JUMPS, CUSPS AND FRACTALS, IN TIME-EVOLUTION PDES

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1. Preliminaries

1.1. **Conventions and notation.** Let $e_n(x) = \frac{1}{\sqrt{2\pi}} e(nx)$. Here and everywhere below, we write the Fourier coefficients of a periodic distribution F, see [18, Chapter 11] or [9, Section 9.3], with one of the usual scalings on $\mathbb{T} = (-\pi, \pi]$, as

$$\widehat{F}(n) = \frac{1}{\sqrt{2\pi}} \langle F, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} F(y) \,\mathrm{d}y.$$

This choice makes either series in the expression

$$F(x) \sim \sum_{n\in\mathbb{Z}} \langle F, e_n \rangle e_n(x) = \sum_{n\in\mathbb{Z}} \widehat{F}(n) \mathrm{e}^{inx},$$

convenient for long calculations.

Recall that $\{e_n\}_{n\in\mathbb{Z}} \subset \mathrm{L}^2(\mathbb{T})$ is an orthonormal basis of eigenfunctions for the Laplacian,

$$-\partial_x^2 : \mathrm{H}^2(\mathbb{T}) \longrightarrow \mathrm{L}^2(\mathbb{T}).$$

Indeed, for all $n \in \mathbb{Z}$,

$$-\partial_x^2 e_n = n^2 e_n.$$

The following function spaces will be considered throughout. The definitions and relevant properties, are given in the next subsection and in the text. In two occasions we will use $(0, \pi]$ not identifying 0 with π, instead of $\mathbb{T}$, as the mapping of the definitions and properties is obvious we omit the details.

- $\mathrm{BV}(\mathbb{T})$ functions of *bounded variation*,
- $\mathrm{AC}(\mathbb{T})$ *absolutely continuous* functions,
- $\mathrm{C}^\alpha(\mathbb{T})$ *Hölder continuous* functions of regularity $\alpha \in (0, 1)$,
- $\mathrm{H}^\alpha(\mathbb{T})$ functions in the L^2 *Sobolev space* with regularity $\alpha \geq 0$,
- $\mathrm{B}_p^\alpha(\mathbb{T})$ distributions in the ℓ^∞-L^p *Besov space* with regularity $\alpha \in \mathbb{R}$ and $1 \leq p \leq \infty$.

1.2. **Connections and properties of the classical function spaces.** Recall the classical definitions of $\mathrm{BV}(\mathbb{T})$ and $\mathrm{AC}(\mathbb{T})$, given in standard analysis monographs such as [15, p.9 and p.47]. We know that

$$f \in \mathrm{AC}(\mathbb{T}) \qquad \Longleftrightarrow \qquad f' \in \mathrm{L}^1(\mathbb{T}).$$

We also know that $f \in \mathrm{BV}(\mathbb{T})$ if and only if f' is a finite Radon measure on the Borel σ-algebra. Moreover, if $f \in \mathrm{BV}(\mathbb{T})$, then

$$f = f_{\mathrm{ac}} + f_{\mathrm{s}},$$

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for $f_{\mathrm{ac}} \in \mathrm{AC}(\mathbb{T})$ with $f'_{\mathrm{ac}} \in \mathrm{L}^\infty(\mathbb{T})$, and f'_{s} singular with support of Lebesgue measure 0. For the proofs of these statements, see the two theorems on [15, p.53].

Problem 1. *Let $f \in \mathrm{BV}(\mathbb{T})$. Show that there exists a constant, such that*

$$|\widehat{f}(n)| \leq \frac{c}{|n|}$$

for all $n \neq 0$.

Solution. Use that

$$\widehat{f}(n) = \frac{\widehat{f'}(n)}{n}$$

and the representation of a bounded variation function given above. □

Let $f : \mathbb{T} \longrightarrow \mathbb{C}$ and $\alpha \geq 0$. We will write $f \in \mathrm{H}^\alpha(\mathbb{T})$, whenever

$$\sum_{n \in \mathbb{Z}} (1+n^2)^\alpha \left| \widehat{f}(n) \right|^2 < \infty.$$

It is easy to see that

$$f' \in \mathrm{H}^\alpha(\mathbb{T}) \iff f \in \mathrm{H}^{\alpha+1}(\mathbb{T}).$$

Let $f : \mathbb{T} \longrightarrow \mathbb{C}$ and $\alpha \in (0,1)$. We will write $f \in \mathrm{C}^\alpha(\mathbb{T})$, whenever

$$\sup_{x \in \mathbb{T}} |f(x)| + \sup_{\substack{x \in \mathbb{T} \\ h \neq 0}} \frac{|f(x+h) - f(x)|}{|h|^\alpha} < \infty.$$

This expression defines a norm in $\mathrm{C}^\alpha(\mathbb{T})$ and makes it a Banach space, but we will neither use nor prove this fact. See [13, §11.3].

Problem 2. *Let $f : (-\pi, \pi] \longrightarrow \mathbb{R}$ be given by*

$$f(x) = |x| \log \frac{1}{|x|}.$$

Show that $f \in \mathrm{C}^\alpha(\mathbb{T})$ for all $0 < \alpha < 1$. Show that f is not a Lipschitz function. Is $f \in \mathrm{AC}(\mathbb{T})$?

Solution. For the first and second parts, use that for fixed $0 < b < 1$,

$$|x| < |x| \log \frac{1}{|x|} < |x|^b$$

in a neighbourhood of $x = 0$. For the third part, note that

$$f'(x) = \operatorname{sgn}(x)[\log \frac{1}{|x|} - 1].$$

Thus $f'(x) \in \mathrm{L}^1(\mathbb{T})$. Therefore, indeed $f \in \mathrm{AC}(\mathbb{T})$. □

Problem 3. *Show that $\mathrm{H}^1(\mathbb{T}) \subseteq \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$. Hint: use the Cauchy-Schwarz inequality. Is $\mathrm{H}^1(\mathbb{T}) = \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$?*

Solution. Let $f \in \mathrm{H}^1(\mathbb{T})$. Then $f' \in \mathrm{L}^2(\mathbb{T})$. Hence $f' \in \mathrm{L}^1(\mathbb{T})$. Then $f \in \mathrm{AC}(\mathbb{T})$. Let $g = f'$. Then,

$$\begin{aligned}|f(x) - f(y)| &= \left|\int_y^x g(z)\,\mathrm{d}z\right| \\ &\le \left(\int_y^x \mathrm{d}z\right)^{\frac{1}{2}} \left(\int_y^x |g(z)|^2 \mathrm{d}z\right)^{\frac{1}{2}} \\ &\le |x-y|^{\frac{1}{2}} \|g\|_{\mathrm{L}^2}.\end{aligned}$$

This ensures that $f \in \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$.

The answer to the second question is "no". For example, $|\cdot|^{\frac{1}{2}} \notin \mathrm{H}^1(\mathbb{T})$ but $|\cdot| \in \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$. □

2. The Schrödinger equation

The study of different modifications of the following Schrödinger's equation, is the main goal of this cursillo. We are interested in the regularity of the solution. Consider

$$\text{(A)} \qquad \begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x \in \mathbb{T} \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in \mathbb{T}. \end{aligned}$$

It is routine to seen that, for $t \in \mathbb{R}$,

$$u(x,t) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{-ij^2 t} \widehat{f}(j) e_j(x) = \sum_{j=-\infty}^{\infty} \mathrm{e}^{-ij^2 t + ijx} \widehat{f}(j).$$

Therefore, the solution does not change its Sobolev norm for any $t \in \mathbb{R}$.

Problem 4. *Let $\alpha \ge 0$. Show that*

$$\|u(\cdot,t)\|_{\mathrm{H}^\alpha(\mathbb{T})} = \|f\|_{\mathrm{H}^\alpha(\mathbb{T})}$$

for all $t \in \mathbb{R}$.

Solution. Use Parseval's identity and the fact that $|\mathrm{e}^{-ij^2 t}| = 1$ for all $t \in \mathbb{R}$. □

Quite remarkably, in this quarter of a Century, it has been discovered that the regularity properties of the solution, beyond the Sobolev scale, are intimately connected with the best approximation of t, in continued fractions. The next theorem illustrates this in a concrete manner. It assembles results first formulated in [11] and [16]. Some of the original proofs were simplified in [17], [14] and [4].

Theorem A. *Let $f \in \mathrm{L}^2(\mathbb{T})$. Let u be the solution to* (A).

a) *If $p, q \in \mathbb{N}$ are co-prime, then*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} f\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset $\mathcal{K} \subset \mathbb{R}$ with complement of measure 0, such that the following property is valid. If $f \in \mathrm{BV}(\mathbb{T})$, then*

$$u(\cdot,t) \in \bigcup_{\epsilon>0} \mathrm{C}^{\frac{1}{2}-\epsilon}(\mathbb{T})$$

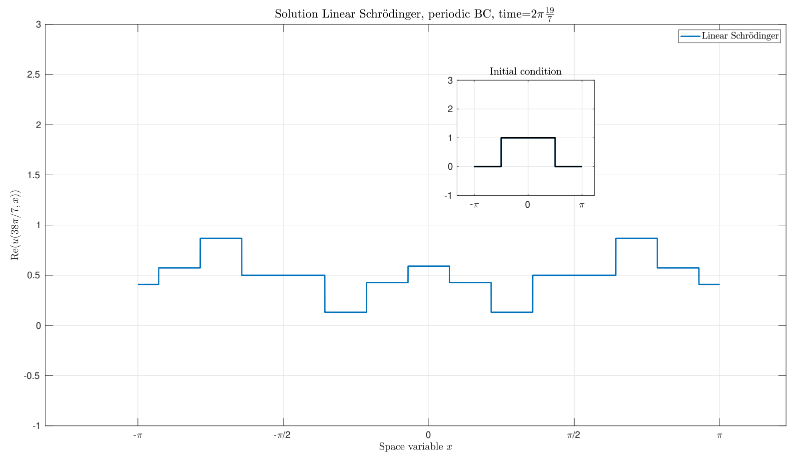

FIGURE 1. Revivals: solution of (A) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t = 2\pi\frac{19}{7}$.

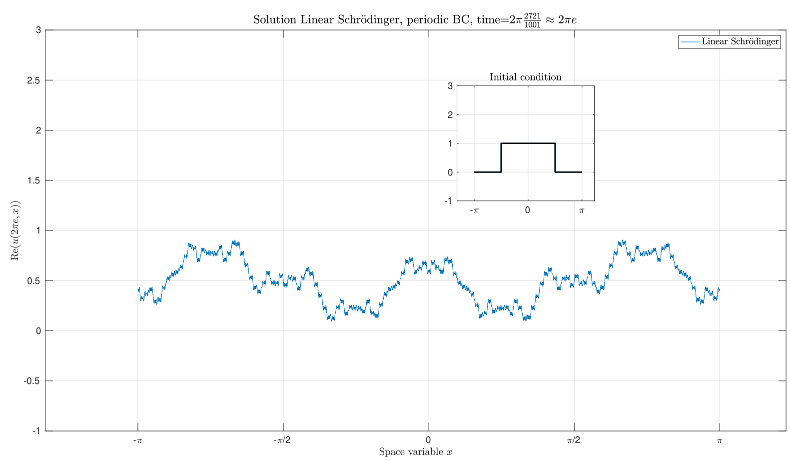

FIGURE 2. Fractality: solution of (A) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t \approx 2\pi\mathrm{e}$.

for all $t \in \mathcal{K}$. Moreover, if additionally $f \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T})$ but

$$f \notin \bigcap_{\epsilon>0} \mathrm{H}^{\frac{1}{2}+\epsilon}(\mathbb{T}), \tag{1}$$

then the graph of $\mathrm{Re}\, u(\cdot,t)$ has fractal dimension $\frac{3}{2}$ for almost all $t \in \mathcal{K}$.

This theorem prescribes that the regularity of the solution in the space variable, changes significantly with time, when seen from a perspective different than that of the Sobolev norm. For example, if f is a step function, the solution is a finite linear combination of step functions whenever $\frac{t}{2\pi} \in \mathbb{Q}$, while it is continuous but "rough" for almost every $\frac{t}{2\pi} \notin \mathbb{Q}$. We can call this a *revivals/fractality dichotomy*, and illustrate this in figures 1 and 2.

2.1. **Proof of Theorem A-a).** The proof of the first statement in Theorem A is as follows.

Let $t = 2\pi\frac{p}{q}$. Take $j \underset{q}{\equiv} m$ so that $\mathrm{e}^{ij^2t} = \mathrm{e}^{im^2t}$. Then,

$$u(x,t) = \frac{1}{2\pi}\sum_{m=0}^{q-1} \mathrm{e}^{-im^2t} \sum_{\substack{j\in\mathbb{Z}\\ j\underset{q}{\equiv} m}} \langle f, \mathrm{e}^{ij(\cdot)}\rangle \mathrm{e}^{ijx}$$

Now,

$$\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i(m-j)\frac{k}{q}} = \begin{cases} q & j \underset{q}{\equiv} m \\ 0 & j \underset{q}{\not\equiv} m. \end{cases} \tag{2}$$

Thus,

$$\sum_{\substack{j\in\mathbb{Z}\\ j\underset{q}{\equiv} m}} \langle f, \mathrm{e}^{ij(\cdot)}\rangle \mathrm{e}^{ijx} = \frac{1}{q}\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i m\frac{k}{q}} \sum_{j\in\mathbb{Z}} \mathrm{e}^{-2\pi i\frac{k}{q}j} \langle f, \mathrm{e}^{ij(\cdot)}\rangle \mathrm{e}^{ijx}$$

$$= \frac{1}{q}\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i m\frac{k}{q}} \sum_{j\in\mathbb{Z}} \Big\langle f\Big(\cdot - \frac{2\pi k}{q}\Big), \mathrm{e}^{ij(\cdot)}\Big\rangle \mathrm{e}^{ijx}.$$

From this, the statement of Theorem A-a) follows.

Problem 5. *Give the proof of the identity* (2)*.*

Solution. If $j \underset{q}{\equiv} m$, then $m - j = nq$ for some $n \in \mathbb{N}$ and hence

$$\sum_{k=0}^{q-1} e^{2\pi i(m-j)\frac{k}{q}} = \sum_{k=0}^{q-1} 1 = q.$$

Otherwise, $j = nq + r$ for some $r \in \{1, \ldots, q-1\}$, so

$$\sum_{k=0}^{q-1} e^{2\pi i(m-j)\frac{k}{q}} = \sum_{k=0}^{q-1} e^{2\pi i r\frac{k}{q}}.$$

Suppose, for simplicity, that $r = 1$ so we want to check that

$$S = \sum_{k=0}^{q-1} e^{2\pi i\frac{k}{q}} = 0.$$

Note that this is the sum of the roots of unit. For the variable $z \in \mathbb{C}$, they are the roots of the polynomial

$$\prod_{k=0}^{q-1} (z - e^{2\pi i\frac{k}{q}}) = z^q - 1.$$

By developing the product on the left hand side, the coefficient for z^{q-1} is $-S$. But the right hand side has no such power of z, so $S = 0$. A similar trick applies to $r > 1$. $\square$

Problem 6. *Let $f \in \mathrm{L}^2(0,\pi)$. Find the (unique) solution to*

$$\begin{aligned} &\partial_t u(x,t) = i\partial_x^2 u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ &\partial_x u(0,t) = \partial_x u(\pi,t) = 0 && t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in (0,\pi). \end{aligned} \tag{3}$$

Give your solution in terms of the Fourier series of f. Now, set

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi]. \end{cases}$$

Find $\tilde{t} > 0$ such that $u(\cdot, \tilde{t})$ has a discontinuity at $x = \frac{\pi}{8}$. Hint: the second part is tougher than you think.

Solution. Any $f \in \mathrm{L}^2(0,\pi)$ can be expanded as

$$f(x) = \sum_{n=0}^{\infty} \widetilde{f}(n) \cos(nx) \qquad x \in (0,\pi)$$

where

$$\widetilde{f}(0) = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \qquad \widetilde{f}(n) = \frac{2}{\pi}\int_0^{\pi} \cos(nx) f(x)\,\mathrm{d}x.$$

Then,

$$u(x,t) = \sum_{n=0}^{\infty} e^{-in^2 t}\widetilde{f}(n)\cos(nx).$$

Consider now the second part of the question.

Step 1. We derive a version of Theorem A-a). From the proof and the previous part, we start with

$$u(x,t) = \sum_{n=0}^{\infty} e^{-in^2 t}\widetilde{f}(n)\cos(nx)$$

and transform into exponential form. Let f_{e} denote the 2π-periodic extension of

$$f_{\mathrm{e}}(x) = \begin{cases} f(x) & x \in [0,\pi] \\ f(-x) & x \in (-\pi, 0) \end{cases}$$

By expressing the cosine in exponential form, doubling the integral of the Fourier coefficients and gathering terms, we get

$$\widetilde{f}(n) = \frac{1}{\pi}\left(\int_{-\pi}^{0} + \int_0^{\pi}\right) f_{\mathrm{e}}(x) e^{-inx}\,\mathrm{d}x = \frac{1}{\pi}\langle f_{\mathrm{e}}, e^{in(\cdot)}\rangle.$$

This is now in the notation of the proof of the theorem and it has the same expression, except that it involves the even extension of the initial data.

Therefore, using exactly the same steps in that poof, we obtain,

Theorem A'-a). *Let $f \in \mathrm{L}^2(0,\pi)$ and consider the solution to the equation* (3)*. Then, for $\tilde{t} = \frac{2\pi p}{q}$ where p and q are co-primes,*

$$u(x,\tilde{t}) = \frac{1}{q}\sum_{k,m=0}^{q-1} e^{2\pi i \frac{km - pm^2}{q}} f_{\mathrm{e}}(x - 2\pi k/q).$$

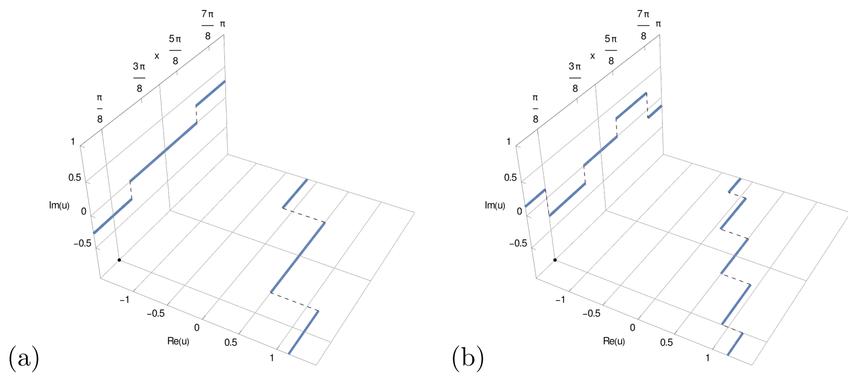

FIGURE 3. (a) $q = 16$ and $p = 1$, (b) $q = 32$ and $p = 1$.

Step 2. With this result at hand, let us now find $\tilde{t}$ such that the RHS of the above revival expression has a discontinuity at $\tilde{t} = \frac{\pi}{8}$. First, note that

$$f_{\mathrm{e}}(x) = \operatorname{sgn}(\cos(x)).$$

Then we need an educated guess.

Taking $q = 16$ gives no discontinuity at $\tilde{x} = \frac{\pi}{8}$ despite of having the correct coefficients to play around with $k = 5$ and $k = 13$ in the above formula. See Figure 3.

Taking $q = 32$ and $p = 1$, gives $\tilde{t} = \frac{\pi}{16}$. Now,

$$\frac{\pi}{8} - \frac{k\pi}{16} = \frac{\pi}{2} + 2n\pi \qquad \Longleftrightarrow \qquad k \equiv_{32} -6 \equiv_{32} 26$$

and

$$\frac{\pi}{8} - \frac{k\pi}{16} = -\frac{\pi}{2} + 2n\pi \qquad \Longleftrightarrow \qquad k \equiv_{32} 10.$$

These are the only contributing terms in the revival summation that give a jump at $\pi/8$ in the case $q = 32$. One is a jump up, the other a jump down. We need to check that these do not cancel out. Octave gives

```
octave:1> m=0:31;
octave:2> sum(exp(i*pi*(26*m-m.^2)/16))
ans =  4.4446 + 6.6518i
octave:3> sum(exp(i*pi*(10*m-m.^2)/16))
ans = -4.4446 - 6.6518i
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Hence, at $k = 26$,

$$A = \sum_{m=0}^{31} e^{-\pi \frac{26m - m^2}{16}} \approx 4.4446 + 6.6518i$$

and, at $k = 10$,

$$B = \sum_{m=0}^{31} e^{-\pi \frac{10m - m^2}{16}} \approx -4.4446 - 6.6518i.$$

Thus, since A is safely away from B and they are both safely away from 0, we know that there is a discontinuity at $x = \frac{\pi}{8}$. See Figure 3. $\square$

2.2. **Besov spaces.** The Fourier coefficients of Hölder continuous functions have a specific behaviour, which can be seen through the scale of Besov spaces. The latter give a more refined criterion for the regularity of a function than the scale of Sobolev spaces. To simplify our notation, we will write $\mathrm{B}^\alpha_p(\mathbb{T})$, for $\alpha \in \mathbb{R}$ and $p \geq 1$, to denote what is normally written as $\mathrm{B}^\alpha_{p,\infty}(\mathbb{T})$, precisely defined as follows.

Let $\chi : \mathbb{R} \longrightarrow [0,1]$ be a C^∞ function, such that

$$\operatorname{supp} \chi = [2^{-1}, 2]$$

and

$$\sum_{j=0}^{\infty} \chi(2^{-j}\xi) = 1$$

for all $\xi \geq 1$. Let the *Littlewood-Paley projections* of a periodic distribution on $\mathbb{T}$, be given by

$$(K_j f)(x) = \sum_{n\in\mathbb{Z}} \chi_j(|n|)\widehat{f}(n)e^{inx},$$

where $\chi_j(\xi) = \chi(2^{-j}\xi)$ for $j \in \mathbb{N}$ and $\chi_0(\xi) = 1 - \sum_{j=1}^{\infty} \chi_j(\xi)$. We write $f \in \mathrm{B}^\alpha_p(\mathbb{T})$, if and only if,

$$\sup_{j=0,1,\ldots} 2^{\alpha j}\|K_j f\|_{\mathrm{L}^p(\mathbb{T})} < \infty.$$

We will be concerned almost exclusively with the case $p = \infty$.

We highlight the following two properties,

$$f' \in \mathrm{B}^\alpha_\infty(\mathbb{T}) \Longleftrightarrow f \in \mathrm{B}^{\alpha+1}_\infty(\mathbb{T}) \tag{4}$$

for all $\alpha \in \mathbb{R}$ and

$$\mathrm{B}^\alpha_\infty(\mathbb{T}) = \mathrm{C}^\alpha(\mathbb{T}), \tag{5}$$

for all $\alpha \in (0,1)$. Let us prove these statements.

Let $g \in \mathcal{S}(\mathbb{R})$ be such that $\mathcal{F}g(\xi) = \chi(\xi)$, where

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,\mathrm{d}x$$

is the *Fourier transform.* Then, $(\mathcal{F}g_j)(\xi) = \chi_j(\xi)$ for $g_j(x) = 2^j g(2^j x)$.

If $f \in \mathcal{S}(\mathbb{R})$, *Poisson's Summation Formula* prescribes that,

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} (\mathcal{F}f)(2\pi n)e^{2\pi inx}$$

for all $x \in \mathbb{R}$. Letting $\tilde{f}(x) = f(2\pi x)$, gives

$$(\mathcal{F}\tilde{f})(\xi) = \frac{1}{2\pi}(\mathcal{F}f)\left(\frac{\xi}{2\pi}\right).$$

Then,

$$\sum_{k\in\mathbb{Z}} f(z+2\pi k) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} (\mathcal{F}f)(k)e^{inz}.$$

Hence, we can represent the projections K_j of any periodic distribution F, as

$$\begin{aligned}(K_jF)(x) &= \sum_{k=-\infty}^{\infty} \chi_j(|k|)\left(\frac{1}{2\pi}\int_{\mathbb{T}} F(y)e^{-iky}\,\mathrm{d}y\right)e^{ikx}\\ &= \int_{\mathbb{T}}\left(\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\chi_j(|k|)e^{ik(x-y)}\right)F(y)\,\mathrm{d}y\\ &= \int_{\mathbb{T}}\left(\sum_{k=-\infty}^{\infty} g_j(x-y+2k\pi)\right)F(y)\,\mathrm{d}y\\ &= \sum_{k=-\infty}^{\infty}\int_{\mathbb{T}} g_j(x-y+2k\pi)F(y-2k\pi)\,\mathrm{d}y\\ &= \int_{\mathbb{R}} g_j(x-y)F(y)\,\mathrm{d}y = (g_j\star F)(x).\end{aligned}$$

for all $x\in\mathbb{R}$. Here the symbol "$\star$" denotes the *convolution on* $\mathbb{R}$.

Now, according to [1, Lemma 2.1, p.52] in the case $p=\infty$, there exists a constant $C>0$ which only depends on r_1, r_2 and λ, ensuring the following estimates. For any function $u\in\mathrm{L}^\infty(\mathbb{R})$, such that

$$\operatorname{supp}(\mathcal{F}u)\subset\lambda\{\xi\in\mathbb{R}\,:\,0<r_1\le|\xi|\le r_2\},$$

we have

$$\frac{\lambda}{C}\|u\|_{\mathrm{L}^\infty(\mathbb{R})}\le\|u'\|_{\mathrm{L}^\infty(\mathbb{R})}\le C\lambda\|u\|_{\mathrm{L}^\infty(\mathbb{R})}. \tag{6}$$

This is some times called *Bernstein's Inequality*.

Problem 7. *Give the proof of* (4)*. Hint: use* (6)*.*

Solution. Take $u=g_j\star F$, $\lambda=2^j$, $r_1=2^{-1}$ and $r_2=2$ in (6). Then, the left hand side inequality yields,

$$2^{(\alpha+1)j}\|K_jF\|_{\mathrm{L}^\infty(\mathbb{T})}\le C2^{\alpha j}\|K_j(F')\|_{\mathrm{L}^\infty(\mathbb{T})}<\infty,$$

for $F'\in B^\alpha_\infty(\mathbb{T})$. Conversely, the right hand side inequality yields,

$$2^{\alpha j}\|K_j(F')\|_{\mathrm{L}^\infty(\mathbb{T})}\le C2^{(\alpha+1)j}\|K_jF\|_{\mathrm{L}^\infty(\mathbb{T})}<\infty,$$

for $F\in B^{\alpha+1}_\infty(\mathbb{T})$. $\square$

Proof of (5)*.* We know that $f\in\mathrm{C}^\alpha(\mathbb{T})$, if and only if $S_1+S_2<\infty$, for

$$S_1=\sup_{x\in\mathbb{T}}|f(x)|$$

and

$$S_2=\sup_{\substack{x\in\mathbb{T}\\h\neq0}}\frac{|f(x+h)-f(x)|}{|h|^\alpha}.$$

Recall that, $f\in\mathrm{B}^\alpha_\infty(\mathbb{T})$, if and only if $R<\infty$, for

$$R=\sup_{j=0,1,\dots}\sup_{x\in\mathbb{T}}2^{\alpha j}|K_jf(x)|.$$

Let $f\in\mathrm{B}^\alpha_\infty(\mathbb{T})$. We show that S_1 and S_2 are finite. Firstly note that,

$$f(x)=\sum_{j=0}^{\infty}K_jf(x).$$

Hence,

$$S_1 \le \sum_{j=0}^{\infty} \|K_j f\|_{\mathrm{L}^\infty(\mathbb{T})} \le \sum_{j=0}^{\infty} \frac{R}{2^{\alpha j}} < \infty.$$

Here we have used that $\alpha > 0$.

Now, if

$$S_3 = \limsup_{h\to 0} \left(\sup_{x\in\mathbb{T}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} \right) < \infty,$$

then $S_2 < \infty$. For $j = 0, 1, \ldots$, let

$$S_4(j) = \limsup_{h\to 0} \left(\sup_{x\in\mathbb{T}} \frac{|K_j(f(x+h)-f(x))|}{|h|^\alpha} \right).$$

Then, on the one hand,

$$S_3 \le \sum_{j=0}^{\infty} S_4(j).$$

On the other hand, by the Mean Value Theorem, for suitable $|h_j| < 2^{-2j}$,

$$\begin{aligned}
S_4(j) &\le \sup_{\substack{x\in\mathbb{T}\\ 0<|h|\le 2^{-2j}}} \frac{|K_j f(x+h) - K_j f(x)|}{|h|^\alpha} \\
&\le \sup_{\substack{x\in\mathbb{T}\\ 0<|h|\le 2^{-2j}}} \frac{|(K_j f)'(x+h_j)||h|}{|h|^\alpha} \\
&= \sup_{0<|h|\le 2^{-2j}} |h|^{1-\alpha} \sup_{x\in\mathbb{T}} |(g_j \star f)'(x+h_j)| \\
&\le 2^{-2j(1-\alpha)} \|(g_j\star f)'\|_{\mathrm{L}^\infty(\mathbb{R})} \\
&\le C 2^j 2^{-2j(1-\alpha)} \|g_j \star f\|_{\mathrm{L}^\infty(\mathbb{R})} \\
&= C 2^{-j(1-\alpha)} 2^{\alpha j} \|K_j f\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\le C R 2^{-j(1-\alpha)}.
\end{aligned}$$

Thus, indeed, $S_3 < \infty$. Here we have used that $1-\alpha > 0$. This confirms that $\mathrm{B}^\alpha_\infty(\mathbb{T}) \subseteq \mathrm{C}^\alpha(\mathbb{T})$.

Now, let us show that $\mathrm{C}^\alpha(\mathbb{T}) \subseteq \mathrm{B}^\alpha_\infty(\mathbb{T})$. Assume that $f \in \mathrm{C}^\alpha(\mathbb{T})$. That is $S_1 < \infty$ and $S_2 < \infty$. Considering f as a periodic function of $x \in \mathbb{R}$, we have

$$S_1 = \sup_{x\in\mathbb{R}} |f(x)| < \infty$$

and

$$S_2 = \sup_{\substack{x\in\mathbb{R}\\ h\ne 0}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty.$$

Our goal is to show that $R < \infty$.

Since $g \in \mathcal{S}(\mathbb{R})$, then there exists a constant $c_3 > 0$, such that

$$|g_j(x)| \le c_3 \frac{2^j}{(1+2^j|x|)^2},$$

for all $x \in \mathbb{R}$. Now for any $\varphi \in \mathbb{R}$, thought of as a constant periodic function, we have that $(g_j \star \varphi)(x) = \varphi \chi_j(0) = 0$ for $j = 1, 2, \ldots$. Then,

$$(g_j \star f)(x) = (g_j \star (f + \varphi))(x)$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{N}$. Thus,

$$\begin{aligned}
|(g_j \star f)(x)| &\leq \int_{\mathbb{R}} |g_j(y)||f(x-y)+\varphi| \,\mathrm{d}y \\
&\leq c_3 2^j \int_{\mathbb{R}} \frac{|f(x-y)+\varphi|}{(1+2^j|y|)^2} \,\mathrm{d}y \\
&= c_3 \int_{\mathbb{R}} \frac{\left|f\left(x-\frac{z}{2^j}\right)+\varphi\right|}{(1+|z|)^2} \,\mathrm{d}z.
\end{aligned}$$

for all $x \in \mathbb{R}$, $\varphi \in \mathbb{R}$ and $j \in \mathbb{N}$.

This gives, taking $\varphi = -f(x)$, that

$$2^{\alpha j}|(g_j \star f)(x)| \leq c_3 2^{\alpha j} \int_{\mathbb{R}} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2} \,\mathrm{d}z = A_j(x) + B_j(x),$$

where we split the integral as follows. The first term is,

$$\begin{aligned}
A_j(x) &= c_3 2^{\alpha j} \int_{-2^j}^{2^j} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2} \,\mathrm{d}z \\
&= c_3 \int_{-2^j}^{2^j} \frac{|z|^\alpha \left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{\left(\frac{|z|}{2^j}\right)^\alpha (1+|z|)^2} \,\mathrm{d}z \\
&\leq c_3 S_2 \int_{-\infty}^{\infty} \frac{|z|^\alpha}{(1+|z|)^2} \,\mathrm{d}z \leq c_4 < \infty,
\end{aligned}$$

for all $j = 1, 2, \ldots$ and $x \in \mathbb{T}$. Here we have used that $0 < \alpha < 1$. The second term is,

$$\begin{aligned}
B_j(x) &= c_3 2^{\alpha j} \int_{|z| \geq 2^j} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2} \,\mathrm{d}z \\
&\leq c_3 2^{\alpha j} 2 S_1 \int_{|z| \geq 2^j} \frac{\mathrm{d}z}{(1+|z|)^2} \\
&\leq c_5 S_1 2^{(\alpha-1)j} \leq c_6 < \infty,
\end{aligned}$$

for all $j = 1, 2, \ldots$ and $x \in \mathbb{T}$. Here we have used that $\alpha < 1$. Then $R \leq c_4 + c_6 < \infty$. This completes the proof of (5). □

Problem 8. *Show that*

$$\mathrm{B}_1^{\alpha_1}(\mathbb{T}) \cap \mathrm{B}_\infty^{\alpha_2}(\mathbb{T}) \subset \mathrm{H}^\alpha(\mathbb{T})$$

for all $\alpha < (\alpha_1 + \alpha_2)/2$.

2.3. **Proof of Theorem A-b) first statement.** We will make use of the next lemma, which is analogous to [11, Corollaries 2.2 and 2.4]. The formulation with only half of the Fourier coefficients that we give here will be useful later on.

Lemma 1. *There exists a subset* $\mathcal{K} \subset \mathbb{R}$ *with complement of measure 0, such that the following holds true for all* $t \in \mathcal{K}$. *Given* $\delta > 0$, *there exists a constant* $C > 0$ *such that*

$$\sup_{x \in \mathbb{T}} \left| \sum_{n=0}^{\infty} \chi_j(n) \mathrm{e}^{in^2 t + inx} \right| \leq C 2^{\frac{j}{2}(1+\delta)}, \tag{7}$$

for all $j = 0, 1, \ldots$.

Proof. According to Dirichlet's Theorem, for every irrational number $a > 0$ there are infinitely many positive integers p, $q \in \mathbb{N}$, such that p and q are co-prime, and

$$\left| a - \frac{p}{q} \right| \leq \frac{1}{q^2}. \tag{8}$$

By virtue of [8, Lemma 4], there exists a constant $c_1 > 0$ such that, if the irreducible fraction $\frac{p}{q}$ satisfies (8), then

$$\begin{aligned} \left| \sum_{n=M}^{N} \mathrm{e}^{2\pi i(an^2+bn)} \right| &= \left| \sum_{k=1}^{N-M} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \\ &\leq c_1 \left(\frac{N-M}{\sqrt{q}} + \sqrt{q} \right) \end{aligned}$$

for all $N \in \mathbb{N}$, $0 < M < N$ and $b \in \mathbb{R}$. Here c_1 is independent of a and b. Take any sequence $\{\omega_n\}$, such that $\omega_n = 0$ for $n < M$ or $n > N$, and

$$\sum_{n=M}^{N} |\omega_{n+1} - \omega_n| \leq d.$$

Since,

$$\begin{aligned} \left| \sum_{n=M}^{N} \omega_n \mathrm{e}^{2\pi i(an^2+bn)} \right| &= \left| \sum_{n=M}^{N} (\omega_{n+1} - \omega_n) \sum_{k=M}^{n} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \\ &\leq \sum_{n=M}^{N} |\omega_{n+1} - \omega_n| \left| \sum_{k=M}^{n} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \\ &\leq d \sup_{n=M,\ldots,N} \left| \sum_{k=M}^{n} \mathrm{e}^{2\pi i(ak^2+bk)} \right|, \end{aligned}$$

then,

$$\left| \sum_{n=M}^{N} \omega_n \mathrm{e}^{2\pi i(an^2+bn)} \right| \leq dc_1 \left(\frac{N-M}{\sqrt{q}} + \sqrt{q} \right). \tag{9}$$

Let $[a_0, a_1, \ldots]$ be the continued fraction expansion of the irrational number a,

$$a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}.$$

Then, the irreducible fractions,

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_{n-1} + \frac{1}{a_n}}}},$$

are such that (8) holds true with $\{p_n\}$ and $\{q_n\}$ increasing sequences. According to the Khinchin-Lévy Theorem, for almost every $a > 0$ the denominators q_n satisfy [12, p.66]

$$\lim_{n \to \infty} \frac{\log q_n}{n} = \rho := \frac{\pi^2}{12 \log 2}.$$

If a is such that this limit exists, then for all $j \in \mathbb{N}$ sufficiently large we can find quotients $\frac{p_n(j)}{q_n(j)}$ with denominators satisfying $q_{n(j)} = 2^{j(1+r_j)}$, where $r_j \to 0$ as

$j \to \infty$. Indeed, we can take $n(j)$ equal to the integer part of $j(\log 2)/\rho$. This choice implies

$$\lim_{j\to\infty} \frac{\log q_{n(j)}}{j} = \log 2.$$

Note that $\{q_{n(j)}\}$ is not a subsequence of $\{q_n\}$, since $\log 2/\rho < 1$ and therefore indices may be repeated, but this does not cause problems.

Let $\mathcal{K}$ be the set of times of the form $t = 2\pi a$, such that the sequence of quotients of $|a|$ satisfies the conditions of the previous paragraph. Let $t \in \mathcal{K}$ and fix $\delta > 0$. Let $J > 0$ be such that $|r_j| < \delta$ for all $j \geq J$. Taking $M = 2^{j-1}$, $N = 2^{j+1}$, $\omega_n = \chi_j(n)$, and

$$d = 2\sup_{\xi\in\mathbb{R}} |\chi'(\xi)|,$$

in (9), yields

$$\begin{aligned}
\sup_{x\in\mathbb{T}} \left| \sum_{n=0}^{\infty} \chi_j(n) \mathrm{e}^{in^2t+inx} \right| &= \sup_{x\in\mathbb{T}} \left| \sum_{n=2^{j-1}}^{2^{j+1}} \chi_j(n) \mathrm{e}^{in^2t+inx} \right| \\
&\leq dc_1 \left(\frac{2^{j+1} - 2^{j-1}}{\sqrt{q_{n_j}}} + \sqrt{q_{n_j}} \right) \\
&\leq dc_1 \left(\frac{2^{j-1}3}{2^{\frac{j}{2}(1-\delta)}} + 2^{\frac{j}{2}(1+\delta)} \right) \\
&\leq c_2 2^{\frac{j}{2}(1+\delta)},
\end{aligned}$$

for all $j \geq J$. This implies (7) for sufficiently large $C > 0$. □

The proof of the first statement of Theorem A-b) is as follows.

Let the periodic distribution,

$$E_t(x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx+in^2t}.$$

Since the Fourier coefficients e^{in^2t} of E_t are unimodular, the series converges in the weak sense of distributions and determines E_t uniquely for all $t \in \mathbb{R}$, [18, Theorems 11.6-1 and 11.6-2]. Moreover, for all $t \in \mathbb{R}$, $E_t \in \mathrm{B}^\beta_\infty(\mathbb{T})$ for all $\beta < -1$. This is a good start.

Let $t \in \mathcal{K}$, with $\mathcal{K} \subset \mathbb{R}$ as in Lemma 1. Then it follows from (7) that in fact the stronger inclusion $E_t \in \mathrm{B}^\beta_\infty(\mathbb{T})$ for all $\beta < -\frac{1}{2}$. Define the periodic distribution H_t by $H_t' = E_t$, namely

$$H(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\mathrm{e}^{inx+in^2t}}{in}.$$

Then, according to (4) and (5), $H_t \in \mathrm{C}^{\beta+1}(\mathbb{T})$ for all $\beta < -\frac{1}{2}$.

Note that

$$\widehat{f}(n) = \frac{1}{2\pi in} \int_{\mathbb{T}} \mathrm{e}^{-iny}\, \mathrm{d}f(y) = \frac{\widehat{\mu}(n)}{in}, \quad n \neq 0,$$

where μ is the Lebesgue-Stieltjes measure associated to f, which satisfies $|\mu|(\mathbb{T}) < \infty$. Then the solution of (A), can be expressed in terms of H_t as follows,

$$\begin{aligned} u(x,t) &= \widehat{f}(0) + \sum_{n\neq 0,\, n=-\infty}^{\infty} \mathrm{e}^{in^2t}\widehat{f}(n)\mathrm{e}^{inx} \\ &= \widehat{f}(0) + \sum_{n\neq 0,\, n=-\infty}^{\infty} \frac{\mathrm{e}^{in^2t}\widehat{\mu}(n)}{in}\mathrm{e}^{inx} \\ &= \widehat{f}(0) + (H_t * \mu)(x). \end{aligned}$$

Here and elsewhere below, "$*$" denotes the *convolution on* $\mathbb{T}$,

$$(H * F)(x) = \frac{1}{2\pi}\int_{\mathbb{T}} H(x-y)F(y)\,\mathrm{d}y,$$

so that $(\widehat{H * F})(n) = \widehat{H}(n)\widehat{F}(n)$. Hence, since μ is a bounded measure, we indeed have $u(\cdot,t) \in \mathrm{C}^{\alpha}(\mathbb{T})$ for all $\alpha < \frac{1}{2}$.

This completes the proof of the first claim in Theorem A-b).

Problem 9. *Show that if* $H \in \mathrm{C}^{\alpha}(\mathbb{T})$ *and* F *is a periodic distribution, then* $(H * F) \in \mathrm{C}^{\alpha}(\mathbb{T})$.

2.4. **Fractal dimension of a graph.** For the second statement in part b) of Theorem A, we first recall the notion of fractal dimension. Then, we give a formula for the dimension of the graph of a function, in terms of its regularity.

Let $g : \mathbb{T} \longrightarrow \mathbb{R}$ be a continuous function. Denote the *graph of* g, by

$$\Gamma = \Big\{(x, g(x)) \in \mathbb{T}\times\mathbb{R} \,:\, x \in \mathbb{T}\Big\}.$$

The *upper Minkowski (or fractal) dimension* of Γ, is defined by the expression

$$\dim_{\mathrm{B}} \Gamma = \limsup_{\varepsilon\to 0} \frac{\log \mathcal{N}(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $\mathcal{N}(\varepsilon)$ is the number of squares that intersect Γ, in a grid (covering $\mathbb{T}\times\mathbb{R}$) made of squares of side ε.

First we recall the classical upper bound, formulated in [6, Corollary 11.2-(a)].

Lemma 2. *If* $g \in \mathrm{C}^{\alpha}(\mathbb{T})$, *then* $\dim_{\mathrm{B}} \Gamma \le 2 - \alpha$.

Proof. We simplify the notation by doing this proof in the interval $[0,1]$ instead of $(-\pi,\pi]$. So we show that the function $h : [0,1] \longrightarrow \mathbb{R}$, given by $h(x) = g(2\pi x - \pi)$, has a graph of fractal dimension less than or equal to $2-\alpha$. The Hölder constant of h is α and the fractal dimension does not change with the scaling of the interval. We adapt the counting function $\mathcal{N}$, accordingly.

Let

$$R_h[a,b] = \sup_{x,y\in[a,b]} |h(x) - h(y)|.$$

Let $0 < \varepsilon < 1$ and let m be the smallest integer greater than or equal to $\frac{1}{\varepsilon}$. Then,

$$\mathcal{N}(\varepsilon) \le 2m + \sum_{k=1}^{m-1} \frac{R_h[k\varepsilon,(k+1)\varepsilon]}{\varepsilon}. \tag{10}$$

Indeed, split the interval $[0,1]$ into sub-intervals $[k\varepsilon,(k+1)\varepsilon]$ all of size ε and consider a mesh of size ε. Then, the number of squares that intersect the graph of h in each sub-interval, is at least

$$\frac{R_h[k\varepsilon,(k+1)\varepsilon]}{\varepsilon}.$$

To find the upper bound in (10), we use the fact that h is continuous. The function may overlap another square, when entering a new sub-interval from below or leaving one from above. Hence, the maximum number of squares that intersect the graph of h, is

$$2+\frac{R_h[k\varepsilon,(k+1)\varepsilon]}{\varepsilon}.$$

This shows (10).

Now, from the hypothesis, it follows that for a suitable constant $c_1>0$,

$$R_h[k\varepsilon,(k+1)\varepsilon]\le c_1\varepsilon^{\alpha}.$$

Then, using the upper bound in (10) and the fact that $m<1+\varepsilon^{-1}$, we have

$$\mathcal{N}(\varepsilon)\le 2m+c_1m\varepsilon^{\alpha-1}\le(1+\varepsilon^{-1})(2+c_1\varepsilon^{\alpha-1})\le c_2\varepsilon^{\alpha-2}.$$

Taking logarithms, gives

$$\frac{\log\mathcal{N}(\varepsilon)}{\log\varepsilon^{-1}}\le\frac{\log c_2}{\log\varepsilon^{-1}}+2-\alpha.$$

Hence, taking the limsup, yields the claim of the lemma. □

Finding bounds, complementary to the one in the above lemma, turns out to be less straightforward. One of the best results currently available was obtained in the paper [5].

Lemma 3. *If* $g\notin \mathrm{B}_1^{\alpha}(\mathbb{T})$, *then* $\dim_{\mathrm{B}}\Gamma\ge 2-\alpha$.

Proof. This lemma is a direct corollary of the stronger [5, Theorem 4.2]. □

2.5. **Proof of Theorem A-b) second statement.** We first show the following lemma, which was first established in [4]. Out proof follows the rather ingenious strategy employed in that paper. The important point of this statement is the fact that the regularity of the real part of the solution (and not only of the solution itself), cannot improve beyond the regularity of the initial data f, for almost all $t\in\mathbb{R}$.

Lemma 4. *Let* $f\in\mathrm{BV}(\mathbb{T})$ *and let*

$$r_0:=\sup\{s>0: f\in \mathrm{H}^s(\mathbb{T})\}.$$

If $r_0\in[\frac{1}{2},1)$, *then there exists a subset* $\mathcal{J}\subset\mathbb{R}$ *with complement of measure 0, such that the following holds true for all* $t\in\mathcal{J}$. *Whenever* $r>r_0$, *we have* $\operatorname{Re}u(\cdot,t)\notin \mathrm{H}^r(\mathbb{T})$.

Proof. Since $\mathrm{H}^s(\mathbb{T})\subset\mathrm{H}^r(\mathbb{T})$ for $r<s$, without loss of generality we will assume that r is such that $r_0<r<\frac{r_0+1}{2}$.

Write the real part of the solution as,

$$\begin{aligned}\operatorname{Re} u(x,t) &= \frac{1}{2}\big(u(x,t)+\overline{u(x,t)}\big)\\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}}\left(\mathrm{e}^{itj^2}\widehat{f}(j)+\mathrm{e}^{-itj^2}\overline{\widehat{f}(-j)}\right)e_j(x).\end{aligned}$$

Then, the conclusion will follow, if we find a sequence $\{J_n\}_{n=1}^{\infty}\subset\mathbb{N}$, such that

$$\lim_{n\to\infty}\sum_{j=1}^{J_n} j^{2r}\left|\mathrm{e}^{itj^2}\widehat{f}(j)+\mathrm{e}^{-itj^2}\overline{\widehat{f}(-j)}\right|^2=\infty.$$

Now,

$$\left|\mathrm{e}^{itj^2}\widehat{f}(j)+\mathrm{e}^{-itj^2}\overline{\widehat{f}(-j)}\right|^2=|\widehat{f}(j)|^2+|\widehat{f}(-j)|^2+2\operatorname{Re}\left(\mathrm{e}^{2itj^2}\widehat{f}(j)\widehat{f}(-j)\right).$$

By hypothesis, we have that

$$\sum_{j\in\mathbb{Z}} j^{2r}|\widehat{f}(j)|^2=\infty.$$

Then,

$$\sum_{j=1}^{\infty} j^{2r}(|\widehat{f}(j)|^2+|\widehat{f}(-j)|^2)=\infty. \tag{11}$$

Let the partial summations

$$S_L(t)=\sum_{j=1}^{L} j^{2r}\mathrm{e}^{2itj^2}\widehat{f}(j)\widehat{f}(-j).$$

If we can find another sequence $\{L_n\}_{n=0}^{\infty}\subset\mathbb{N}$, such that $S_{L_n}(t)$ converges as $n\to\infty$, this will ensure that

$$2\operatorname{Re}\left[\sum_{j=1}^{L_n} j^{2r}\left(\mathrm{e}^{2itj^2}\widehat{f}(j)\widehat{f}(-j)\right)\right]$$

converges, and we will be able to find J_n from this convergence and the divergence (11). Note that we do need to argue through the subsequence J_n, as there might be some cancellations in the intermediate terms, preventing a "uniform divergence".

So our next goal is to show that the sequence L_n exists. Since S_L is 2π-periodic, without loss of generality we can assume that $t\in(-\pi,\pi]$. Write $S_L(t)$ in the expanded form,

$$S_L(t)=\sum_{m=1}^{\infty}\mathrm{e}^{itm}\left(\sum_{\substack{1\le j\le L\\ 2j^2=m}} j^{2r}\widehat{f}(j)\widehat{f}(-j)\right).$$

Since $j^2=\frac{m}{2}$ has at most two (integer) solutions, the sum inside the bracket has at most 2 terms for every positive integer m and it is equal to 0 for most of them. If

$$\sum_{j=1}^{\infty} j^{4r}\left|\widehat{f}(j)\widehat{f}(-j)\right|^2<\infty, \tag{12}$$

then $S = \lim_{L\to\infty} S_L$ is the limit of a Fourier series, convergent in the norm of $\mathrm{L}^2(-\pi,\pi)$. Should this happen, by virtue of Carleson's theorem [10, Theorem 3.6.15], there would exists a subset $\mathcal{J}\subset\mathbb{R}$ whose complement is of measure 0, satisfying the following. For all $t\in\mathcal{J}$, there is a subsequence $\{L_n\}\subset\mathbb{N}$, such that the limit

$$\lim_{n\to\infty} S_{L_n}(t) = S(t)$$

converges pointwise. Therefore, according to the previous paragraph, $\mathcal{J}$ would be the needed subset in the statement of the lemma.

So we complete the proof by showing that (12) holds true. Since $f\in\mathrm{H}^{\frac{1}{2}}(\mathbb{T})$, then

$$|\widehat{f}(-j)|^2 \le c_2 j^{-2}$$

for all $j\in\mathbb{N}$. Let $s<r_0$. Since $f\in\mathrm{H}^s(\mathbb{T})$, then

$$\sum_{j=1}^{\infty} j^{2s}|\widehat{f}(j)|^2 < \infty.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{\infty} j^{4r}\left|\widehat{f}(j)\widehat{f}(-j)\right|^2 &\le c_2\sum_{j=1}^{\infty} j^{4r-2-2s}\left(j^{2s}|\widehat{f}(j)|^2\right)\\ &\le c_2\sum_{j=1}^{\infty} j^{2s}|\widehat{f}(j)|^2 < \infty,\end{aligned}$$

for any $r<\frac{s+1}{2}$. Taking $s<r_0$ close enough to r_0, we can always get r to satisfy both, this latter condition which ensures (12), and the assumption in the beginning of the proof. $\square$

To complete the proof of Theorem A-b), we proceed as follows.

Let D denote the upper Minkowski dimension of the graph of $\operatorname{Re} u(\cdot,t)$. By virtue of the first statement in Theorem A-b) and Lemma 2, it follows that $D\le\frac{3}{2}$ for all $t\in\mathcal{K}$.

Now, take $r_0=\frac{1}{2}$ in the previous lemma. According to it and to the statement of Problem 8, the condition (1) imposed on f implies that $\operatorname{Re} u(\cdot,t)\notin\mathrm{B}^r_1(\mathbb{T})$ whenever $r>\frac{1}{2}$, for all $t\in\mathcal{K}\cap\mathcal{J}$. Thus, by virtue of Lemma 3, we also have the complementary bound $D\ge\frac{3}{2}$ for all such t.

This completes the proof of Theorem A.

Problem 10. *Let u be the solution to the time-evolution equation* (3) *from Problem 6. Show that there exists $\tilde{t}>0$ such that $u(\cdot,\tilde{t})$ is continuous. Hint: this is easier than you think.*

Solution. From what we have discovered so far, a solution to the equation with Neumann boundary conditions is a solution to

$$\begin{aligned}&\partial_t u = i\partial_x^2 u\\ &u(0,t)=u(2\pi,t),\ \partial_x u(0,t)=\partial_x u(2\pi,t)\\ &u(x,0)=f_{\mathrm{e}}(x)\end{aligned}$$

for $x\in\mathbb{T}$. This is the equation from Theorem A. Indeed, the Fourier expansions of the solutions match exactly. Then, for the answer to this problem, just invoke the theorem part b) first statement. Since f_{e} is piecewise constant, it is of bounded variation in $\mathbb{T}$. Therefore, the answer is immediate: for almost all $\tilde{t}$ the solution is C^α for all $0 \leq \alpha < 1/2$ and hence it is continuous. □

3. Changing the boundary conditions

Let $b \in \mathbb{R} \setminus \{1\}$. We now consider

$$\text{(B)} \qquad \begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ &u'(a,t) = \frac{b}{1-b} u(a,t) && a = 0,\pi \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in (0,\pi) \end{aligned}$$

which is a modification from the periodic boundary conditions of (A) to so-called Robin boundary conditions. Our goal is to determine to what extend, this modification supports any form of revival and or fractality.

If $b = 0$ we obtain the Neumann boundary conditions. So, Theorem B below generalises the solutions to the problems 6 and 10 from the previous section. For $b \in (0,1)$, the part a) of this theorem, was established in [2] and the PhD dissertation [7]. The extension to the other real b that we give here does not present any technical improvement from that case. As far as I am aware, the part b) of the theorem is new.

Let any function $h : [0,\pi] \longrightarrow \mathbb{C}$. We denote by h_e the 2π-periodic extension of the even function

$$h_\mathrm{e}(x) = \begin{cases} h(x) & x \in [0,\pi] \\ h(-x) & x \in (-\pi,0). \end{cases}$$

That is, the *even extension* of h to $\mathbb{R}$. Likewise, we denote by h_o the 2π-periodic extension of the odd function

$$h_\mathrm{o}(x) = \begin{cases} h(x) & x \in [0,\pi] \\ -h(-x) & x \in (-\pi,0). \end{cases}$$

That is, the *odd extension* of h to $\mathbb{R}$.

The next conventions will simplify the arguments below. Let

$$A_b = \frac{2b}{(1-b)\big(e^{2\pi\frac{b}{1-b}} - 1\big)}$$

for $b \neq 0$ and $A_0 = \frac{1}{\pi}$. We denote with the unambiguous symbol ϕ, the function $\phi : \mathbb{R} \longrightarrow \mathbb{C}$, which is the 2π-periodic extension of

$$\phi(x) = \sqrt{A_b}\, \mathrm{e}^{\frac{b}{1-b}x},$$

from $x \in [0,2\pi]$ to $\mathbb{R}$. We remark that ϕ, regarded as a function of $\mathbb{T}$, has a discontinuity at 0 but it is C^∞ at $\pm\pi$. Also, note that ϕ satisfies the boundary conditions of (B) and that $\|\phi\|_{L^2(0,\pi)} = 1$.

For any $f \in \mathrm{L}^2(0,\pi)$, we will consider an associated function $g : \mathbb{T} \longrightarrow \mathbb{C}$, given by the expression

$$g = \sqrt{\pi}\phi * (f_\mathrm{o} - f_\mathrm{e}). \tag{13}$$

Here, I clarify that g is a function on the whole torus and the convolution is, as previously, also on $\mathbb{T}$.

Theorem B. *Let $f \in \mathrm{L}^2(0,\pi)$ and let g be as in the expression* (13). *Let u be the solution to* (B).