, g\rangle g(x) + \sum_{j=1}^{q-1} c_j(p,q) f^{\mathrm{e}}\left(x - \frac{\pi j}{q}\right) + \sum_{j=1}^{q-1} d_j(p,q)[g * (f^{\mathrm{o}} - f^{\mathrm{e}})]\left(x - \frac{\pi j}{q}\right)$$

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[3] *Proc Royal Soc A* **477**, 20210241 (2021)

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Theorem B

$b = 0.35 \qquad t = 2\pi \frac{p}{q}$

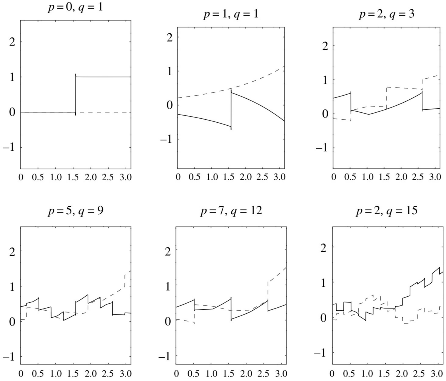

Figure 5. Real (solid) and imaginary (dashed) part of the solution of Robin's problem (5.1) with $b = 0.35$ at rational times $t = 2\pi p/q$.

Theorem B

$b = 0.6 \quad t = 2\pi \frac{p}{q}$

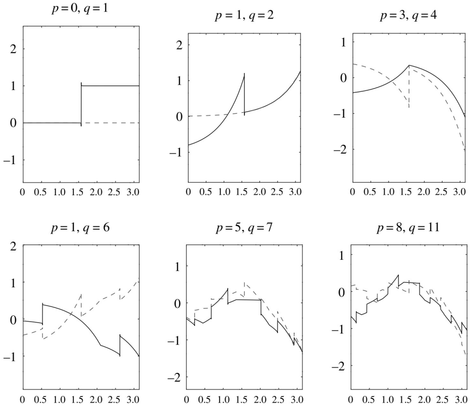

Figure 6. Real (solid) and imaginary (dashed) parts of the solution of Robin's problem (5.1) with $b = 0.6$ at rational times $t = 2\pi p/q$.

Theorem B

$b = 0.6 \quad t \notin 2\pi\mathbb{Q}$

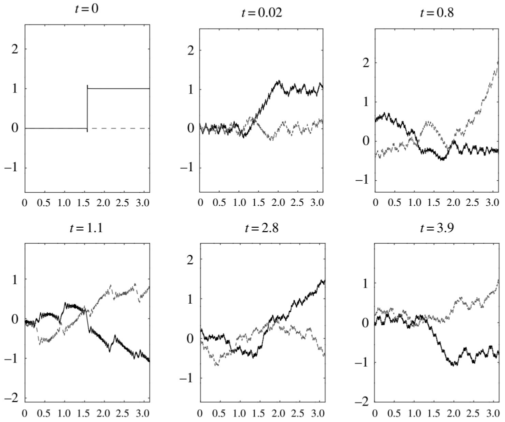

Figure 8. Real (solid) and imaginary (dashed) parts of the solution of Robin's problem (5.1) with $b = 0.6$ at generic times.

Adding a potential to the right hand side

Schrödinger with potential

complex $V \in \mathrm{H}^2$ with $\|V\|_\infty < \frac{3}{2}$ or real $V \in \mathrm{BV}$ with $\|V\|_\infty$ arbitrary

$$\begin{aligned} & i\partial_t u(x,t) = -\partial_x^2 u(x,t) + V(x)u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ & u(0,t) = u(\pi,t) = 0 && t \in \mathbb{R} \\ & u(x,0) = f(x) && x \in (0,\pi) \end{aligned} \tag{C}$$

[6] *Z Anal Awend* **43**, 401–416 (2024)

Schrödinger with potential

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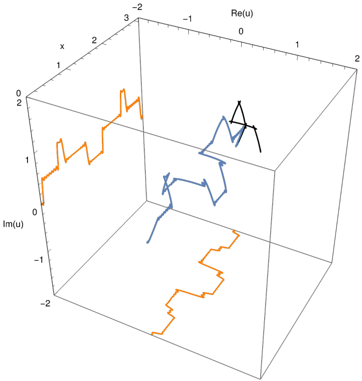
Theorem C.[6]
For all $t \neq 0$, there is $w(\cdot,t) \in \mathrm{C}^0(0,\pi)$ ensuring the following. If $p, q \in \mathbb{Z}$ are co-prime, then

$$u\Big(x, 2\pi\frac{p}{q}\Big) = w\Big(x, 2\pi\frac{p}{q}\Big) + \frac{\mathrm{e}^{-2\pi i \langle V\rangle \frac{p}{q}}}{q} \sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i \frac{mk - m^2 p}{q}} f^{\circ}\Big(x - 2\pi\frac{k}{q}\Big)$$

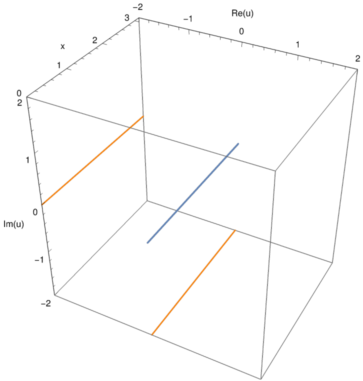
[6] *Z Anal Awend* **43**, 401–416 (2024)

Theorem C

$f(x) = \chi_{[\frac{3\pi}{8}, \frac{5\pi}{8}]}(x)$ and $V(x) = 2q\cos(2x)$. Approximation 100 modes for $q = 0$.



$u(x, t = \frac{2\pi}{5})$



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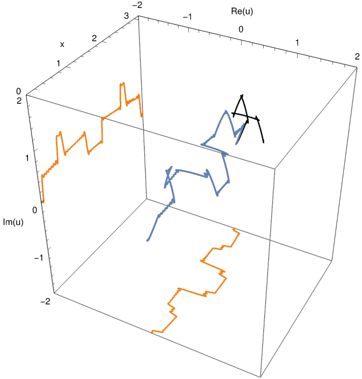
Theorem C

$f(x) = \chi_{[\frac{3\pi}{8},\frac{5\pi}{8}]}(x)$ and $V(x) = 2q\cos(2x)$. Approximation 100 modes for $q = \frac{i}{4}$.



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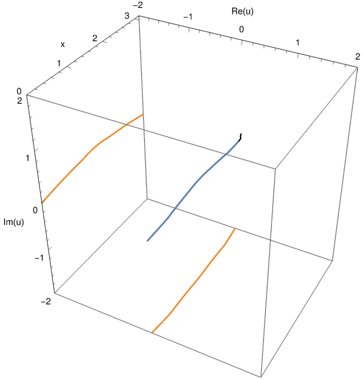

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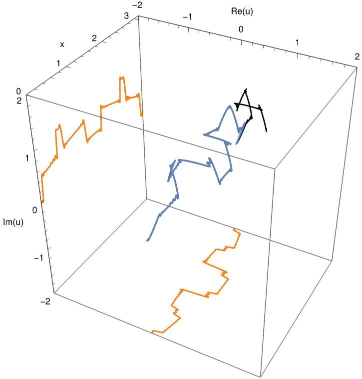
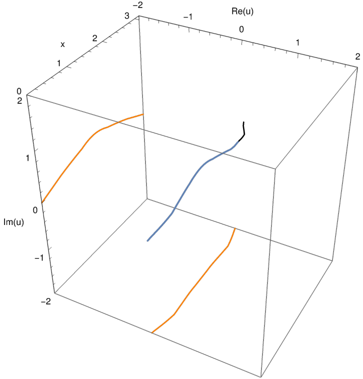


$u(x, t = \frac{2\pi}{5})$



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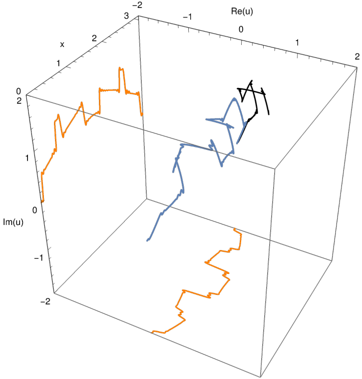
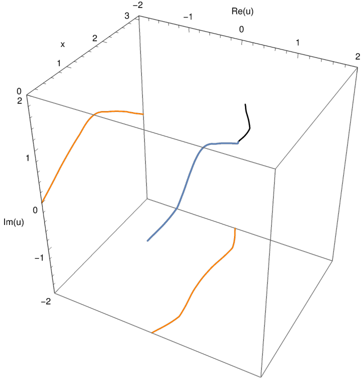


$u(x, t = \frac{2\pi}{5})$



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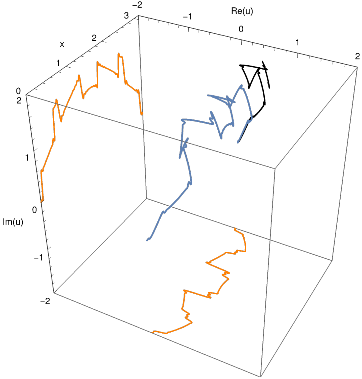

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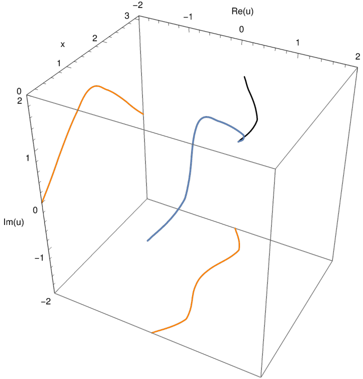

$w(x, t = \frac{2\pi}{5})$

Seemingly more drastic changes

The linear Benjamin-Ono equation[8]

$$\begin{aligned} &\partial_t u(x,t) = \mathcal{H}\partial_x^2 u(x,t) && x \in \mathbb{T} \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in \mathbb{T} \end{aligned} \tag{D}$$

$$\mathcal{H}g(x) = \frac{1}{2\pi}\ \mathrm{p.\,v.}\!\int_{-\pi}^{\pi} \cot\frac{x-y}{2}\, g(y)\, \mathrm{d}y.$$

[7] *arXiv:2501.01322*

[8] G. Chen, P. J. Olver, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **469**, 20120407 (2013) and *Stud Appl Math.* **147**, 1209 (2021)

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$$\mathcal{H}g(x) = \frac{1}{2\pi}\ \mathrm{p.\,v.}\int_{-\pi}^{\pi} \cot\frac{x-y}{2}\, g(y)\,\mathrm{d}y.$$

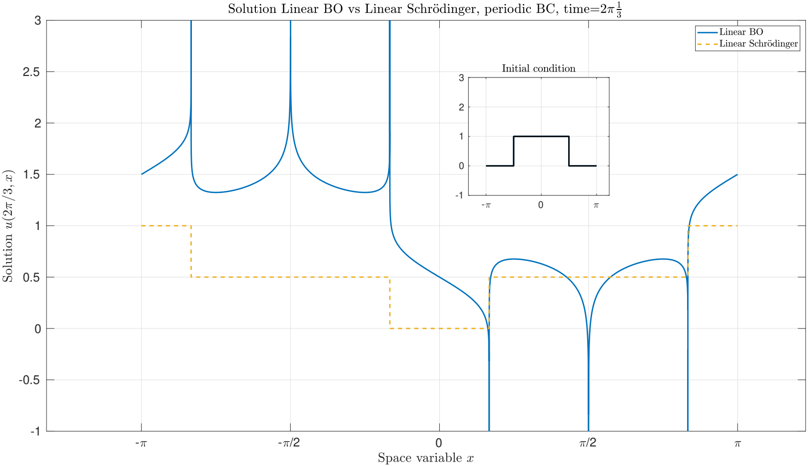
Theorem D.[7]
Let $f : \mathbb{T} \longrightarrow \mathbb{R}$. If $f \in \mathrm{L}^2(\mathbb{T})$ and p, $q \in \mathbb{N}$ co-prime,

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\,\mathrm{Re}\left[\sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i\frac{km+pm^2}{q}}(I + i\mathcal{H})f\Big(x - 2\pi\frac{k}{q}\Big)\right]$$

[7] *arXiv:2501.01322*

[8] G. Chen, P. J. Olver, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **469**, 20120407 (2013) and *Stud Appl Math.* **147**, 1209 (2021)

Solution at $t \in 2\pi\frac{p}{q}$



The complementary statement

Theorem D - b).[7]

Let $f : \mathbb{T} \longrightarrow \mathbb{R}$.

b1) If $f \in \mathrm{BV}(\mathbb{T})$, then

$$u(\cdot,t) \in \bigcup_{\alpha \in [0,\frac{1}{2})} \mathrm{C}^{\alpha}(\mathbb{T})$$

for almost all $t \in \mathbb{R}$.

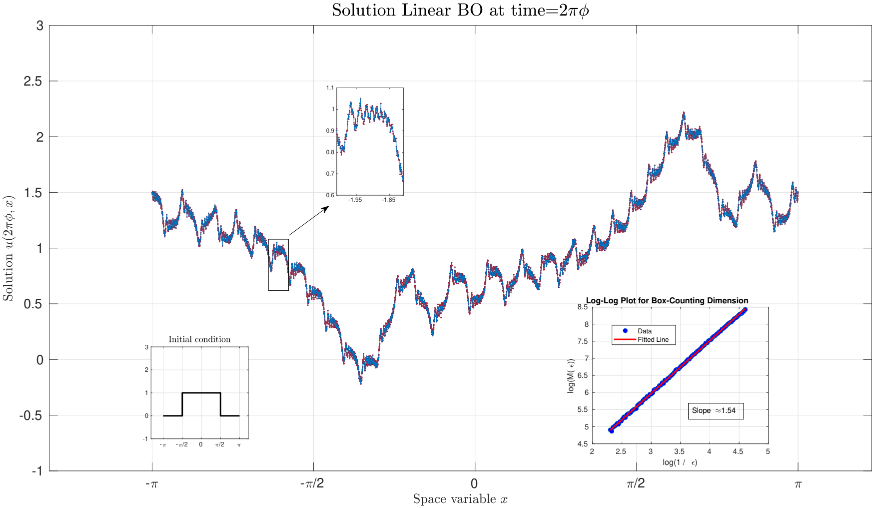
b2) If $f \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T}) \setminus \mathrm{H}^{r_0}(\mathbb{T})$ for some $r_0 \in [\frac{1}{2}, 1)$, then

$$u(\cdot,t) \notin \bigcup_{r > r_0} \mathrm{H}^{r}(\mathbb{T})$$

for almost all $t \in \mathbb{R}$.

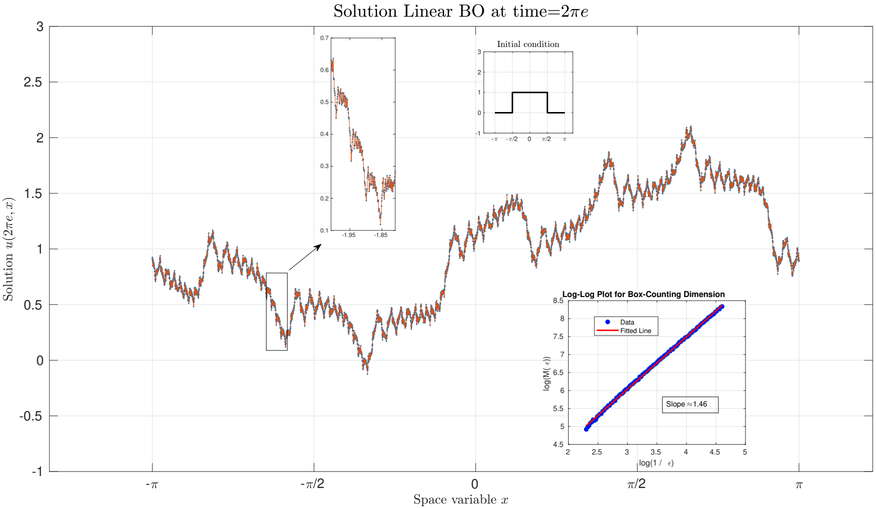
[7] *arXiv:2501.01322*

Fractality



Solution Linear BO at time=$2\pi\phi$
Solution $u(2\pi\phi, x)$
Space variable x
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-π
-π/2
0
π/2
π
Initial condition
-1.95
-1.85
Log-Log Plot for Box-Counting Dimension
Data
Fitted Line
log(M(ϵ))
log(1 / ϵ)
Slope ≈1.54

Fractality



Solution Linear BO at time=$2\pi e$
Solution $u(2\pi e, x)$
Space variable x
-π
-π/2
0
π/2
π
3
2.5
2
1.5
1
0.5
0
-0.5
-1
Initial condition
Log-Log Plot for Box-Counting Dimension
Data
Fitted Line
Slope ≈1.46
log(M(ϵ))
log(1 / ϵ)

Details.

Tuesday

Pocket proof of a)

Write again (A)

$$\begin{aligned} &\partial_t u = (-i)Hu & &H : \mathrm{H}^2(\mathbb{T}) \longrightarrow \mathrm{L}^2(\mathbb{T}) \\ &u|_{t=0} = f \in \mathrm{L}^2(\mathbb{T}) & &Hu = -\partial_x^2 u \end{aligned} \tag{1}$$

Pocket proof of a)

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For $t \in \mathbb{R}$ the solution is

$$u(x,t) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \mathrm{e}^{ij(\cdot)} \rangle \mathrm{e}^{ijx}$$

Pocket proof of a)

Write again (A)

$$
\begin{aligned}
&\partial_t u = (-i)Hu && H : \mathrm{H}^2(\mathbb{T}) \longrightarrow \mathrm{L}^2(\mathbb{T}) \\
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$$

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$$
u(x,t) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \mathrm{e}^{ij(\cdot)} \rangle \mathrm{e}^{ijx}
$$

For $\tilde{t} = \frac{2\pi p}{q}$ take $j \underset{q}{\equiv} m$ so that $\mathrm{e}^{ij^2 \tilde{t}} = \mathrm{e}^{im^2 \tilde{t}}$,

Pocket proof of a)

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$$u(x,t) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \mathrm{e}^{ij(\cdot)} \rangle \mathrm{e}^{ijx}$$

For $\tilde{t} = \frac{2\pi p}{q}$ take $j \underset{q}{\equiv} m$ so that $\mathrm{e}^{ij^2\tilde{t}} = \mathrm{e}^{im^2\tilde{t}}$, then

$$u(x,\tilde{t}) = \frac{1}{2\pi} \sum_{m=0}^{q-1} \mathrm{e}^{-im^2\tilde{t}} \sum_{\substack{j \in \mathbb{Z} \\ j \underset{q}{\equiv} m}} \langle f, \mathrm{e}^{ij(\cdot)} \rangle \mathrm{e}^{ijx}$$

Pocket proof of a)

For $\tilde{t} = \frac{2\pi p}{q}$ take $j \underset{q}{\equiv} m$ so that $\mathrm{e}^{ij^2\tilde{t}} = \mathrm{e}^{im^2\tilde{t}}$, then

$$u(x,\tilde{t}) = \frac{1}{2\pi} \sum_{m=0}^{q-1} \mathrm{e}^{-im^2\tilde{t}} \underbrace{\sum_{\substack{j\in\mathbb{Z} \\ j \underset{q}{\equiv} m}} \langle f, \mathrm{e}^{ij(\cdot)} \rangle \mathrm{e}^{ijx}}_{T}$$

Pocket proof of a)

For $\tilde{t} = \frac{2\pi p}{q}$ take $j \underset{q}{\equiv} m$ so that $e^{ij^2\tilde{t}} = e^{im^2\tilde{t}}$, then

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Then

$$u\left(x,\tilde{t}\right) = \frac{1}{q} \sum_{k,m=0}^{q-1} e^{2\pi i \frac{km - pm^2}{q}} f\left(x - 2\pi \frac{k}{q}\right)$$

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$$\sup_{x\in\mathbb{T}} \left| \sum_{j\in\mathbb{Z}} \chi\big(2^{-n}|j|\big)\mathrm{e}^{ijx - ij^2 t} \right| \leq C 2^{\frac{n}{2}(1+\delta)}$$

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$\therefore t \notin \mathcal{T} \Rightarrow H_t \in \mathrm{B}^{\alpha}_{\infty,\infty}(\mathbb{T}),\ \ \forall \alpha < \frac{1}{2}$

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Further comments Theorem A - b)

Second claim: $f \in \mathrm{H}^{\frac{1}{2}} \setminus \mathrm{H}^{\frac{1}{2}+}(\mathbb{T}) \Rightarrow \operatorname{Re} u(\cdot, t)$ has rough graph with dim$= \frac{3}{2}$

$$\dim_{\mathrm{B}} = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}(\varepsilon)}{\log \frac{1}{\varepsilon}}$$

$\mathcal{N}(\varepsilon) =$ min number of elements of a covering by balls diameter ε

[10] A. Deliu, B. Jawerth, *Constructive Approximation* **8** (1992)

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$\Rightarrow u(\cdot,t) \notin \mathrm{B}^{\frac{1}{2}+}_{1,\infty}(\mathbb{T}) \Rightarrow$[10] $\dim_{\mathrm{B}} \geq \frac{3}{2}$

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Friday

Pocket proof of Theorem D

The Hilbert transform

Let $g : \mathbb{T} \longrightarrow \mathbb{R}$ be such that $\hat{g}(0) = 0$. For $e_n(x) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{inx}$,

$$\mathcal{H}g(x) = i \sum_{n=1}^{\infty} \left[\hat{g}(-n) e_{-n}(x) - \hat{g}(n) e_n(x) \right] \qquad \text{and}$$

$$g(x) = \sum_{n=1}^{\infty} \left[\hat{g}(-n) e_{-n}(x) + \hat{g}(n) e_n(x) \right]$$

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$\Rightarrow$

$$g(x) = \overline{\left(\sum_{n=1}^{\infty} \hat{g}(n)e_n(x)\right)} + \sum_{n=1}^{\infty} \hat{g}(n)e_n(x) = \mathrm{Re}\left[(I + i\mathcal{H})g(x)\right]$$

Pocket proof of Theorem D

The RHS is a diagonal operator

$$\mathcal{H}\partial_x^2\phi = \lambda\phi \quad \Longleftrightarrow \quad \begin{aligned} &\phi(x) = Ce_k(x) \\ &\lambda = \lambda_k = \begin{cases} -ik^2 & k \leq 0 \\ ik^2 & k \geq 0 \end{cases} \end{aligned}$$

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$$\begin{aligned} u(x,t) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{\lambda_k t}\hat{f}(k)e_k(x) \\ &= \sum_{k=1}^{\infty}\left[\mathrm{e}^{-ik^2 t}\hat{f}(-k)e_{-k}(x) + \mathrm{e}^{ik^2 t}\hat{f}(k)e_k(x)\right] \end{aligned}$$

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Solution via Hilbert transform of Schrödinger

WOLG take $f : \mathbb{T} \longrightarrow \mathbb{R}$ such that $\hat{f}(0) = 0$

Pocket proof of Theorem D

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WOLG take $f : \mathbb{T} \longrightarrow \mathbb{R}$ such that $\hat{f}(0) = 0$
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$$\begin{aligned} &\partial_t v(x,t) = (-i)\partial_x^2 v(x,t) && x \in \mathbb{T} \quad t \in \mathbb{R} \\ &v(x,0) = f(x) && x \in \mathbb{T} \end{aligned} \tag{1}$$

... note the minus sign.

Pocket proof of Theorem D

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$$\begin{aligned} u(x,t) &= \overline{\left(\sum_{k=1}^{\infty} \mathrm{e}^{ik^2 t} \hat{f}(k) e_k(x)\right)} + \sum_{k=1}^{\infty} \mathrm{e}^{ik^2 t} \hat{f}(k) e_k(x) \\ &= \frac{\overline{(I + i\mathcal{H})v(x,t)}}{2} + \frac{(I + i\mathcal{H})v(x,t)}{2} \\ &= \mathrm{Re}\left[(I + i\mathcal{H})v(x,t)\right] \end{aligned}$$

Pocket proof of Theorem D

Wrap up

From Theorem A - a) changing sign of p in time variable $\Rightarrow$

$$v\left(x, \frac{\pi p}{q}\right) = \sum_{j=1}^{q-1} \widetilde{c}_j(p,q) f\left(x - \frac{\pi j}{q}\right)$$

Pocket proof of Theorem D

Wrap up

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$\Rightarrow$

$$i\mathcal{H}v\left(x, \frac{\pi p}{q}\right) = \sum_{j=1}^{q-1} i\widetilde{c_j}(p,q) \mathcal{H}f\left(x - \frac{\pi j}{q}\right)$$

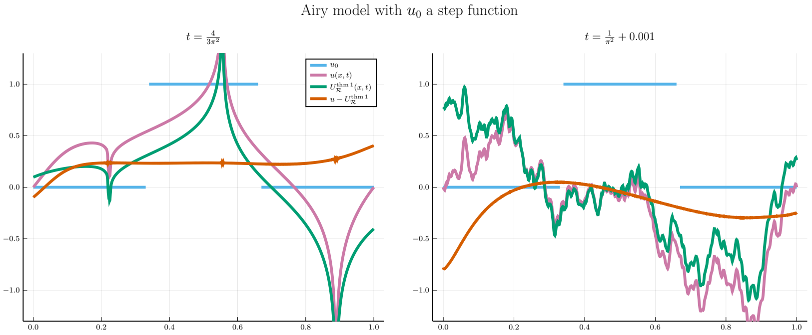
$\Rightarrow$ Theorem D.

Perspective

Airy[11]

Crucial: Dirichlet boundary conditions

$$
\begin{aligned}
&u_t(x,t) + u_{xxx}(x,t) = 0 \\
&u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = u_x(0,t) \\
&u(x,0) = u_0(x)
\end{aligned}
$$

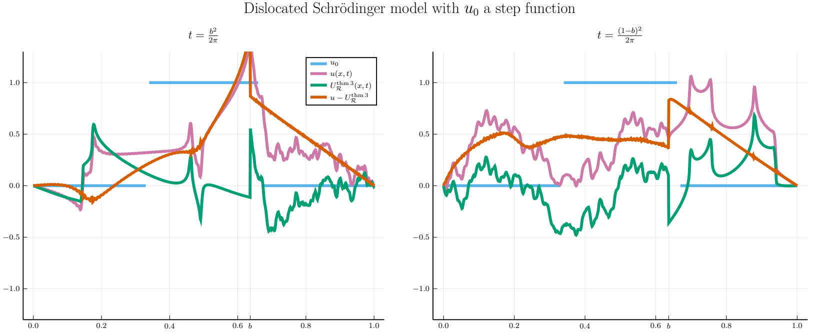


[11] *arXiv: 2403.01117*

Dislocated Laplacian[11]

Fix $b \in (0,1)$

$$u_t(x,t) + iu_{xx}(x,t) = 0 \quad x \in (0,b)$$
$$u_t(x,t) - iu_{xx}(x,t) = 0 \quad x \in (b,1)$$
$$u(0,t) = u(1,t) = 0$$
$$u(b^+,t) = u(b^-,t), \quad u_x(b^+,t) = -u_x(b^-,t)$$
$$u(x,0) = u_0(x)$$



[11] *arXiv: 2403.01117*

End of cursillo