JUMPS, CUSPS AND FRACTALS, IN TIME-EVOLUTION PDES

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1. Preliminaries

1.1. **Conventions and notation.** Let $e_n(x) = \frac{1}{\sqrt{2\pi}} e(nx)$. Here and everywhere below, we write the Fourier coefficients of a periodic distribution F, see [18, Chapter 11] or [9, Section 9.3], with one of the usual scalings on $\mathbb{T} = (-\pi, \pi]$, as

$$\widehat{F}(n) = \frac{1}{\sqrt{2\pi}} \langle F, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} F(y)\, \mathrm{d}y.$$

This choice makes either series in the expression

$$F(x) \sim \sum_{n\in\mathbb{Z}} \langle F, e_n \rangle e_n(x) = \sum_{n\in\mathbb{Z}} \widehat{F}(n) \mathrm{e}^{inx},$$

convenient for long calculations.

Recall that $\{e_n\}_{n\in\mathbb{Z}} \subset \mathrm{L}^2(\mathbb{T})$ is an orthonormal basis of eigenfunctions for the Laplacian,

$$-\partial_x^2 : \mathrm{H}^2(\mathbb{T}) \longrightarrow \mathrm{L}^2(\mathbb{T}).$$

Indeed, for all $n \in \mathbb{Z}$,

$$-\partial_x^2 e_n = n^2 e_n.$$

The following function spaces will be considered throughout. The definitions and relevant properties, are given in the next subsection and in the text. In two occasions we will use $(0, \pi]$ not identifying 0 with π, instead of $\mathbb{T}$, as the mapping of the definitions and properties is obvious we omit the details.

- $\mathrm{BV}(\mathbb{T})$ functions of *bounded variation*,
- $\mathrm{AC}(\mathbb{T})$ *absolutely continuous* functions,
- $\mathrm{C}^\alpha(\mathbb{T})$ *Hölder continuous* functions of regularity $\alpha \in (0, 1)$,
- $\mathrm{H}^\alpha(\mathbb{T})$ functions in the L^2 *Sobolev space* with regularity $\alpha \geq 0$,
- $\mathrm{B}_p^\alpha(\mathbb{T})$ distributions in the ℓ^∞-L^p *Besov space* with regularity $\alpha \in \mathbb{R}$ and $1 \leq p \leq \infty$.

1.2. **Connections and properties of the classical function spaces.** Recall the classical definitions of $\mathrm{BV}(\mathbb{T})$ and $\mathrm{AC}(\mathbb{T})$, given in standard analysis monographs such as [15, p.9 and p.47]. We know that

$$f \in \mathrm{AC}(\mathbb{T}) \quad \iff \quad f' \in \mathrm{L}^1(\mathbb{T}).$$

We also know that $f \in \mathrm{BV}(\mathbb{T})$ if and only if f' is a finite Radon measure on the Borel σ-algebra. Moreover, if $f \in \mathrm{BV}(\mathbb{T})$, then

$$f = f_{\mathrm{ac}} + f_{\mathrm{s}},$$

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for $f_{\mathrm{ac}} \in \mathrm{AC}(\mathbb{T})$ with $f_{\mathrm{ac}}' \in \mathrm{L}^{\infty}(\mathbb{T})$, and f_{s}' singular with support of Lebesgue measure 0. For the proofs of these statements, see the two theorems on [15, p.53].

Problem 1. *Let $f \in \mathrm{BV}(\mathbb{T})$. Show that there exists a constant, such that*

$$|\widehat{f}(n)| \leq \frac{c}{|n|}$$

for all $n \neq 0$.

Let $f : \mathbb{T} \longrightarrow \mathbb{C}$ and $\alpha \geq 0$. We will write $f \in \mathrm{H}^{\alpha}(\mathbb{T})$, whenever

$$\sum_{n\in\mathbb{Z}} (1+n^2)^{\alpha} \left|\widehat{f}(n)\right|^2 < \infty.$$

It is easy to see that

$$f' \in \mathrm{H}^{\alpha}(\mathbb{T}) \iff f \in \mathrm{H}^{\alpha+1}(\mathbb{T}).$$

Let $f : \mathbb{T} \longrightarrow \mathbb{C}$ and $\alpha \in (0,1)$. We will write $f \in \mathrm{C}^{\alpha}(\mathbb{T})$, whenever

$$\sup_{x\in\mathbb{T}} |f(x)| + \sup_{\substack{x\in\mathbb{T} \\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^{\alpha}} < \infty.$$

This expression defines a norm in $\mathrm{C}^{\alpha}(\mathbb{T})$ and makes it a Banach space, but we will neither use nor prove this fact. See [13, §11.3].

Problem 2. *Let $f : (-\pi, \pi] \longrightarrow \mathbb{R}$ be given by*

$$f(x) = |x| \log \frac{1}{|x|}.$$

Show that $f \in \mathrm{C}^{\alpha}(\mathbb{T})$ for all $0 < \alpha < 1$. Show that f is not a Lipschitz function. Is $f \in \mathrm{AC}(\mathbb{T})$?

Problem 3. *Show that $\mathrm{H}^1(\mathbb{T}) \subseteq \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$. Hint: use the Cauchy-Schwarz inequality. Is $\mathrm{H}^1(\mathbb{T}) = \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$?*

2. The Schrödinger equation

The study of different modifications of the following Schrödinger's equation, is the main goal of this cursillo. We are interested in the regularity of the solution. Consider

$$\text{(A)} \qquad \begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x \in \mathbb{T} \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in \mathbb{T}. \end{aligned}$$

It is routine to seen that, for $t \in \mathbb{R}$,

$$u(x,t) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{-ij^2 t} \widehat{f}(j) e_j(x) = \sum_{j=-\infty}^{\infty} \mathrm{e}^{-ij^2 t + ijx} \widehat{f}(j).$$

Therefore, the solution does not change its Sobolev norm for any $t \in \mathbb{R}$.

Problem 4. *Let $\alpha \geq 0$. Show that*

$$\|u(\cdot,t)\|_{\mathrm{H}^{\alpha}(\mathbb{T})} = \|f\|_{\mathrm{H}^{\alpha}(\mathbb{T})}$$

for all $t \in \mathbb{R}$.

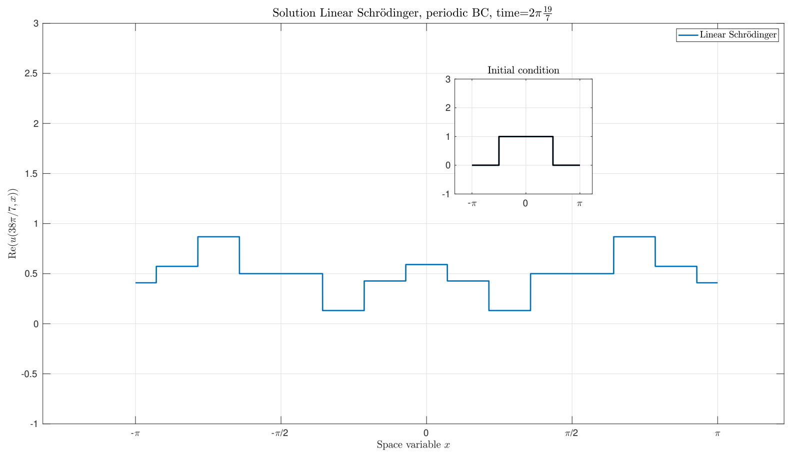

FIGURE 1. Revivals: solution of (A) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t = 2\pi\frac{19}{7}$.

Quite remarkably, in this quarter of a Century, it has been discovered that the regularity properties of the solution, beyond the Sobolev scale, are intimately connected with the best approximation of t, in continued fractions. The next theorem illustrates this in a concrete manner. It assembles results first formulated in [11] and [16]. Some of the original proofs were simplified in [17], [14] and [4].

Theorem A. *Let $f \in \mathrm{L}^2(\mathbb{T})$. Let u be the solution to* (A).

a) *If $p, q \in \mathbb{N}$ are co-prime, then*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i\frac{km-pm^2}{q}} f\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset $\mathcal{K} \subset \mathbb{R}$ with complement of measure 0, such that the following property is valid. If $f \in \mathrm{BV}(\mathbb{T})$, then*

$$u(\cdot, t) \in \bigcup_{\epsilon>0} \mathrm{C}^{\frac{1}{2}-\epsilon}(\mathbb{T})$$

for all $t \in \mathcal{K}$. Moreover, if additionally $f \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T})$ but

$$f \notin \bigcap_{\epsilon>0} \mathrm{H}^{\frac{1}{2}+\epsilon}(\mathbb{T}), \tag{1}$$

then the graph of $\mathrm{Re}\, u(\cdot, t)$ has fractal dimension $\frac{3}{2}$ for almost all $t \in \mathcal{K}$.

This theorem prescribes that the regularity of the solution in the space variable, changes significantly with time, when seen from a perspective different than that of the Sobolev norm. For example, if f is a step function, the solution is a finite linear combination of step functions whenever $\frac{t}{2\pi} \in \mathbb{Q}$, while it is continuous but "rough" for almost every $\frac{t}{2\pi} \notin \mathbb{Q}$. We can call this a *revivals/fractality dichotomy*, and illustrate this in figures 1 and 2.

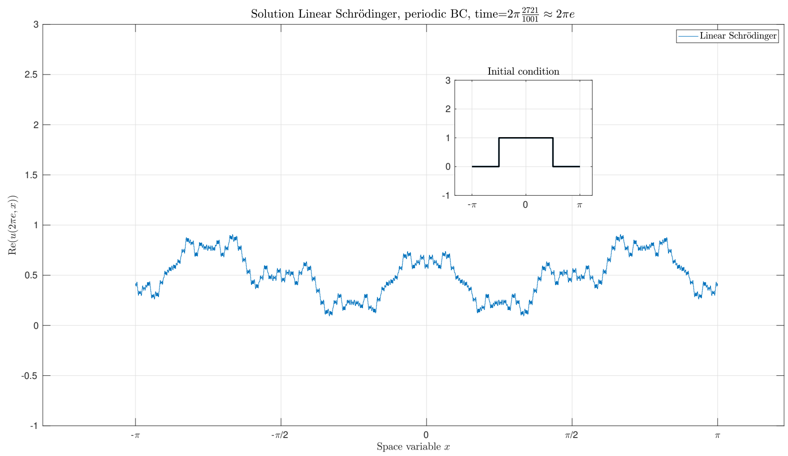

FIGURE 2. Fractality: solution of (A) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t \approx 2\pi\mathrm{e}$.

2.1. **Proof of Theorem A-a).** The proof of the first statement in Theorem A is as follows.

Let $t = 2\pi\frac{p}{q}$. Take $j \underset{q}{\equiv} m$ so that $\mathrm{e}^{ij^2t} = \mathrm{e}^{im^2t}$. Then,

$$u(x,t) = \frac{1}{2\pi}\sum_{m=0}^{q-1}\mathrm{e}^{-im^2t}\sum_{\substack{j\in\mathbb{Z}\\ j\underset{q}{\equiv}m}}\langle f,\mathrm{e}^{ij(\cdot)}\rangle\mathrm{e}^{ijx}$$

Now,

$$\sum_{k=0}^{q-1}\mathrm{e}^{2\pi i(m-j)\frac{k}{q}} = \begin{cases} q & j\underset{q}{\equiv}m \\ 0 & j\underset{q}{\not\equiv}m. \end{cases} \tag{2}$$

Thus,

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\ j\underset{q}{\equiv}m}}\langle f,\mathrm{e}^{ij(\cdot)}\rangle\mathrm{e}^{ijx} &= \frac{1}{q}\sum_{k=0}^{q-1}\mathrm{e}^{2\pi im\frac{k}{q}}\sum_{j\in\mathbb{Z}}\mathrm{e}^{-2\pi i\frac{k}{q}j}\langle f,\mathrm{e}^{ij(\cdot)}\rangle\mathrm{e}^{ijx} \\ &= \frac{1}{q}\sum_{k=0}^{q-1}\mathrm{e}^{2\pi im\frac{k}{q}}\sum_{j\in\mathbb{Z}}\Big\langle f\Big(\cdot-\frac{2\pi k}{q}\Big),\mathrm{e}^{ij(\cdot)}\Big\rangle\mathrm{e}^{ijx}.\end{aligned}$$

From this, the statement of Theorem A-a) follows.

Problem 5. *Give the proof of the identity* (2).

Problem 6. *Let* $f\in\mathrm{L}^2(0,\pi)$. *Find the (unique) solution to*

$$\begin{aligned} &\partial_t u(x,t) = i\partial_x^2 u(x,t) && x\in(0,\pi) \quad t\in\mathbb{R} \\ &\partial_x u(0,t) = \partial_x u(\pi,t) = 0 && t\in\mathbb{R} \\ &u(x,0) = f(x) && x\in(0,\pi). \end{aligned} \tag{3}$$

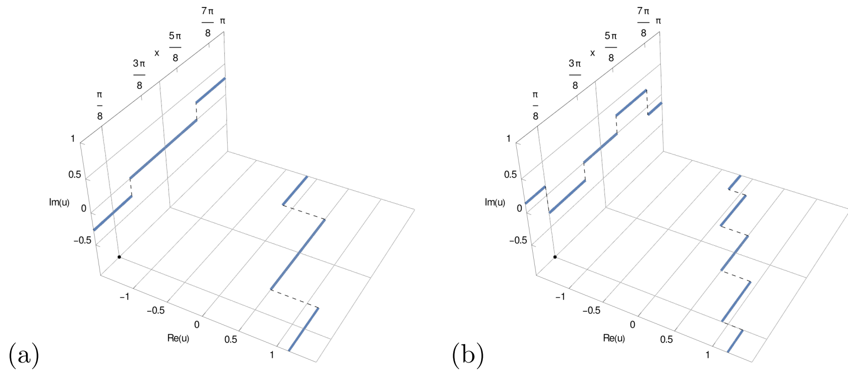

FIGURE 3. (a) $q = 16$ and $p = 1$, (b) $q = 32$ and $p = 1$.

Give your solution in terms of the Fourier series of f. Now, set

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi]. \end{cases}$$

Find $\tilde{t} > 0$ such that $u(\cdot, \tilde{t})$ has a discontinuity at $x = \frac{\pi}{8}$. Hint: the second part is tougher than you think.

2.2. **Besov spaces.** The Fourier coefficients of Hölder continuous functions have a specific behaviour, which can be seen through the scale of Besov spaces. The latter give a more refined criterion for the regularity of a function than the scale of Sobolev spaces. To simplify our notation, we will write $\mathrm{B}^\alpha_p(\mathbb{T})$, for $\alpha \in \mathbb{R}$ and $p \geq 1$, to denote what is normally written as $\mathrm{B}^\alpha_{p,\infty}(\mathbb{T})$, precisely defined as follows.

Let $\chi : \mathbb{R} \longrightarrow [0, 1]$ be a C^∞ function, such that

$$\operatorname{supp} \chi = [2^{-1}, 2]$$

and

$$\sum_{j=0}^{\infty} \chi(2^{-j}\xi) = 1$$

for all $\xi \geq 1$. Let the *Littlewood-Paley projections* of a periodic distribution on $\mathbb{T}$, be given by

$$(K_j f)(x) = \sum_{n \in \mathbb{Z}} \chi_j(|n|) \widehat{f}(n) e^{inx},$$

where $\chi_j(\xi) = \chi(2^{-j}\xi)$ for $j \in \mathbb{N}$ and $\chi_0(\xi) = 1 - \sum_{j=1}^{\infty} \chi_j(\xi)$. We write $f \in \mathrm{B}^\alpha_p(\mathbb{T})$, if and only if,

$$\sup_{j=0,1,\ldots} 2^{\alpha j} \|K_j f\|_{\mathrm{L}^p(\mathbb{T})} < \infty.$$

We will be concerned almost exclusively with the case $p = \infty$.

We highlight the following two properties,

$$f' \in \mathrm{B}^\alpha_\infty(\mathbb{T}) \Longleftrightarrow f \in \mathrm{B}^{\alpha+1}_\infty(\mathbb{T}) \tag{4}$$

for all $\alpha \in \mathbb{R}$ and

$$\mathrm{B}^\alpha_\infty(\mathbb{T}) = \mathrm{C}^\alpha(\mathbb{T}), \tag{5}$$

for all $\alpha \in (0, 1)$. Let us prove these statements.

Let $g \in \mathcal{S}(\mathbb{R})$ be such that $\mathcal{F}g(\xi) = \chi(\xi)$, where

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} \,\mathrm{d}x$$

is the *Fourier transform.* Then, $(\mathcal{F}g_j)(\xi) = \chi_j(\xi)$ for $g_j(x) = 2^j g(2^j x)$.

If $f \in \mathcal{S}(\mathbb{R})$, *Poisson's Summation Formula* prescribes that,

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} (\mathcal{F}f)(2\pi n) e^{2\pi i n x}$$

for all $x \in \mathbb{R}$. Letting $\tilde{f}(x) = f(2\pi x)$, gives

$$(\mathcal{F}\tilde{f})(\xi) = \frac{1}{2\pi} (\mathcal{F}f)\left(\frac{\xi}{2\pi}\right).$$

Then,

$$\sum_{k\in\mathbb{Z}} f(z + 2\pi k) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} (\mathcal{F}f)(k) e^{inz}.$$

Hence, we can represent the projections K_j of any periodic distribution F, as

$$\begin{aligned}
(K_j F)(x) &= \sum_{k=-\infty}^{\infty} \chi_j(|k|) \left(\frac{1}{2\pi} \int_{\mathbb{T}} F(y) e^{-iky} \,\mathrm{d}y \right) e^{ikx} \\
&= \int_{\mathbb{T}} \left(\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \chi_j(|k|) e^{ik(x-y)} \right) F(y) \,\mathrm{d}y \\
&= \int_{\mathbb{T}} \left(\sum_{k=-\infty}^{\infty} g_j(x - y + 2k\pi) \right) F(y) \,\mathrm{d}y \\
&= \sum_{k=-\infty}^{\infty} \int_{\mathbb{T}} g_j(x - y + 2k\pi) F(y - 2k\pi) \,\mathrm{d}y \\
&= \int_{\mathbb{R}} g_j(x-y) F(y) \,\mathrm{d}y = (g_j \star F)(x).
\end{aligned}$$

for all $x \in \mathbb{R}$. Here the symbol "$\star$" denotes the *convolution on* $\mathbb{R}$.

Now, according to [1, Lemma 2.1, p.52] in the case $p = \infty$, there exists a constant $C > 0$ which only depends on r_1, r_2 and λ, ensuring the following estimates. For any function $u \in \mathrm{L}^\infty(\mathbb{R})$, such that

$$\operatorname{supp}(\mathcal{F}u) \subset \lambda\{\xi \in \mathbb{R} \,:\, 0 < r_1 \leq |\xi| \leq r_2\},$$

we have

$$\frac{\lambda}{C} \|u\|_{\mathrm{L}^\infty(\mathbb{R})} \leq \|u'\|_{\mathrm{L}^\infty(\mathbb{R})} \leq C\lambda \|u\|_{\mathrm{L}^\infty(\mathbb{R})}. \tag{6}$$

This is some times called *Bernstein's Inequality.*

Problem 7. *Give the proof of* (4). *Hint: use* (6).

Proof of (5). We know that $f \in \mathrm{C}^\alpha(\mathbb{T})$, if and only if $S_1 + S_2 < \infty$, for

$$S_1 = \sup_{x\in\mathbb{T}} |f(x)|$$

and

$$S_2 = \sup_{\substack{x\in\mathbb{T}\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\alpha}.$$

Recall that, $f \in \mathrm{B}^\alpha_\infty(\mathbb{T})$, if and only if $R<\infty$, for

$$R = \sup_{j=0,1,\dots} \sup_{x\in\mathbb{T}} 2^{\alpha j}|K_j f(x)|.$$

Let $f \in \mathrm{B}^\alpha_\infty(\mathbb{T})$. We show that S_1 and S_2 are finite. Firstly note that,

$$f(x) = \sum_{j=0}^\infty K_j f(x).$$

Hence,

$$S_1 \le \sum_{j=0}^\infty \|K_j f\|_{\mathrm{L}^\infty(\mathbb{T})} \le \sum_{j=0}^\infty \frac{R}{2^{\alpha j}} < \infty.$$

Here we have used that $\alpha > 0$.

Now, if

$$S_3 = \limsup_{h\to 0}\left(\sup_{x\in\mathbb{T}} \frac{|f(x+h)-f(x)|}{|h|^\alpha}\right) < \infty,$$

then $S_2<\infty$. For $j=0,1,\dots$, let

$$S_4(j) = \limsup_{h\to 0}\left(\sup_{x\in\mathbb{T}} \frac{|K_j(f(x+h)-f(x))|}{|h|^\alpha}\right).$$

Then, on the one hand,

$$S_3 \le \sum_{j=0}^\infty S_4(j).$$

On the other hand, by the Mean Value Theorem, for suitable $|h_j|<2^{-2j}$,

$$\begin{aligned}
S_4(j) &\le \sup_{\substack{x\in\mathbb{T}\\ 0<|h|\le 2^{-2j}}} \frac{|K_j f(x+h)-K_j f(x)|}{|h|^\alpha} \\
&\le \sup_{\substack{x\in\mathbb{T}\\ 0<|h|\le 2^{-2j}}} \frac{|(K_j f)'(x+h_j)||h|}{|h|^\alpha} \\
&= \sup_{0<|h|\le 2^{-2j}} |h|^{1-\alpha} \sup_{x\in\mathbb{T}} |(g_j \star f)'(x+h_j)| \\
&\le 2^{-2j(1-\alpha)}\|(g_j\star f)'\|_{\mathrm{L}^\infty(\mathbb{R})} \\
&\le C2^j 2^{-2j(1-\alpha)}\|g_j\star f\|_{\mathrm{L}^\infty(\mathbb{R})} \\
&= C2^{-j(1-\alpha)}2^{\alpha j}\|K_j f\|_{\mathrm{L}^\infty(\mathbb{T})} \\
&\le CR2^{-j(1-\alpha)}.
\end{aligned}$$

Thus, indeed, $S_3<\infty$. Here we have used that $1-\alpha>0$. This confirms that $\mathrm{B}^\alpha_\infty(\mathbb{T}) \subseteq \mathrm{C}^\alpha(\mathbb{T})$.

Now, let us show that $\mathrm{C}^\alpha(\mathbb{T}) \subseteq \mathrm{B}^\alpha_\infty(\mathbb{T})$. Assume that $f\in \mathrm{C}^\alpha(\mathbb{T})$. That is $S_1<\infty$ and $S_2<\infty$. Considering f as a periodic function of $x\in\mathbb{R}$, we have

$$S_1 = \sup_{x\in\mathbb{R}} |f(x)| < \infty$$

and

$$S_2 = \sup_{\substack{x\in\mathbb{R}\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty.$$

Our goal is to show that $R<\infty$.

Since $g \in \mathcal{S}(\mathbb{R})$, then there exists a constant $c_3>0$, such that

$$|g_j(x)| \le c_3 \frac{2^j}{(1+2^j|x|)^2},$$

for all $x\in\mathbb{R}$. Now for any $\varphi\in\mathbb{R}$, thought of as a constant periodic function, we have that $(g_j \star \varphi)(x) = \varphi\chi_{_j}(0) = 0$ for $j=1,2,\ldots$. Then,

$$(g_j \star f)(x) = (g_j \star (f+\varphi))(x)$$

for all $x\in\mathbb{R}$ and $j\in\mathbb{N}$. Thus,

$$\begin{aligned} |(g_j\star f)(x)| &\le \int_{\mathbb{R}} |g_j(y)||f(x-y)+\varphi|\,\mathrm{d}y \\ &\le c_3 2^j \int_{\mathbb{R}} \frac{|f(x-y)+\varphi|}{(1+2^j|y|)^2}\,\mathrm{d}y \\ &= c_3 \int_{\mathbb{R}} \frac{\left|f\left(x-\frac{z}{2^j}\right)+\varphi\right|}{(1+|z|)^2}\,\mathrm{d}z. \end{aligned}$$

for all $x\in\mathbb{R}$, $\varphi\in\mathbb{R}$ and $j\in\mathbb{N}$.

This gives, taking $\varphi=-f(x)$, that

$$2^{\alpha j}|(g_j\star f)(x)| \le c_3 2^{\alpha j}\int_{\mathbb{R}} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2}\,\mathrm{d}z = A_j(x)+B_j(x),$$

where we split the integral as follows. The first term is,

$$\begin{aligned} A_j(x) &= c_3 2^{\alpha j}\int_{-2^j}^{2^j} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2}\,\mathrm{d}z \\ &= c_3 \int_{-2^j}^{2^j} \frac{|z|^\alpha\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{\left(\frac{|z|}{2^j}\right)^\alpha(1+|z|)^2}\,\mathrm{d}z \\ &\le c_3 S_2 \int_{-\infty}^{\infty} \frac{|z|^\alpha}{(1+|z|)^2}\,\mathrm{d}z \le c_4 < \infty, \end{aligned}$$

for all $j=1,2,\ldots$ and $x\in\mathbb{T}$. Here we have used that $0<\alpha<1$. The second term is,

$$\begin{aligned} B_j(x) &= c_3 2^{\alpha j}\int_{|z|\ge 2^j} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2}\,\mathrm{d}z \\ &\le c_3 2^{\alpha j} 2S_1 \int_{|z|\ge 2^j} \frac{\mathrm{d}z}{(1+|z|)^2} \\ &\le c_5 S_1 2^{(\alpha-1)j} \le c_6 < \infty, \end{aligned}$$

for all $j=1,2,\ldots$ and $x\in\mathbb{T}$. Here we have used that $\alpha<1$. Then $R\le c_4+c_6<\infty$. This completes the proof of (5). $\square$

Problem 8. *Show that*

$$\mathrm{B}_1^{\alpha_1}(\mathbb{T}) \cap \mathrm{B}_\infty^{\alpha_2}(\mathbb{T}) \subset \mathrm{H}^\alpha(\mathbb{T})$$

for all $\alpha < (\alpha_1 + \alpha_2)/2$.

2.3. **Proof of Theorem A-b) first statement.** We will make use of the next lemma, which is analogous to [11, Corollaries 2.2 and 2.4]. The formulation with only half of the Fourier coefficients that we give here will be useful later on.

Lemma 1. *There exists a subset* $\mathcal{K} \subset \mathbb{R}$ *with complement of measure 0, such that the following holds true for all* $t \in \mathcal{K}$. *Given* $\delta > 0$, *there exists a constant* $C > 0$ *such that*

$$\sup_{x\in\mathbb{T}} \left| \sum_{n=0}^{\infty} \chi_j(n) \mathrm{e}^{in^2t+inx} \right| \leq C 2^{\frac{j}{2}(1+\delta)}, \tag{7}$$

for all $j = 0, 1, \ldots$.

Proof. According to Dirichlet's Theorem, for every irrational number $a > 0$ there are infinitely many positive integers p, $q \in \mathbb{N}$, such that p and q are co-prime, and

$$\left| a - \frac{p}{q} \right| \leq \frac{1}{q^2}. \tag{8}$$

By virtue of [8, Lemma 4], there exists a constant $c_1 > 0$ such that, if the irreducible fraction $\frac{p}{q}$ satisfies (8), then

$$\begin{aligned} \left| \sum_{n=M}^{N} \mathrm{e}^{2\pi i(an^2+bn)} \right| &= \left| \sum_{k=1}^{N-M} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \\ &\leq c_1 \left(\frac{N-M}{\sqrt{q}} + \sqrt{q} \right) \end{aligned}$$

for all $N \in \mathbb{N}$, $0 < M < N$ and $b \in \mathbb{R}$. Here c_1 is independent of a and b. Take any sequence $\{\omega_n\}$, such that $\omega_n = 0$ for $n < M$ or $n > N$, and

$$\sum_{n=M}^{N} |\omega_{n+1} - \omega_n| \leq d.$$

Since,

$$\begin{aligned} \left| \sum_{n=M}^{N} \omega_n \mathrm{e}^{2\pi i(an^2+bn)} \right| &= \left| \sum_{n=M}^{N} (\omega_{n+1} - \omega_n) \sum_{k=M}^{n} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \\ &\leq \sum_{n=M}^{N} |\omega_{n+1} - \omega_n| \left| \sum_{k=M}^{n} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \\ &\leq d \sup_{n=M,\ldots,N} \left| \sum_{k=M}^{n} \mathrm{e}^{2\pi i(ak^2+bk)} \right|, \end{aligned}$$

then,

$$\left| \sum_{n=M}^{N} \omega_n \mathrm{e}^{2\pi i(an^2+bn)} \right| \leq d c_1 \left(\frac{N-M}{\sqrt{q}} + \sqrt{q} \right). \tag{9}$$

Let $[a_0, a_1, \ldots]$ be the continued fraction expansion of the irrational number a,

$$a = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}.$$

Then, the irreducible fractions,

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_{n-1} + \frac{1}{a_n}}}},$$

are such that (8) holds true with $\{p_n\}$ and $\{q_n\}$ increasing sequences. According to the Khinchin-Lévy Theorem, for almost every $a > 0$ the denominators q_n satisfy [12, p.66]

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \rho := \frac{\pi^2}{12 \log 2}.$$

If a is such that this limit exists, then for all $j \in \mathbb{N}$ sufficiently large we can find quotients $\frac{p_n(j)}{q_n(j)}$ with denominators satisfying $q_{n(j)} = 2^{j(1+r_j)}$, where $r_j \to 0$ as $j \to \infty$. Indeed, we can take $n(j)$ equal to the integer part of $j(\log 2)/\rho$. This choice implies

$$\lim_{j\to\infty} \frac{\log q_{n(j)}}{j} = \log 2.$$

Note that $\{q_{n(j)}\}$ is not a subsequence of $\{q_n\}$, since $\log 2/\rho < 1$ and therefore indices may be repeated, but this does not cause problems.

Let $\mathcal{K}$ be the set of times of the form $t = 2\pi a$, such that the sequence of quotients of $|a|$ satisfies the conditions of the previous paragraph. Let $t \in \mathcal{K}$ and fix $\delta > 0$. Let $J > 0$ be such that $|r_j| < \delta$ for all $j \geq J$. Taking $M = 2^{j-1}$, $N = 2^{j+1}$, $\omega_n = \chi_j(n)$, and

$$d = 2 \sup_{\xi\in\mathbb{R}} |\chi'(\xi)|,$$

in (9), yields

$$\begin{aligned} \sup_{x\in\mathbb{T}} \left| \sum_{n=0}^{\infty} \chi_j(n) \mathrm{e}^{in^2 t + inx} \right| &= \sup_{x\in\mathbb{T}} \left| \sum_{n=2^{j-1}}^{2^{j+1}} \chi_j(n) \mathrm{e}^{in^2 t + inx} \right| \\ &\leq dc_1 \left(\frac{2^{j+1} - 2^{j-1}}{\sqrt{q_{n_j}}} + \sqrt{q_{n_j}} \right) \\ &\leq dc_1 \left(\frac{2^{j-1} 3}{2^{\frac{j}{2}(1-\delta)}} + 2^{\frac{j}{2}(1+\delta)} \right) \\ &\leq c_2 2^{\frac{j}{2}(1+\delta)}, \end{aligned}$$

for all $j \geq J$. This implies (7) for sufficiently large $C > 0$. □

The proof of the first statement of Theorem A-b) is as follows.

Let the periodic distribution,

$$E_t(x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx + in^2 t}.$$

Since the Fourier coefficients $\mathrm{e}^{in^2 t}$ of E_t are unimodular, the series converges in the weak sense of distributions and determines E_t uniquely for all $t \in \mathbb{R}$, [18, Theorems

11.6-1 and 11.6-2]. Moreover, for all $t \in \mathbb{R}$, $E_t \in \mathrm{B}^{\beta}_{\infty}(\mathbb{T})$ for all $\beta < -1$. This is a good start.

Let $t \in \mathcal{K}$, with $\mathcal{K} \subset \mathbb{R}$ as in Lemma 1. Then it follows from (7) that in fact the stronger inclusion $E_t \in \mathrm{B}^{\beta}_{\infty}(\mathbb{T})$ for all $\beta < -\frac{1}{2}$. Define the periodic distribution H_t by $H_t' = E_t$, namely

$$H(t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\mathrm{e}^{inx + in^2 t}}{in}.$$

Then, according to (4) and (5), $H_t \in \mathrm{C}^{\beta+1}(\mathbb{T})$ for all $\beta < -\frac{1}{2}$.

Note that

$$\widehat{f}(n) = \frac{1}{2\pi in} \int_{\mathbb{T}} \mathrm{e}^{-iny} \, \mathrm{d}f(y) = \frac{\widehat{\mu}(n)}{in}, \quad n \neq 0,$$

where μ is the Lebesgue-Stieltjes measure associated to f, which satisfies $|\mu|(\mathbb{T}) < \infty$. Then the solution of (A), can be expressed in terms of H_t as follows,

$$\begin{aligned} u(x,t) &= \widehat{f}(0) + \sum_{n \neq 0,\, n=-\infty}^{\infty} \mathrm{e}^{in^2 t} \widehat{f}(n) \mathrm{e}^{inx} \\ &= \widehat{f}(0) + \sum_{n \neq 0,\, n=-\infty}^{\infty} \frac{\mathrm{e}^{in^2 t} \widehat{\mu}(n)}{in} \mathrm{e}^{inx} \\ &= \widehat{f}(0) + (H_t * \mu)(x). \end{aligned}$$

Here and elsewhere below, "$*$" denotes the *convolution on* $\mathbb{T}$,

$$(H * F)(x) = \frac{1}{2\pi} \int_{\mathbb{T}} H(x-y) F(y) \, \mathrm{d}y,$$

so that $(\widehat{H * F})(n) = \widehat{H}(n) \widehat{F}(n)$. Hence, since μ is a bounded measure, we indeed have $u(\cdot, t) \in \mathrm{C}^{\alpha}(\mathbb{T})$ for all $\alpha < \frac{1}{2}$.

This completes the proof of the first claim in Theorem A-b).

Problem 9. *Show that if* $H \in \mathrm{C}^{\alpha}(\mathbb{T})$ *and* F *is a periodic distribution, then* $(H * F) \in \mathrm{C}^{\alpha}(\mathbb{T})$.

2.4. **Fractal dimension of a graph.** For the second statement in part b) of Theorem A, we first recall the notion of fractal dimension. Then, we give a formula for the dimension of the graph of a function, in terms of its regularity.

Let $g : \mathbb{T} \longrightarrow \mathbb{R}$ be a continuous function. Denote the *graph of* g, by

$$\Gamma = \Big\{ (x, g(x)) \in \mathbb{T} \times \mathbb{R} \, : \, x \in \mathbb{T} \Big\}.$$

The *upper Minkowski (or fractal) dimension* of Γ, is defined by the expression

$$\dim_{\mathrm{B}} \Gamma = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $\mathcal{N}(\varepsilon)$ is the number of squares that intersect Γ, in a grid (covering $\mathbb{T} \times \mathbb{R}$) made of squares of side ε.

First we recall the classical upper bound, formulated in [6, Corollary 11.2-(a)].

Lemma 2. *If* $g \in \mathrm{C}^{\alpha}(\mathbb{T})$, *then* $\dim_{\mathrm{B}} \Gamma \leq 2 - \alpha$.

Proof. We simplify the notation by doing this proof in the interval $[0,1]$ instead of $(-\pi,\pi]$. So we show that the function $h:[0,1]\longrightarrow\mathbb{R}$, given by $h(x)=g(2\pi x-\pi)$, has a graph of fractal dimension less than or equal to $2-\alpha$. The Hölder constant of h is α and the fractal dimension does not change with the scaling of the interval. We adapt the counting function $\mathcal{N}$, accordingly.

Let

$$R_h[a,b]=\sup_{x,y\in[a,b]}|h(x)-h(y)|.$$

Let $0<\varepsilon<1$ and let m be the smallest integer greater than or equal to $\frac{1}{\varepsilon}$. Then,

$$\mathcal{N}(\varepsilon)\leq 2m+\sum_{k=1}^{m-1}\frac{R_h[k\varepsilon,(k+1)\varepsilon]}{\varepsilon}. \tag{10}$$

Indeed, split the interval $[0,1]$ into sub-intervals $[k\varepsilon,(k+1)\varepsilon]$ all of size ε and consider a mesh of size ε. Then, the number of squares that intersect the graph of h in each sub-interval, is at least

$$\frac{R_h[k\varepsilon,(k+1)\varepsilon]}{\varepsilon}.$$

To find the upper bound in (10), we use the fact that h is continuous. The function may overlap another square, when entering a new sub-interval from below or leaving one from above. Hence, the maximum number of squares that intersect the graph of h, is

$$2+\frac{R_h[k\varepsilon,(k+1)\varepsilon]}{\varepsilon}.$$

This shows (10).

Now, from the hypothesis, it follows that for a suitable constant $c_1>0$,

$$R_h[k\varepsilon,(k+1)\varepsilon]\leq c_1\varepsilon^{\alpha}.$$

Then, using the upper bound in (10) and the fact that $m<1+\varepsilon^{-1}$, we have

$$\mathcal{N}(\varepsilon)\leq 2m+c_1m\varepsilon^{\alpha-1}\leq(1+\varepsilon^{-1})(2+c_1\varepsilon^{\alpha-1})\leq c_2\varepsilon^{\alpha-2}.$$

Taking logarithms, gives

$$\frac{\log\mathcal{N}(\varepsilon)}{\log\varepsilon^{-1}}\leq\frac{\log c_2}{\log\varepsilon^{-1}}+2-\alpha.$$

Hence, taking the limsup, yields the claim of the lemma. □

Finding bounds, complementary to the one in the above lemma, turns out to be less straightforward. One of the best results currently available was obtained in the paper [5].

Lemma 3. *If* $g\notin \mathrm{B}_1^{\alpha}(\mathbb{T})$, *then* $\dim_{\mathrm{B}}\Gamma\geq 2-\alpha$.

Proof. This lemma is a direct corollary of the stronger [5, Theorem 4.2]. □

2.5. **Proof of Theorem A-b) second statement.** We first show the following lemma, which was first established in [4]. Out proof follows the rather ingenious strategy employed in that paper. The important point of this statement is the fact that the regularity of the real part of the solution (and not only of the solution itself), cannot improve beyond the regularity of the initial data f, for almost all $t \in \mathbb{R}$.

Lemma 4. *Let $f \in \mathrm{BV}(\mathbb{T})$ and let*

$$r_0 := \sup\{s > 0 : f \in \mathrm{H}^s(\mathbb{T})\}.$$

If $r_0 \in [\frac{1}{2}, 1)$, then there exists a subset $\mathcal{J} \subset \mathbb{R}$ with complement of measure 0, such that the following holds true for all $t \in \mathcal{J}$. Whenever $r > r_0$, we have $\operatorname{Re} u(\cdot, t) \notin \mathrm{H}^r(\mathbb{T})$.

Proof. Since $\mathrm{H}^s(\mathbb{T}) \subset \mathrm{H}^r(\mathbb{T})$ for $r < s$, without loss of generality we will assume that r is such that $r_0 < r < \frac{r_0+1}{2}$.

Write the real part of the solution as,

$$\begin{aligned}\operatorname{Re} u(x,t) &= \frac{1}{2}\big(u(x,t) + \overline{u(x,t)}\big) \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}} \left(\mathrm{e}^{itj^2}\widehat{f}(j) + \mathrm{e}^{-itj^2}\overline{\widehat{f}(-j)}\right) e_j(x).\end{aligned}$$

Then, the conclusion will follow, if we find a sequence $\{J_n\}_{n=1}^{\infty} \subset \mathbb{N}$, such that

$$\lim_{n\to\infty}\sum_{j=1}^{J_n} j^{2r}\left|\mathrm{e}^{itj^2}\widehat{f}(j) + \mathrm{e}^{-itj^2}\overline{\widehat{f}(-j)}\right|^2 = \infty.$$

Now,

$$\left|\mathrm{e}^{itj^2}\widehat{f}(j) + \mathrm{e}^{-itj^2}\overline{\widehat{f}(-j)}\right|^2 = |\widehat{f}(j)|^2 + |\widehat{f}(-j)|^2 + 2\operatorname{Re}\left(\mathrm{e}^{2itj^2}\widehat{f}(j)\widehat{f}(-j)\right).$$

By hypothesis, we have that

$$\sum_{j\in\mathbb{Z}} j^{2r}|\widehat{f}(j)|^2 = \infty.$$

Then,

$$\sum_{j=1}^{\infty} j^{2r}(|\widehat{f}(j)|^2 + |\widehat{f}(-j)|^2) = \infty. \tag{11}$$

Let the partial summations

$$S_L(t) = \sum_{j=1}^{L} j^{2r}\mathrm{e}^{2itj^2}\widehat{f}(j)\widehat{f}(-j).$$

If we can find another sequence $\{L_n\}_{n=0}^{\infty} \subset \mathbb{N}$, such that $S_{L_n}(t)$ converges as $n \to \infty$, this will ensure that

$$2\operatorname{Re}\left[\sum_{j=1}^{L_n} j^{2r}\left(\mathrm{e}^{2itj^2}\widehat{f}(j)\widehat{f}(-j)\right)\right]$$

converges, and we will be able to find J_n from this convergence and the divergence (11). Note that we do need to argue through the subsequence J_n, as there might be some cancellations in the intermediate terms, preventing a "uniform divergence".

So our next goal is to show that the sequence L_n exists. Since S_L is 2π-periodic, without loss of generality we can assume that $t \in (-\pi, \pi]$. Write $S_L(t)$ in the expanded form,

$$S_L(t) = \sum_{m=1}^{\infty} \mathrm{e}^{itm} \left(\sum_{\substack{1 \le j \le L \\ 2j^2 = m}} j^{2r} \widehat{f}(j)\widehat{f}(-j) \right).$$

Since $j^2 = \frac{m}{2}$ has at most two (integer) solutions, the sum inside the bracket has at most 2 terms for every positive integer m and it is equal to 0 for most of them. If

$$\sum_{j=1}^{\infty} j^{4r} \left| \widehat{f}(j)\widehat{f}(-j) \right|^2 < \infty, \tag{12}$$

then $S = \lim_{L\to\infty} S_L$ is the limit of a Fourier series, convergent in the norm of $\mathrm{L}^2(-\pi,\pi)$. Should this happen, by virtue of Carleson's theorem [10, Theorem 3.6.15], there would exists a subset $\mathcal{J} \subset \mathbb{R}$ whose complement is of measure 0, satisfying the following. For all $t \in \mathcal{J}$, there is a subsequence $\{L_n\} \subset \mathbb{N}$, such that the limit

$$\lim_{n\to\infty} S_{L_n}(t) = S(t)$$

converges pointwise. Therefore, according to the previous paragraph, $\mathcal{J}$ would be the needed subset in the statement of the lemma.

So we complete the proof by showing that (12) holds true. Since $f \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T})$, then

$$|\widehat{f}(-j)|^2 \le c_2 j^{-2}$$

for all $j \in \mathbb{N}$. Let $s < r_0$. Since $f \in \mathrm{H}^s(\mathbb{T})$, then

$$\sum_{j=1}^{\infty} j^{2s} |\widehat{f}(j)|^2 < \infty.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{\infty} j^{4r} \left| \widehat{f}(j)\widehat{f}(-j) \right|^2 &\le c_2 \sum_{j=1}^{\infty} j^{4r-2-2s} \left(j^{2s} |\widehat{f}(j)|^2 \right) \\ &\le c_2 \sum_{j=1}^{\infty} j^{2s} |\widehat{f}(j)|^2 < \infty,\end{aligned}$$

for any $r < \frac{s+1}{2}$. Taking $s < r_0$ close enough to r_0, we can always get r to satisfy both, this latter condition which ensures (12), and the assumption in the beginning of the proof. $\square$

To complete the proof of Theorem A-b), we proceed as follows.

Let D denote the upper Minkowski dimension of the graph of $\operatorname{Re} u(\cdot, t)$. By virtue of the first statement in Theorem A-b) and Lemma 2, it follows that $D \le \frac{3}{2}$ for all $t \in \mathcal{K}$.

Now, take $r_0 = \frac{1}{2}$ in the previous lemma. According to it and to the statement of Problem 8, the condition (1) imposed on f implies that $\operatorname{Re} u(\cdot, t) \notin \mathrm{B}_1^r(\mathbb{T})$ whenever $r > \frac{1}{2}$, for all $t \in \mathcal{K} \cap \mathcal{J}$. Thus, by virtue of Lemma 3, we also have the complementary bound $D \ge \frac{3}{2}$ for all such t.

This completes the proof of Theorem A.

Problem 10. *Let u be the solution to the time-evolution equation* (3) *from Problem 6. Show that there exists $\tilde{t} > 0$ such that $u(\cdot, \tilde{t})$ is continuous. Hint: this is easier than you think.*

3. Changing the boundary conditions

Let $b \in \mathbb{R} \setminus \{1\}$. We now consider

$$\text{(B)} \qquad \begin{aligned} & i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ & u'(a,t) = \frac{b}{1-b} u(a,t) && a = 0, \pi \quad t \in \mathbb{R} \\ & u(x,0) = f(x) && x \in (0,\pi) \end{aligned}$$

which is a modification from the periodic boundary conditions of (A) to so-called Robin boundary conditions. Our goal is to determine to what extend, this modification supports any form of revival and or fractality.

If $b = 0$ we obtain the Neumann boundary conditions. So, Theorem B below generalises the solutions to the problems 6 and 10 from the previous section. For $b \in (0,1)$, the part a) of this theorem, was established in [2] and the PhD dissertation [7]. The extension to the other real b that we give here does not present any technical improvement from that case. As far as I am aware, the part b) of the theorem is new.

Let any function $h : [0,\pi] \longrightarrow \mathbb{C}$. We denote by h_{e} the 2π-periodic extension of the even function

$$h_{\mathrm{e}}(x) = \begin{cases} h(x) & x \in [0,\pi] \\ h(-x) & x \in (-\pi, 0). \end{cases}$$

That is, the *even extension* of h to $\mathbb{R}$. Likewise, we denote by h_{o} the 2π-periodic extension of the odd function

$$h_{\mathrm{o}}(x) = \begin{cases} h(x) & x \in [0,\pi] \\ -h(-x) & x \in (-\pi, 0). \end{cases}$$

That is, the *odd extension* of h to $\mathbb{R}$.

The next conventions will simplify the arguments below. Let

$$A_b = \frac{2b}{(1-b)\left(e^{2\pi \frac{b}{1-b}} - 1\right)}$$

for $b \neq 0$ and $A_0 = \frac{1}{\pi}$. We denote with the unambiguous symbol ϕ, the function $\phi : \mathbb{R} \longrightarrow \mathbb{C}$, which is the 2π-periodic extension of

$$\phi(x) = \sqrt{A_b}\, \mathrm{e}^{\frac{b}{1-b} x},$$

from $x \in [0, 2\pi]$ to $\mathbb{R}$. We remark that ϕ, regarded as a function of $\mathbb{T}$, has a discontinuity at 0 but it is C^∞ at $\pm\pi$. Also, note that ϕ satisfies the boundary conditions of (B) and that $\|\phi\|_{L^2(0,\pi)} = 1$.

For any $f \in \mathrm{L}^2(0,\pi)$, we will consider an associated function $g : \mathbb{T} \longrightarrow \mathbb{C}$, given by the expression

$$g = \sqrt{\pi}\phi * (f_{\mathrm{o}} - f_{\mathrm{e}}). \tag{13}$$

Here, I clarify that g is a function on the whole torus and the convolution is, as previously, also on $\mathbb{T}$.

Theorem B. *Let $f \in \mathrm{L}^2(0,\pi)$ and let g be as in the expression* (13). *Let u be the solution to* (B).

a) *If $p, q \in \mathbb{N}$ are co-prime, then*

$$u\Big(x, \frac{2\pi p}{q}\Big) = \mathrm{e}^{2\pi i \frac{pb^2}{q(1-b)^2}} \langle f, \phi\rangle_{\mathrm{L}^2(0,\pi)} \phi(x) + \\ \frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{km-pm^2}{q}} (f_{\mathrm{e}} + g)\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset $\mathcal{K} \subset \mathbb{R}$ with complement of measure 0, such that the following property is valid. If $f \in \mathrm{BV}([0,\pi])$, then*

$$u(\cdot, t) \in \bigcup_{\epsilon>0} \mathrm{C}^{\frac{1}{2}-\epsilon}([0,\pi])$$

for all $t \in \mathcal{K}$. Moreover, if additionally $f_{\mathrm{e}} \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T})$ but

$$f_{\mathrm{e}} \notin \bigcap_{\epsilon>0} \mathrm{H}^{\frac{1}{2}+\epsilon}(\mathbb{T}),$$

then the graph of $\mathrm{Re}\, u(\cdot,t)$ has fractal dimension $\frac{3}{2}$ for almost all $t \in \mathcal{K}$.

In the statement a) of this theorem, we highlight the contribution of f_{e} and of g to the revival formula, separately. The reason for this is that g is more regular than f, therefore it does not make a contribution of the fractal part of the solution in the statement b). This will be seen in the proof below.

Problem 11. *Show that the expressions*

$$\mathrm{e}^{2\pi i \frac{pb^2}{q(1-b)^2}} \langle f, \phi\rangle_{\mathrm{L}^2(0,\pi)} \phi(x)$$

and

$$\frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{km-pm^2}{q}} g\Big(x - 2\pi\frac{k}{q}\Big)\right],$$

cancel out from the statement a) of this theorem, in the case $b = 0$. Compare with your answer to Problem 6.

3.1. **Proof of Theorem B.** Let $L : \mathrm{Dom}(L) \longrightarrow \mathrm{L}^2(0,\pi)$ be the self-adjoint operator $L = -\partial_x^2$ with domain

$$\mathrm{Dom}(L) = \mathrm{H}^2(0,\pi) \cap \left\{ g \in \mathrm{H}^1(0,\pi) \,:\, g'(0) = \frac{b}{1-b} g(0) \text{ and } g'(\pi) = \frac{b}{1-b} g(\pi) \right\}.$$

We first give the eigenfunctions and the spectrum of L.

To start with, note that

$$L\phi = -\frac{b^2}{(1-b)^2}\phi.$$

For $j \in \mathbb{N}$, let

$$\Lambda_j = \frac{b - j(1-b)i}{b + j(1-b)i}$$

and

$$\phi_j(x) = e_j(x) - \Lambda_j e_{-j}(x).$$

Then, $\|\phi_j\|_{\mathrm{L}^2(0,\pi)} = 1$. Indeed, note that $|\Lambda_j| = 1$. Moreover, $\phi_j \in \mathrm{Dom}(L)$ and $L\phi_j = j^2\phi_j$.

The family $\{\phi\} \cup \{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathrm{L}^2(0,\pi)$. Therefore, we know that

$$\mathrm{Spec}(L) = \left\{ -\frac{b^2}{(1-b)^2} \right\} \cup \{j^2\}_{j=1}^{\infty}.$$

Problem 12. *Show that the family of eigenvectors $\{\phi\} \cup \{\phi_j\}_{j=1}^{\infty}$ is indeed an orthonormal basis of $\mathrm{L}^2(0,\pi)$. Hint: the important point here is to show that they are a complete family.*

Then, for all $f \in \mathrm{L}^2(0,\pi)$. The solution to (B) is

$$u(x,t) = \mathrm{e}^{i\frac{b^2}{(1-b)^2}t} \langle f, \phi \rangle_{\mathrm{L}^2(0,\pi)} + \sum_{j=1}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \phi_j \rangle_{\mathrm{L}^2(0,\pi)} \phi_j(x).$$

The crucial point in the proof of Theorem B is the following lemma, which gives a different representation of the second term on the right hand side of this expression.

Lemma 5. *Let $f \in \mathrm{L}^2(0,\pi)$ and let g be given by (13). Let*

$$U(x,t) = \sum_{j=1}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \phi_j \rangle_{\mathrm{L}^2(0,\pi)} \phi_j(x).$$

Then $U = U_1 + U_2$, where

$$U_1(x,t) = \sum_{k \in \mathbb{Z}} \widehat{f_{\mathrm{e}}}(k) \mathrm{e}^{-ik^2 t + ikx}$$

and

$$U_2(x,t) = \sum_{k \in \mathbb{Z}} \widehat{g}(k) \mathrm{e}^{-ik^2 t + ikx}.$$

Proof. Begin by re-writing

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \left(\frac{1-\Lambda_j}{2} \cos(jx) + i \frac{1+\Lambda_j}{2} \sin(jx) \right).$$

Then

$$\langle f, \phi_j \rangle_{\mathrm{L}^2(0,\pi)} = \frac{1-\Lambda_j}{2} a_j + \frac{1+\overline{\Lambda_j}}{2i} b_j,$$

where a_j are the cosine Fourier coefficients of f and b_j are the sine Fourier coefficients of f, in $(0,\pi)$.

Let $\phi_{\mathrm{r}}(x) = \phi(2\pi - x)$. That is the reflection of ϕ about π. Note that,

$$\widehat{\phi}(j) = \frac{b}{2\,(b - j(1-b)i)} \text{ and } \widehat{\phi_{\mathrm{r}}}(j) = \frac{b}{2\,(b + j(1-b)i)}.$$

Then, re-writing U with the above representation of the eigenfunctions and inner products, gives

$$U(x,t) = \frac{1}{2} \sum_{k=1}^{5} v_k(x,t)$$

where each of the v_k are the solution to (A), with corresponding initial conditions

$$v_1(x,0) = 2f_{\mathrm{e}}(x), \qquad v_2(x,0) = -(\phi_{\mathrm{r}} + \phi) * f_{\mathrm{e}}(x), \qquad v_3(x,0) = (\phi_{\mathrm{r}} - \phi) * f_{\mathrm{e}}(x)$$
$$v_4(x,0) = (\phi - \phi_{\mathrm{r}}) * f_{\mathrm{o}}(x) \quad \text{and} \quad v_5(x,0) = (\phi_{\mathrm{r}} + \phi) * f_{\mathrm{o}}(x).$$

Once this expression is obtained, we observe that $U_1(x,t) = \frac{1}{2}v_1(x,t)$ and

$$U_2(x,t) = \frac{1}{2}\sum_{k=2}^{5} v_k(x,t).$$

$\square$

Problem 13. *Give the proof of Theorem B-a), using Lemma 5.*

We now present the proof of Theorem B-b). Observe that $\phi \in \mathrm{C}^\infty(0,\pi)$, so we can ignore this correction in the solution and concentrate on U.

For the first claim, let $f \in \mathrm{BV}(0,\pi)$. Then f_e, f_o and hence $f_\mathrm{o} - f_\mathrm{e}$, are in $\mathrm{BV}(\mathbb{T})$. To see this, note that the even and odd extensions can, at most, introduce a jump discontinuity at $x = 0$ and $x = \pm\pi$.

Since U_1 is a solution to (A) with initial condition f_e, according to Theorem A-b), we know that $U_1(\cdot,t) \in \mathrm{C}^\alpha(\mathbb{T})$ for all $\alpha < \frac{1}{2}$, provided $t \in \mathcal{K}$. Likewise, being a convolution, $g \in \mathrm{BV}(\mathbb{T})$ and U_2 is the solution to (A) with this initial condition. Then, also $U_2(\cdot,t) \in \mathrm{C}^\alpha(\mathbb{T})$. Hence, the first statement in Theorem B-b) is valid.

Note that, seen as a function of $\mathbb{T}$, $\phi \in \mathrm{BV}(\mathbb{T})$. Indeed it has a jump discontinuity at $x = 0$ for $b \neq 0$, but it is C^∞ at all other points $x \in \mathbb{T}$. But we are not using this fact in the previous paragraph.

Consider now the second claim made in Theorem B-b).

Assume the hypothesis,

$$f_\mathrm{e} \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T}) \setminus \bigcap_{\epsilon>0} \mathrm{H}^{\frac{1}{2}+\epsilon}(\mathbb{T}).$$

By virtue of Theorem A-b), it then follows that the graph of $\operatorname{Re} U_1(\cdot,t) : \mathbb{T} \longrightarrow \mathbb{C}$ has fractal dimension equal to $\frac{3}{2}$ for almost all $t \in \mathcal{K}$. But since f_e is even, also $U_1(\cdot,t)$ is even. Then, necessarily, $\operatorname{Re} U_1(\cdot,t) : (0,\pi) \longrightarrow \mathbb{C}$ has also fractal dimension equal to $\frac{3}{2}$ for almost all $t \in \mathcal{K}$.

We complete the proof by showing that $\operatorname{Re} U_2(\cdot,t) \in \mathrm{C}^1(\mathbb{T})$. We only know that

$$f_\mathrm{o} \in \mathrm{BV}(\mathbb{T})$$

and so,

$$(\widehat{f_\mathrm{o} - f_\mathrm{e}})(k) = O\left(|k|^{-1}\right)$$

as $|k| \to \infty$. See Problem 1. But we have, $\phi \in \mathrm{H}^{\frac{1}{2}-\epsilon}(\mathbb{T})$ for all $\epsilon > 0$. Hence,

$$\widehat{\phi}(k) = o\left(|k|^{-1}\right)$$

as $|k| \to \infty$. Since

$$\widehat{g}(k) = \sqrt{\pi}\widehat{\phi}(k)\ (\widehat{f_\mathrm{o} - f_\mathrm{e}})(k),$$

we thus gather that

$$\widehat{g}(k) = o\left(|k|^{-2}\right)$$

as $|k| \to \infty$. Therefore,

$$g' \in \mathrm{B}^\alpha_\infty(\mathbb{T}) = \mathrm{C}^\alpha(\mathbb{T})$$

for all $\alpha < \frac{1}{2}$. Thus, g is continuously differentiable and hence $U_2(\cdot,t)$ for all $t \in \mathbb{R}$ too. So, $\operatorname{Re} U_2(\cdot,t)$ is continuously differentiable for all $t \in \mathbb{R}$.

By Lemma 2, this implies that the graph of $\operatorname{Re} U_2(\cdot,t)$ has fractal dimension equal to 1, both, as a function from $\mathbb{T}$ and as a function from the restricted segment $(0,\pi)$. This completes the proof of Theorem B.

4. Adding a potential

Let $V : (0, \pi] \longrightarrow \mathbb{C}$ be a potential, satisfying either of the following conditions

- $V \in \mathrm{H}^2(0, \pi)$ and $\|V\|_{\mathrm{L}^\infty(0,\pi)} < \frac{3}{2}$; or
- $V : (0, \pi] \longrightarrow \mathbb{R}$, $V \in \mathrm{BV}(0, \pi)$ and $V \in \mathrm{L}^\infty(0, \pi)$.

In this short section we describe the revival property for the equation,

$$\text{(C)} \qquad \begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) + V(x)u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ &u(0,t) = u(\pi,t) = 0 && t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in (0,\pi) \end{aligned}$$

The proof of the next theorem can be found in [3].

Theorem C. *Let $f \in \mathrm{L}^2(0, \pi)$ and let u be the solution to* (C)*. Then, for all $t \in \mathbb{R}$, there exists $w(\cdot, t) \in \mathrm{C}^0(0, \pi)$ ensuring the following. If $p, q \in \mathbb{Z}$ are co-prime, then*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = w\Big(x, 2\pi\frac{p}{q}\Big) + \frac{\mathrm{e}^{-2\pi i \langle V\rangle \frac{p}{q}}}{q} \sum_{m=0}^{q-1} \left(\sum_{k=1}^{q-1} \mathrm{e}^{2\pi i \frac{mk - m^2 p}{q}} f_{\mathrm{o}}\Big(x - 2\pi\frac{k}{q}\Big) \right).$$

Here $\langle V\rangle = \frac{1}{\pi}\langle V, 1\rangle$ is the *mean of* V.

According to this theorem, the revivals property still holds true for the boundary-value problem (C), modulo a continuous correction.

Problem 14. *Give the proof of Theorem C for $V(x) = c$, where $c \in \mathbb{C}$. Hint: note that* (C) *has Dirichlet boundary conditions and that c does not need to satisfy the hypotheses on V stated in the bullet points.*

5. Multiplying by Hilbert's transform

In this final section we consider the striking example of the linear Benjamin-Ono equation, where a seemingly substantial change to the right hand side of the equation, still supports the revivals/fractality dichotomy that we described in Section 2.

Let the *Hilbert transform* of any $g : \mathbb{T} \longrightarrow \mathbb{C}$, be given by the expression

$$\mathcal{H}g(x) = \frac{1}{2\pi}\ \mathrm{p.\,v.}\!\int_{-\pi}^{\pi} \cot\frac{x-y}{2} g(y)\,\mathrm{d}y,$$

assuming that the principal value integral exists. An expression of $\mathcal{H}g$ for $g \in \mathrm{L}^2(\mathbb{T})$, in terms of the Fourier coefficients, shows that $\mathcal{H} : \mathrm{L}^2(\mathbb{T}) \longrightarrow \mathrm{L}^2(\mathbb{T})$ is a bounded operator.

Let $f : \mathbb{T} \longrightarrow \mathbb{R}$. The *linear Benjamin-Ono (BO) equation*, is the time-evolution problem

$$\text{(D)} \qquad \begin{aligned} &\partial_t u(x,t) = \mathcal{H}\partial_x^2 u(x,t) && x \in \mathbb{T} \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in \mathbb{T}. \end{aligned}$$

As we shall see next, this equation supports a revivals/fractality dichotomy that combines properties observed in all the models discussed previously, except that it gives rise to cusps of a very specific type in the solution. This is made precise in the next theorem, whose conclusion differs from that of Theorem A only in the first statement. Note that the solution of (D) are real-valued, but the second statement of the part b) is not an immediate consequence of any well-posedness of this equation in the Sobolev norms.

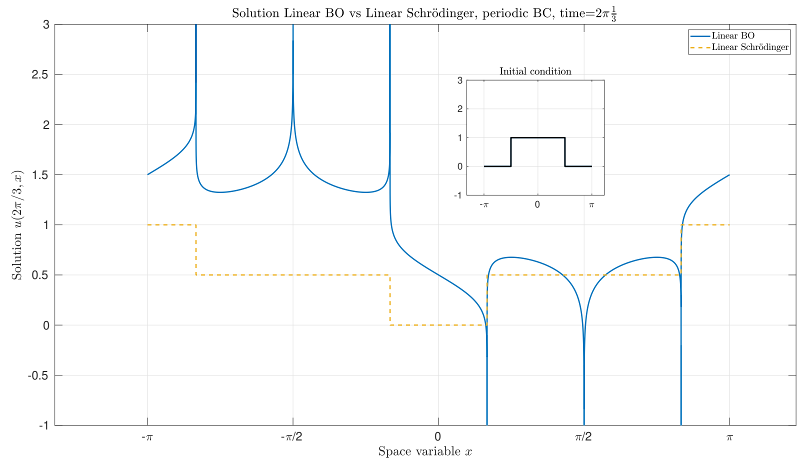

FIGURE 4. Solution of (D) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t = 2\pi\frac{1}{3}$ superimposed on the real and imaginary parts of the solution of (A) at $-t$. The cusp singularities in the solution of (D) correspond to jump singularities in either part of the solution of (A).

Theorem D.

a) *Let $f \in \mathrm{L}^2(\mathbb{T})$ be real-valued. Then, for all co-prime $p, q \in \mathbb{N}$,*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{m=0}^{q-1} \mathrm{Re}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i\frac{km+pm^2}{q}}(I + i\mathcal{H})f\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset $\mathcal{K} \subset \mathbb{R}$ with complement of measure 0, such that the following holds true. If $f \in \mathrm{BV}(\mathbb{T})$, then*

$$u(\cdot, t) \in \bigcup_{\epsilon>0} \mathrm{C}^{\frac{1}{2}-\epsilon}(\mathbb{T})$$

for all $t \in \mathcal{K}$. Moreover, if additionally $f \in \mathrm{H}^{\frac{1}{2}}(\mathbb{T})$ but

$$f \notin \bigcap_{\epsilon>0} \mathrm{H}^{\frac{1}{2}+\epsilon}(\mathbb{T}),$$

then the graph of $u(\cdot, t)$ has fractal dimension $\frac{3}{2}$ for almost all $t \in \mathcal{K}$.

The first statement shows that, if f has a jump discontinuity, then the solution at times $t \in 2\pi\mathrm{Q}$ will have logarithmic singularities. See the Figure 4. By contrast, quite remarkably, the solution is continuous for almost all other t. A contrast between the Schrödinger and this equation, is the fact that, for all initial data of bounded variation, the solution to (A) is bounded but this is not the case for the linear BO equation. I suspect that there is an interesting structural connection between the two, using BMO spaces.

Problem 15. *Show that*

$$\mathcal{H}\mathbb{1}_{[a,b]}(x) = \frac{1}{\pi}\log\left|\frac{\sin\left(\frac{x-a}{2}\right)}{\sin\left(\frac{x-b}{2}\right)}\right|,$$

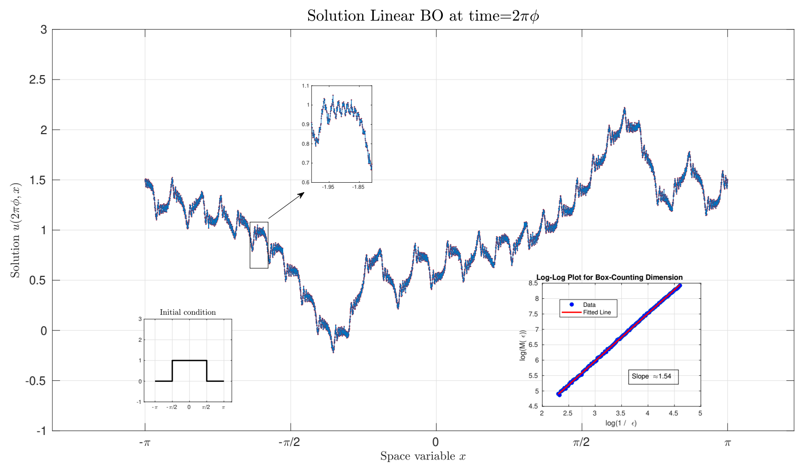

FIGURE 5. Solution for $\frac{t}{2\pi}$ a rational approximation of $\phi \sim \frac{p}{q}$ for $p = F_{16} = 2584$ and $q = F_{15} = 1597$. Note that $|\phi - \frac{p}{q}| < 1.7\times10^{-6}$. The estimate of the box counting dimension is $D = 1.54$.

for $a, b \in \mathbb{T}$ *with* $-\pi \leq a < b < \pi$.

5.1. **Proof of Theorem D.** Firstly, note that

$$\mathcal{H}e_n = \begin{cases} i & \text{for } n < 0 \\ 0 & \text{for } n = 0 \\ -i & \text{for } n > 0. \end{cases}$$

Problem 16. *Compute* $\mathcal{H}e_n$ *to verify the previous claim.*

Then, $\mathcal{H}$ and $-\partial_x^2$ have the same orthonormal basis of eigenfunctions. Hence,

$$\mathcal{H}g(x) = i\sum_{n=1}^{\infty}[\hat{g}(-n)\mathrm{e}^{-inx} - \hat{g}(n)\mathrm{e}^{inx}] \tag{14}$$

for all $g \in \mathrm{L}^2(\mathbb{T})$ and

$$\mathcal{H}\partial_x^2 g(x) = i\sum_{n=1}^{\infty} n^2[\hat{g}(n)\mathrm{e}^{inx} - \hat{g}(-n)\mathrm{e}^{-inx}]$$

for all $g \in \mathrm{H}^2(\mathbb{T})$. In fact the latter is the domain of the integro-differential operator $\mathcal{H}\partial_x^2$. This implies that, for any $f \in \mathrm{L}^2(\mathbb{T})$, the solution to (D) is given by the expression

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{inx}\mathrm{e}^{in|n|t}\widehat{f}(n) \\ &= \widehat{f}(0) + \sum_{n=1}^{\infty}[\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n) + \mathrm{e}^{-inx}\mathrm{e}^{-in^2t}\widehat{f}(-n)]. \end{aligned} \tag{15}$$

Since f is real-valued, $\widehat{f}(-n) = \overline{\widehat{f}(n)}$ and so

$$\mathrm{e}^{-inx}\mathrm{e}^{-in^2t}\widehat{f}(-n) = \overline{\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n)}.$$

Hence, u is also real-valued and

$$u(x,t) = \widehat{f}(0) + 2\,\mathrm{Re}\left[\sum_{n=1}^{\infty} \mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n)\right]. \tag{16}$$

This representation provides the link between the solutions of (D) and (A), as stated in the next lemma.

Lemma 6. *Let $f \in \mathrm{L}^2(\mathbb{T})$ be real-valued. Let $v(x,t)$ denote the solution to* (A) *with initial datum f. Then, the solutions to* (D)*, is given by the expression*

$$u(x,t) = \mathrm{Re}\left[v(x,-t) + i\mathcal{H}v(x,t)\right]. \tag{17}$$

Proof. Since $\overline{\widehat{f}(n)} = \widehat{f}(-n)$, the solution to (A) with datum f is given by

$$v(x,-t) = \widehat{f}(0) + \sum_{n=1}^{\infty}[\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n) + \mathrm{e}^{-inx}\mathrm{e}^{in^2t}\overline{\widehat{f}(n)}].$$

Here note that we have changed the sign of t. Then, using (14), we have

$$\sum_{n=1}^{\infty} \mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n) = \frac{v(x,-t) - \widehat{f}(0) + i\mathcal{H}v(x,t)}{2}.$$

Replacing the sum of this expression and of its conjugate into equation (16) gives the relation (17). □

Note that the expression (17) can be written in operator form as

$$\mathrm{e}^{\mathcal{H}\partial_x^2 t}f = \widehat{f}(0) + 2\,\mathrm{Re}\left(\mathrm{e}^{-i\partial_x^2 t}\Pi f - \widehat{f}(0)\right) = 2\,\mathrm{Re}\left(\Pi\mathrm{e}^{-i\partial_x^2 t}f\right) - \widehat{f}(0), \tag{18}$$

where $\Pi f(x) = \sum_{n=0}^{\infty}\hat{f}(n)\mathrm{e}^{inx}$ is the *Szegö projector*. Indeed, the latter commutes with both $\mathcal{H}$ and $-\partial_x^2$.

The combination of Lemma 6 and Theorem A, gives Theorem D as follows.

Proof of Theorem D-b). Consider the first part, first. The periodic distribution

$$\tilde{E}_t = \sum_{n=1}^{\infty}\left[\mathrm{e}^{inx+in^2t} + \mathrm{e}^{-inx-in^2t}\right]$$

gives the solution to (D), via $u = (\tilde{E}_t * f)$. From here, the proof is verbatim the proof of the first statement in Theorem A-b), except for the change in the sign of the exponent in the negative Fourier coefficients of E_t. This also shows that the set $\mathcal{K}$ of both theorems is the same.

For the second part we proceed as follows. By hypothesis, $f \notin \mathrm{H}^{r_0}(\mathbb{T})$ for some $r_0 \in [\frac{1}{2}, 1)$. Since f is real-valued, then $f = \Pi f + \overline{\Pi f} - \widehat{f}(0)$. Hence, $\Pi f \notin \mathrm{H}^{r_0}(\mathbb{T})$. Thus, according to part b) of Theorem A, for almost all $t \in \mathcal{K}$,

$$\mathrm{Re}\left(\mathrm{e}^{-i\partial_x^2 t}\Pi f\right) \notin \bigcup_{r>r_0} \mathrm{H}^r(\mathbb{T}).$$

Hence, by virtue of (18), we have

$$u(\cdot,t)+\widehat{f}(0)=2\,\mathrm{Re}\left(\mathrm{e}^{-i\partial_x^2 t}\Pi f\right)\notin\bigcup_{r>r_0}\mathrm{H}^r(\mathbb{T}),$$

for all such t. $\square$

Problem 17. *Complete the proof of Theorem D-a).*

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations.* Springer-Verlag, Berlin, 2011.

[2] L. Boulton, G. Farmakis, and B. Pelloni. Beyond periodic revivals for linear dispersive PDEs. *Proceedings of the Royal Society A*, 477:10241, 2021.

[3] L. Boulton, G. Farmakis, and B. Pelloni. The phenomenon of revivals on complex potential Schrödinger's equation. *Zeitschrift für Analysis und ihre Anwendungen*, 43:401–416, 2024.

[4] V. Chousionis, M.B. Erdoğan, and N. Tzirakis. Fractal solutions of linear and nonlinear dispersive partial differential equations. *Proceedings of the London Mathematical Society*, 110(3):543–564, 2014.

[5] A. Deliu and B. Jawerth. Geometrical dimension versus smoothness. *Constructive Approximation*, 8:211–222, 1992.

[6] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley & Sons, New York, 1990.

[7] G. Farmakis. *Revivals in Time Evolution Problems.* PhD Thesis, Heriot-Watt University, Edinburgh, 2022.

[8] H. Fiedler, W. Jurkat, and O. Körner. Asymptotic expansions of finite theta series. *Acta Arithmetica*, 32(2):129–146, 1977.

[9] G. Folland. *Fourier Analysis and its Applications.* Brooks/Cole Publishing Company, California, 1992.

[10] L. Grafakos. *Classical Fourier Analysis.* Springer Verlag, New York, 2008.

[11] L. Kapitanski and I. Rodnianski. Does a quantum particle know the time? In *Emerging applications of number theory*, pages 355–371. Springer, 1999.

[12] A.YA. Khinchin. *Continued Fractions.* The University of Chicago Press, Chicago and London, 1964.

[13] G. Leoni. *A First Course in Sobolev Spaces.* The American Mathematical Society, Providence, Rhode Island, 2009.

[14] P.J. Olver. Dispersive quantization. *The American Mathematical Monthly*, 117(7):599–610, 2010.

[15] F. Riesz and B. Sz.-Nagy. *Functional Analysis.* Dover Publications, New York, 1990.

[16] I. Rodnianski. Fractal solutions of the Schrödinger equation. *Contemporary Mathematics*, 255:181–188, 2000.

[17] M. Taylor. The Schrödinger equation on spheres. *Pacific journal of mathematics*, 209(1):145–155, 2003.

[18] A.C. Zemanian. *Distribution Theory and Transform Analysis.* Dover Publications, New York, 1987.